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Research Article

Fuzzy Stability of Quadratic Functional Equations

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The fuzzy stability problems for the Cauchy additive functional equation and the Jensen additive functional equation in fuzzy Banach spaces have been investigated by Moslehian et al. In this paper, we prove the generalized Hyers-Ulam stability of the following quadratic functional equations $f(x+y)+f(x-y)=2f(x)+2f(y)$ and $f(ax+by)+f(ax-by)=2a^2f(x)+2b^2f(y)$ $(a,b\in\mathbb{R}\backslash\{0\}, a\neq\pm1)$ in fuzzy Banach spaces.

1. Introduction and Preliminaries

Katsaras [1] defined a fuzzy norm on a vector space to construct a fuzzy vector topological structure on the space. Some mathematicians have defined fuzzy norms on a vector space from various points of view [2–4]. In particular, Bag and Samanta [5], following Cheng and Mordeson [6], gave an idea of fuzzy norm in such a manner that the corresponding fuzzy metric is of Kramosil and Michálek type [7]. They established a decomposition theorem of a fuzzy norm into a family of crisp norms and investigated some properties of fuzzy normed spaces [8].

We use the definition of fuzzy normed spaces given in [5, 9, 10] to investigate a fuzzy version of the generalized Hyers-Ulam stability for the quadratic functional equations

$$f(x+y)+f(x-y)=2f(x)+2f(y), \tag{1.1}$$
$$f(ax+by)+f(ax-by)=2a^2f(x)+2b^2f(y) \tag{1.2}$$

in the fuzzy normed vector space setting, where a,b are nonzero real numbers with $a\neq\pm1$.

Definition 1.1 (see [5, 9, 10]). Let X be a real vector space. A function $N : X \times \mathbb{R} \to [0,1]$ is called a *fuzzy norm* on X if, for all $x, y \in X$ and all $s, t \in \mathbb{R}$,

(N_1) $N(x,t) = 0$ for $t \leq 0$,

(N_2) $x = 0$ if and only if $N(x,t) = 1$ for all $t > 0$,

(N_3) $N(cx,t) = N(x, t/|c|)$ if $c \neq 0$,

(N_4) $N(x+y, s+t) \geq \min\{N(x,s), N(y,t)\}$,

(N_5) $N(x,\cdot)$ is a nondecreasing function of $\mathbb{R}$ and $\lim_{t\to\infty} N(x,t) = 1$,

(N_6) for $x \neq 0$, $N(x,\cdot)$ is continuous on $\mathbb{R}$.

The pair (X, N) is called a *fuzzy normed vector space*.

The properties of fuzzy normed vector spaces and examples of fuzzy norms are given in [9, 10].

Definition 1.2 (see [5, 9, 10]). Let (X, N) be a fuzzy normed vector space. A sequence $\{x_n\}$ in X is said to *be convergent* or *converges* if there exists an $x \in X$ such that $\lim_{n\to\infty} N(x_n - x, t) = 1$ for all $t > 0$. In this case, x is called the *limit* of the sequence $\{x_n\}$ and we denote it by N-$\lim_{n\to\infty} x_n = x$.

Definition 1.3 (see [5, 9, 10]). Let (X, N) be a fuzzy normed vector space. A sequence $\{x_n\}$ in X is called *Cauchy* if for each $\varepsilon > 0$ and each $t > 0$ there exists an $n_0 \in \mathbb{N}$ such that, for all $n \geq n_0$ and all $p > 0$, we have $N(x_{n+p} - x_n, t) > 1 - \varepsilon$.

It is well known that every convergent sequence in a fuzzy normed vector space is Cauchy. If each Cauchy sequence is convergent, then the fuzzy norm is said to be *complete* and the fuzzy normed vector space is called a *fuzzy Banach space*.

We say that a mapping $f : X \to Y$ between fuzzy normed vector spaces X and Y is continuous at a point $x_0 \in X$ if, for each sequence $\{x_n\}$ converging to x_0 in X, the sequence $\{f(x_n)\}$ converges to $f(x_0)$. If $f : X \to Y$ is continuous at each $x \in X$, then $f : X \to Y$ is said to be *continuous* on X (see [8]).

The stability problem of functional equations is originated from a question of Ulam [11] concerning the stability of group homomorphisms. Hyers [12] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' theorem was generalized by Aoki [13] for additive mappings and by Th. M. Rassias [14] for linear mappings by considering an unbounded Cauchy difference. The paper of Th. M. Rassias [14] has provided a lot of influence in the development of what we call *generalized Hyers-Ulam stability* of functional equations. A generalization of the Th. M. Rassias theorem was obtained by Găvruţa [15] by replacing the unbounded Cauchy difference by a general control function in the spirit of Th. M. Rassias' approach.

A square norm on an inner product space satisfies the parallelogram equality

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2. \tag{1.3}$$

The functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y) \tag{1.4}$$

is called a *quadratic functional equation*. In particular, every solution of the quadratic functional equation is said to be a *quadratic mapping*. A generalized Hyers-Ulam stability problem for the quadratic functional equation was proved by Skof [16] for mappings $f : X \to Y$, where X is a normed space and Y is a Banach space. Cholewa [17] noticed that the theorem of Skof is still true if the relevant domain X is replaced by an Abelian group. In [18], Czerwik proved the generalized Hyers-Ulam stability of the quadratic functional equation. During the last two decades, a number of papers and research monographs have been published on various generalizations and applications of the generalized Hyers-Ulam stability to a number of functional equations and mappings (see [19–31]).

This paper is organized as follows. In Section 2, we prove the generalized Hyers-Ulam stability of the quadratic functional equation (1.1) in fuzzy Banach spaces. In Section 3, we prove the generalized Hyers-Ulam stability of the quadratic functional equation (1.2) in fuzzy Banach spaces.

Throughout this paper, assume that X is a vector space and that (Y, N) is a fuzzy Banach space. Let a, b be nonzero real numbers with $a \neq \pm 1$.

2. Generalized Hyers-Ulam Stability of the Quadratic Functional Equation (1.1)

In this section, we prove the generalized Hyers-Ulam stability of the quadratic functional equation (1.1) in fuzzy Banach spaces.

Theorem 2.1. *Let $\varphi : X^2 \to [0, \infty)$ be a function such that*

$$\tilde{\varphi}(x, y) := \sum_{n=0}^{\infty} 4^{-n} \varphi(2^n x, 2^n y) < \infty \tag{2.1}$$

for all $x, y \in X$. Let $f : X \to Y$ be a mapping with $f(0) = 0$ such that

$$\lim_{t \to \infty} N(f(x+y) + f(x-y) - 2f(x) - 2f(y), t\varphi(x, y)) = 1 \tag{2.2}$$

uniformly on $X \times X$. Then $Q(x) := N\text{-}\lim_{n \to \infty}(f(2^n x)/4^n)$ exists for each $x \in X$ and defines a quadratic mapping $Q : X \to Y$ such that if for some $\delta > 0$, $\alpha > 0$

$$N(f(x+y) + f(x-y) - 2f(x) - 2f(y), \delta\varphi(x, y)) \geq \alpha \tag{2.3}$$

for all $x, y \in X$, then

$$N\left(f(x) - Q(x), \frac{\delta}{4}\tilde{\varphi}(x, x)\right) \geq \alpha \tag{2.4}$$

for all $x \in X$.

Furthermore, the quadratic mapping $Q : X \to Y$ is a unique mapping such that

$$\lim_{t\to\infty} N\left(f(x) - Q(x), t\widetilde{\varphi}(x,x)\right) = 1 \tag{2.5}$$

uniformly on X.

Proof. For a given $\varepsilon > 0$, by (2.2), we can find some $t_0 > 0$ such that

$$N\left(f(x+y) + f(x-y) - 2f(x) - 2f(y), t\varphi(x,y)\right) \geq 1 - \varepsilon \tag{2.6}$$

for all $t \geq t_0$. By induction on n, we show that

$$N\left(4^n f(x) - f(2^n x), t\sum_{k=0}^{n-1} 4^{n-k-1}\varphi\left(2^k x, 2^k x\right)\right) \geq 1 - \varepsilon \tag{2.7}$$

for all $t \geq t_0$, all $x \in X$, and all $n \in \mathbb{N}$.

Letting $y = x$ in (2.6), we get

$$N\left(4f(x) - f(2x), t\varphi(x,x)\right) \geq 1 - \varepsilon \tag{2.8}$$

for all $x \in X$ and all $t \geq t_0$. So we get (2.7) for $n = 1$.

Assume that (2.7) holds for $n \in \mathbb{N}$. Then

$$\begin{aligned}
&N\left(4^{n+1} f(x) - f\left(2^{n+1}x\right), t\sum_{k=0}^{n} 4^{n-k}\varphi\left(2^k x, 2^k x\right)\right) \\
&\quad \geq \min\left\{N\left(4^{n+1} f(x) - 4f(2^n x), t_0\sum_{k=0}^{n-1} 4^{n-k}\varphi\left(2^k x, 2^k x\right)\right),\right. \\
&\qquad\qquad \left. N\left(4f(2^n x) - f\left(2^{n+1}x\right), t_0\varphi(2^n x, 2^n x)\right)\right\} \\
&\quad \geq \min\{1-\varepsilon, 1-\varepsilon\} = 1 - \varepsilon.
\end{aligned} \tag{2.9}$$

This completes the induction argument. Letting $t = t_0$ and replacing n and x by p and $2^n x$ in (2.7), respectively, we get

$$N\left(\frac{f(2^n x)}{4^n} - \frac{f(2^{n+p}x)}{4^{n+p}}, \frac{t_0}{4^{n+p}}\sum_{k=0}^{p-1} 4^{p-k-1}\varphi\left(2^{n+k}x, 2^{n+k}x\right)\right) \geq 1 - \varepsilon \tag{2.10}$$

for all integers $n \geq 0$, $p > 0$.

It follows from (2.1) and the equality

$$\sum_{k=0}^{p-1} 4^{-n-k-1} \varphi\left(2^{n+k}x, 2^{n+k}x\right) = \frac{1}{4} \sum_{k=n}^{n+p-1} 4^{-k} \varphi\left(2^k x, 2^k x\right) \tag{2.11}$$

that for a given $\delta > 0$ there is an $n_0 \in \mathbb{N}$ such that

$$\frac{t_0}{2} \sum_{k=n}^{n+p-1} 4^{-k} \varphi\left(2^k x, 2^k x\right) < \delta \tag{2.12}$$

for all $n \geq n_0$ and $p > 0$. Now we deduce from (2.10) that

$$\begin{aligned} N\left(\frac{f(2^n x)}{4^n} - \frac{f(2^{n+p}x)}{4^{n+p}}, \delta\right) &\geq N\left(\frac{f(2^n x)}{4^n} - \frac{f(2^{n+p}x)}{4^{n+p}}, \frac{t_0}{4^{n+p}} \sum_{k=0}^{p-1} 4^{p-k-1} \varphi\left(2^{n+k}x, 2^{n+k}x\right)\right) \\ &\geq 1 - \varepsilon \end{aligned} \tag{2.13}$$

for all $n \geq n_0$ and all $p > 0$. Thus the sequence $\{f(2^n x)/4^n\}$ is Cauchy in Y. Since Y is a fuzzy Banach space, the sequence $\{f(2^n x)/4^n\}$ converges to some $Q(x) \in Y$. So we can define a mapping $Q : X \to Y$ by $Q(x) := N\text{-}\lim_{n\to\infty}(f(2^n x)/4^n)$; namely, for each $t > 0$ and $x \in X$, $\lim_{n\to\infty} N(f(2^n x)/4^n - Q(x), t) = 1$.

Let $x, y \in X$. Fix $t > 0$ and $0 < \varepsilon < 1$. Since $\lim_{n\to\infty} 4^{-n}\varphi(2^n x, 2^n y) = 0$, there is an $n_1 > n_0$ such that $t_0\varphi(2^n x, 2^n y) < 4^n t/4$ for all $n \geq n_1$. Hence for all $n \geq n_1$, we have

$$\begin{aligned} &N(Q(x+y) + Q(x-y) - 2Q(x) - 2Q(y), t) \\ &\quad\geq \min\left\{ N\left(Q(x+y) - 4^{-n} f(2^n x + 2^n y), \frac{t}{8}\right), N\left(Q(x-y) - 4^{-n} f(2^n x - 2^n y), \frac{t}{8}\right), \right. \\ &\qquad N\left(2Q(x) - 4^{-n} \cdot 2f(2^n x), \frac{t}{4}\right), N\left(2Q(y) - 4^{-n} \cdot 2f(2^n y), \frac{t}{4}\right), \\ &\qquad \left. N\left(f(2^n(x+y)) - f(2^n(x-y)) - 2f(2^n x) - 2f(2^n y), \frac{4^n t}{4}\right)\right\}. \end{aligned} \tag{2.14}$$

The first four terms on the right-hand side of the above inequality tend to 1 as $n \to \infty$, and the fifth term is greater than

$$N(f(2^n(x+y)) + f(2^n(x-y)) - 2f(2^n x) - 2f(2^n y), t_0\varphi(2^n x, 2^n y)), \tag{2.15}$$

which is greater than or equal to $1 - \varepsilon$. Thus

$$N(Q(x+y) + Q(x-y) - 2Q(x) - 2Q(y), t) \geq 1 - \varepsilon \tag{2.16}$$

for all $t > 0$. Since $N(Q(x+y)+Q(x-y)-2Q(x)-2Q(y),t) = 1$ for all $t > 0$, by (N_2), $Q(x+y)+Q(x-y)-2Q(x)-2Q(y) = 0$ for all $x \in X$. Thus the mapping $Q : X \to Y$ is quadratic, that is, $Q(x+y)+Q(x-y) = 2Q(x)+2Q(y)$ for all $x, y \in X$.

Now let, for some positive δ and α, (2.3) hold. Let

$$\varphi_n(x,y) := \sum_{k=0}^{n-1} 4^{-k-1}\varphi(2^n x, 2^n y) \tag{2.17}$$

for all $x, y \in X$. Let $x \in X$. By the same reasoning as in the beginning of the proof, one can deduce from (2.3) that

$$N\left(4^n f(x) - f(2^n x), \delta\sum_{k=0}^{n-1} 4^{n-k-1}\varphi\left(2^k x, 2^k x\right)\right) \geq \alpha \tag{2.18}$$

for all positive integers n. Let $t > 0$. We have

$$N(f(x)-Q(x), \delta\varphi_n(x,x)+t) \geq \min\left\{N\left(f(x) - \frac{f(2^n x)}{4^n}, \delta\varphi_n(x,x)\right), N\left(\frac{f(2^n x)}{4^n} - Q(x), t\right)\right\}. \tag{2.19}$$

Combining (2.18) and (2.19) and the fact that $\lim_{n\to\infty} N(f(2^n x)/4^n - Q(x), t) = 1$, we observe that

$$N(f(x) - Q(x), \delta\varphi_n(x,x) + t) \geq \alpha \tag{2.20}$$

for large enough $n \in \mathbb{N}$. Thanks to the continuity of the function $N(f(x) - Q(x), \cdot)$, we see that $N(f(x) - Q(x), (\delta/4)\widetilde{\varphi}(x,x) + t) \geq \alpha$. Letting $t \to 0$, we conclude that

$$N\left(f(x) - Q(x), \frac{\delta}{4}\widetilde{\varphi}(x,x)\right) \geq \alpha. \tag{2.21}$$

To end the proof, it remains to prove the uniqueness assertion. Let T be another quadratic mapping satisfying (2.5). Fix $c > 0$. Given that $\varepsilon > 0$, by (2.5) for Q and T, we can find some $t_0 > 0$ such that

$$\begin{aligned} N\left(f(x) - Q(x), \frac{t}{2}\widetilde{\varphi}(x,x)\right) &\geq 1 - \varepsilon, \\ N\left(f(x) - T(x), \frac{t}{2}\widetilde{\varphi}(x,x)\right) &\geq 1 - \varepsilon \end{aligned} \tag{2.22}$$

for all $x \in X$ and all $t \geq 2t_0$. Fix some $x \in X$ and find some integer n_0 such that

$$t_0 \sum_{k=n}^{\infty} 4^{-k}\varphi\left(2^k x, 2^k x\right) < \frac{c}{2} \tag{2.23}$$

for all $n \geq n_0$. Since

$$\begin{aligned}
\sum_{k=n}^{\infty} 4^{-k}\varphi\left(2^k x, 2^k x\right) &= \frac{1}{4^n}\sum_{k=n}^{\infty} 4^{-(k-n)}\varphi\left(2^{k-n}(2^n x), 2^{k-n}(2^n x)\right) \\
&= \frac{1}{4^n}\sum_{m=0}^{\infty} 4^{-m}\varphi(2^m(2^n x), 2^m(2^n x)) \\
&= \frac{1}{4^n}\widetilde{\varphi}(2^n x, 2^n x),
\end{aligned} \tag{2.24}$$

we have

$$\begin{aligned}
&N(Q(x) - T(x), c) \\
&\quad \geq \min\left\{N\left(\frac{f(2^n x)}{4^n} - Q(x), \frac{c}{2}\right), N\left(T(x) - \frac{f(2^n x)}{4^n}, \frac{c}{2}\right)\right\} \\
&\quad = \min\left\{N\left(f(2^n x) - Q(2^n x), 4^{n-1}2c\right), N\left(T(2^n x) - f(2^n x), 4^{n-1}2c\right)\right\} \\
&\quad \geq \min\left\{N\left(f(2^n x) - Q(2^n x), 4^n t_0\sum_{k=n}^{\infty} 4^{-k}\varphi\left(2^k x, 2^k x\right)\right),\right. \\
&\qquad\qquad \left. N\left(T(2^n x) - f(2^n x), 4^n t_0\sum_{k=n}^{\infty} 4^{-k}\varphi\left(2^k x, 2^k x\right)\right)\right\} \\
&\quad = \min\{N(f(2^n x) - Q(2^n x), t_0\widetilde{\varphi}(2^n x, 2^n x)), N(T(2^n x) - f(2^n x), t_0\widetilde{\varphi}(2^n x, 2^n x))\} \\
&\quad \geq 1 - \varepsilon.
\end{aligned} \tag{2.25}$$

It follows that $N(Q(x) - T(x), c) = 1$ for all $c > 0$. Thus $Q(x) = T(x)$ for all $x \in X$. □

Corollary 2.2. *Let $\theta \geq 0$ and let p be a real number with $0 < p < 2$. Let $f : X \to Y$ be a mapping with $f(0) = 0$ such that*

$$\lim_{t\to\infty} N(f(x+y) + f(x-y) - 2f(x) - 2f(y), t\theta(\|x\|^p + \|y\|^p)) = 1 \tag{2.26}$$

uniformly on $X \times X$. Then $Q(x) := N\text{-}\lim_{n\to\infty}(f(2^n x)/4^n)$ exists for each $x \in X$ and defines a quadratic mapping $Q : X \to Y$ such that if for some $\delta > 0$, $\alpha > 0$

$$N(f(x+y) + f(x-y) - 2f(x) - 2f(y), \delta\theta(\|x\|^p + \|y\|^p)) \geq \alpha \tag{2.27}$$

for all $x, y \in X$, then

$$N\left(f(x) - Q(x), \frac{2\delta\theta}{4 - 2^p}\|x\|^p\right) \geq \alpha \tag{2.28}$$

for all $x \in X$.

Furthermore, the quadratic mapping $Q : X \to Y$ is a unique mapping such that

$$\lim_{t\to\infty} N\left(f(x) - Q(x), \frac{8}{4-2^p} t\theta\|x\|^p\right) = 1 \tag{2.29}$$

uniformly on X.

Proof. Define $\varphi(x, y) := \theta(\|x\|^p + \|y\|^p)$ and apply Theorem 2.1 to get the result. □

Similarly, we can obtain the following. We will omit the proof.

Theorem 2.3. *Let $\varphi : X^2 \to [0,\infty)$ be a function such that*

$$\tilde{\varphi}(x, y) := \sum_{n=1}^{\infty} 4^n \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}\right) < \infty \tag{2.30}$$

for all $x, y \in X$. Let $f : X \to Y$ be a mapping satisfying (2.2) *and $f(0) = 0$. Then $Q(x) := N\text{-}\lim_{n\to\infty} 4^n f(x/2^n)$ exists for each $x \in X$ and defines a quadratic mapping $Q : X \to Y$ such that if for some $\delta > 0, \alpha > 0$*

$$N(f(x + y) + f(x - y) - 2f(x) - 2f(y), \delta\varphi(x, y)) \geq \alpha \tag{2.31}$$

for all $x, y \in X$, then

$$N\left(f(x) - Q(x), \frac{\delta}{4}\tilde{\varphi}(x, x)\right) \geq \alpha \tag{2.32}$$

for all $x \in X$.

Furthermore, the quadratic mapping $Q : X \to Y$ is a unique mapping such that

$$\lim_{t\to\infty} N(f(x) - Q(x), t\tilde{\varphi}(x, x)) = 1 \tag{2.33}$$

uniformly on X.

Corollary 2.4. *Let $\theta \geq 0$ and let p be a real number with $p > 2$. Let $f : X \to Y$ be a mapping satisfying* (2.26) *and $f(0) = 0$. Then $Q(x) := N\text{-}\lim_{n\to\infty} 4^n f(x/2^n)$ exists for each $x \in X$ and defines a quadratic mapping $Q : X \to Y$ such that if for some $\delta > 0, \alpha > 0$*

$$N(f(x + y) + f(x - y) - 2f(x) - 2f(y), \delta\theta(\|x\|^p + \|y\|^p)) \geq \alpha \tag{2.34}$$

for all $x, y \in X$, then

$$N\left(f(x) - Q(x), \frac{2\delta\theta}{2^p - 4}\|x\|^p\right) \geq \alpha \tag{2.35}$$

for all $x \in X$.

Furthermore, the quadratic mapping $Q : X \to Y$ is a unique mapping such that

$$\lim_{t\to\infty} N\left(f(x) - Q(x), \frac{8}{2^p - 4} t\theta\|x\|^p\right) = 1 \tag{2.36}$$

uniformly on X.

Proof. Define $\varphi(x, y) := \theta(\|x\|^p + \|y\|^p)$ and apply Theorem 2.3 to get the result. □

3. Generalized Hyers-Ulam Stability of the Quadratic Functional Equation (1.2)

In this section, we prove the generalized Hyers-Ulam stability of the quadratic functional equation (1.2) in fuzzy Banach spaces.

Lemma 3.1. *Let V and W be real vector spaces. If a mapping $f : V \to W$ satisfies $f(0) = 0$ and*

$$f(ax + by) + f(ax - by) = 2a^2 f(x) + 2b^2 f(y) \tag{3.1}$$

for all $x, y \in V$, then the mapping $f : V \to W$ is quadratic, that is,

$$f(x + y) + f(x - y) = 2f(x) + 2f(y) \tag{3.2}$$

for all $x, y \in V$.

Proof. Assume that $f : V \to W$ satisfies (3.1).

Letting $y = 0$ in (3.1), we get

$$2f(ax) = 2a^2 f(x) \tag{3.3}$$

for all $x \in V$.

Letting $x = 0$ in (3.1), we get

$$f(by) + f(-by) = 2b^2 f(y) \tag{3.4}$$

for all $y \in V$. Replacing y by $-y$ in (3.4), we get

$$f(-by) + f(by) = 2b^2 f(-y) \tag{3.5}$$

for all $y \in V$. It follows from (3.4) and (3.5) that $f(-y) = f(y)$ for all $y \in V$. So

$$2f(by) = 2b^2 f(y) \tag{3.6}$$

for all $y \in V$. Thus

$$f(ax+by) + f(ax-by) = 2a^2 f(x) + 2b^2 f(y) = 2f(ax) + 2f(by) \tag{3.7}$$

for all $x, y \in V$. Replacing ax and by by z and w in (3.7), respectively, we get

$$f(z+w) + f(z-w) = 2f(z) + 2f(w) \tag{3.8}$$

for all $z, w \in V$, as desired. □

Theorem 3.2. *Let $\varphi : X^2 \to [0, \infty)$ be a function such that*

$$\widetilde{\varphi}(x,0) := \sum_{n=0}^{\infty} a^{-2n} \varphi(a^n x, 0) < \infty \tag{3.9}$$

for all $x \in X$. Let $f : X \to Y$ be a mapping with $f(0) = 0$ such that

$$\lim_{t \to \infty} N\Big(f(ax+by) + f(ax-by) - 2a^2 f(x) - 2b^2 f(y), t\varphi(x,y)\Big) = 1 \tag{3.10}$$

uniformly on $X \times X$. Then $Q(x) := N\text{-}\lim_{n\to\infty}(f(a^n x)/a^{2n})$ exists for each $x \in X$ and defines a quadratic mapping $Q : X \to Y$ such that if for some $\delta > 0, \alpha > 0$

$$N\Big(f(ax+by) + f(ax-by) - 2a^2 f(x) - 2b^2 f(y), \delta\varphi(x,y)\Big) \ge \alpha \tag{3.11}$$

for all $x, y \in X$, then

$$N\left(f(x) - Q(x), \frac{\delta}{a^2}\widetilde{\varphi}(x,0)\right) \ge \alpha \tag{3.12}$$

for all $x \in X$.

Furthermore, the quadratic mapping $Q : X \to Y$ is a unique mapping such that

$$\lim_{t\to\infty} N(f(x) - Q(x), t\widetilde{\varphi}(x,0)) = 1 \tag{3.13}$$

uniformly on X.

Proof. For a given $\varepsilon > 0$, by (3.10), we can find some $t_0 > 0$ such that

$$N\Big(f(ax+by) + f(ax-by) - 2a^2 f(x) - 2b^2 f(y), t\varphi(x,y)\Big) \ge 1 - \varepsilon \tag{3.14}$$

for all $t \geq 2t_0$. By induction on n, we show that

$$N\left(a^{2n} f(x) - f(a^n x), \frac{t}{2}\sum_{k=0}^{n-1} a^{2n-2k-2}\varphi\left(a^k x, 0\right)\right) \geq 1 - \varepsilon \tag{3.15}$$

for all $t \geq 2t_0$, all $x \in X$, and all $n \in \mathbb{N}$.

Letting $y = 0$ in (3.14), we get

$$N\left(2f(ax) - 2a^2 f(x), t\varphi(x, 0)\right) \geq 1 - \varepsilon \tag{3.16}$$

for all $x \in X$ and all $t \geq 2t_0$. So we get (3.15) for $n = 1$.

Assume that (3.15) holds for $n \in \mathbb{N}$. Then

$$\begin{aligned} N&\left(a^{2n+2} f(x) - f\left(a^{n+1} x\right), \frac{t}{2}\sum_{k=0}^{n} a^{2n-2k}\varphi\left(a^k x, 0\right)\right) \\ &\geq \min\left\{N\left(a^{2n+2} f(x) - a^2 f(a^n x), t_0\sum_{k=0}^{n-1} a^{2n-2k}\varphi(a^n x, 0)\right),\right. \\ &\qquad\qquad \left. N\left(a^2 f(a^n x) - f\left(a^{n+1} x\right), t_0\varphi(a^n x, 0)\right)\right\} \\ &\geq \min\{1 - \varepsilon, 1 - \varepsilon\} = 1 - \varepsilon. \end{aligned} \tag{3.17}$$

This completes the induction argument. Letting $t = t_0$ and replacing n and x by p and $a^n x$ in (3.15), respectively, we get

$$N\left(\frac{f(a^n x)}{a^{2n}} - \frac{f(a^{n+p} x)}{a^{2n+2p}}, \frac{t_0}{a^{2n+2p}}\sum_{k=0}^{p-1} a^{2p-2k-2}\varphi\left(a^{n+k} x, 0\right)\right) \geq 1 - \varepsilon \tag{3.18}$$

for all integers $n \geq 0, p > 0$.

It follows from (3.9) and the equality

$$\sum_{k=0}^{p-1} a^{-2n-2k-2}\varphi\left(a^{n+k} x, 0\right) = \frac{1}{a^2}\sum_{k=n}^{n+p-1} a^{-2k}\varphi\left(a^k x, 0\right) \tag{3.19}$$

that for a given $\delta > 0$ there is an $n_0 \in \mathbb{N}$ such that

$$\frac{t_0}{a^2}\sum_{k=n}^{n+p-1} a^{-2k}\varphi\left(a^k x, 0\right) < \delta \tag{3.20}$$

for all $n \geq n_0$ and $p > 0$. Now we deduce from (3.18) that

$$N\left(\frac{f(a^n x)}{a^{2n}} - \frac{f(a^{n+p}x)}{a^{2n+2p}}, \delta\right) \geq N\left(\frac{f(a^n x)}{a^{2n}} - \frac{f(a^{n+p}x)}{a^{2n+2p}}, \frac{t_0}{a^{2n+2p}}\sum_{k=0}^{p-1} a^{2p-2k-2}\varphi\left(a^{n+k}x, 0\right)\right)$$
$$\geq 1 - \varepsilon \tag{3.21}$$

for each n $\geq n_0$ and all $p > 0$. Thus the sequence $\{f(a^n x)/a^{2n}\}$ is Cauchy in Y. Since Y is a fuzzy Banach space, the sequence $\{f(a^n x)/a^{2n}\}$ converges to some $Q(x) \in Y$. So we can define a mapping $Q : X \to Y$ by $Q(x) := N\text{-}\lim_{n\to\infty}(f(a^n x)/a^{2n})$; namely, for each $t > 0$ and $x \in X$, $\lim_{n\to\infty} N(f(a^n x)/a^{2n} - Q(x), t) = 1$.

Let $x, y \in X$. Fix $t > 0$ and $0 < \varepsilon < 1$. Since $\lim_{n\to\infty} a^{-2n}\varphi(a^n x, 0) = 0$, there is an $n_1 > n_0$ such that $t_0\varphi(a^n x, 0) < a^{2n}t/4$ for all $n \geq n_1$. Hence for each $n \geq n_1$, we have

$$\begin{aligned}
&N\left(Q(ax+by) + Q(ax-by) - 2a^2Q(x) - 2b^2Q(y), t\right)\\
&\quad\geq \min\Bigg\{N\left(Q(ax+by) - a^{-2n}f(a^n \cdot ax + a^n by), \frac{t}{8}\right),\\
&\qquad N\left(Q(ax-by) - a^{-2n}f(a^n \cdot ax - a^n by), \frac{t}{8}\right),\\
&\qquad N\left(2a^2Q(x) - a^{-2n} \cdot 2a^2 f(a^n x), \frac{t}{4}\right), N\left(2b^2Q(y) - a^{-2n} \cdot 2b^2 f(a^n y), \frac{t}{4}\right),\\
&\qquad N\left(f(a^n(ax+by)) - f(a^n(ax-by)) - 2a^2 f(a^n x) - 2b^2 f(a^n y), \frac{a^{2n}t}{4}\right)\Bigg\}.
\end{aligned} \tag{3.22}$$

The first four terms on the right-hand side of the above inequality tend to 1 as $n \to \infty$, and the fifth term is greater than

$$N\left(f(a^n(ax+by)) + f(a^n(ax-by)) - 2a^2 f(a^n x) - 2b^2 f(a^n y), t_0\varphi(a^n x, 0)\right), \tag{3.23}$$

which is greater than or equal to $1 - \varepsilon$. Thus

$$N\left(Q(ax+by) + Q(ax-by) - 2a^2Q(x) - 2b^2Q(y), t\right) \geq 1 - \varepsilon \tag{3.24}$$

for all $t > 0$. Since $N(Q(ax+by) + Q(ax-by) - 2a^2Q(x) - 2b^2Q(y), t) = 1$ for all $t > 0$, by (N_2), $Q(ax+by) + Q(ax-by) - 2a^2Q(x) - 2b^2Q(y) = 0$ for all $x \in X$. By Lemma 3.1, the mapping $Q : X \to Y$ is quadratic.

Now let, for some positive δ and α, (3.18) hold. Let

$$\varphi_n(x, 0) := \sum_{k=0}^{n-1} a^{-2k-2}\varphi\left(a^k x, 0\right) \tag{3.25}$$

for all $x \in X$. Let $x \in X$. By the same reasoning as in the beginning of the proof, one can deduce from (3.18) that

$$N\left(a^{2n} f(x) - f(a^n x), \delta \sum_{k=0}^{n-1} a^{2n-2k-2} \varphi\left(a^k x, 0\right)\right) \geq \alpha \tag{3.26}$$

for all positive integers n. Let $t > 0$. We have

$$N(f(x) - Q(x), \delta\varphi_n(x,0) + t) \geq \min\left\{N\left(f(x) - \frac{f(a^n x)}{a^{2n}}, \delta\varphi_n(x,0)\right), N\left(\frac{f(a^n x)}{a^{2n}} - Q(x), t\right)\right\}. \tag{3.27}$$

Combining (3.26) and (3.27) and the fact that $\lim_{n\to\infty} N(f(a^n x)/a^{2n} - Q(x), t) = 1$, we observe that

$$N(f(x) - Q(x), \delta\varphi_n(x,0) + t) \geq \alpha \tag{3.28}$$

for large enough $n \in \mathbb{N}$. Thanks to the continuity of the function $N(f(x) - Q(x), \cdot)$, we see that $N(f(x) - Q(x), (\delta/a^2)\widetilde{\varphi}(x,0) + t) \geq \alpha$. Letting $t \to 0$, we conclude that

$$N\left(f(x) - Q(x), \frac{\delta}{a^2}\widetilde{\varphi}(x,0)\right) \geq \alpha. \tag{3.29}$$

To end the proof, it remains to prove the uniqueness assertion. Let T be another quadratic mapping satisfying (3.1) and (3.13). Fix $c > 0$. Given that $\varepsilon > 0$, by (3.13) for Q and T, we can find some $t_0 > 0$ such that

$$\begin{aligned} &N\left(f(x) - Q(x), \frac{t}{2}\widetilde{\varphi}(x,0)\right) \geq 1 - \varepsilon, \\ &N\left(f(x) - T(x), \frac{t}{2}\widetilde{\varphi}(x,0)\right) \geq 1 - \varepsilon \end{aligned} \tag{3.30}$$

for all $x \in X$ and all $t \geq 2t_0$. Fix some $x \in X$ and find some integer n_0 such that

$$t_0 \sum_{k=n}^{\infty} a^{-2k} \varphi\left(a^k x, 0\right) < \frac{c}{2} \tag{3.31}$$

for all $n \geq n_0$. Since

$$\begin{aligned}\sum_{k=n}^{\infty} a^{-2k}\varphi\left(a^k x, 0\right) &= \frac{1}{a^{2n}}\sum_{k=n}^{\infty} a^{-2(k-n)}\varphi\left(a^{k-n}(a^n x), 0\right) \\ &= \frac{1}{a^{2n}}\sum_{m=0}^{\infty} a^{-2m}\varphi(a^m(a^n x), 0) \\ &= \frac{1}{a^{2n}}\widetilde{\varphi}(a^n x, 0),\end{aligned} \tag{3.32}$$

we have

$$\begin{aligned}&N(Q(x)-T(x),c) \\ &\quad\geq \min\left\{N\left(\frac{f(a^n x)}{a^{2n}} - Q(x), \frac{c}{2}\right), N\left(T(x) - \frac{f(a^n x)}{a^{2n}}, \frac{c}{2}\right)\right\} \\ &\quad= \min\left\{N\left(f(a^n x) - Q(a^n x), a^{2n-2}2c\right), N\left(T(a^n x) - f(a^n x), a^{2n-2}2c\right)\right\} \\ &\quad\geq \min\left\{N\left(f(a^n x) - Q(a^n x), a^{2n}t_0\sum_{k=n}^{\infty} a^{-2k}\varphi\left(a^k x, 0\right)\right),\right. \\ &\qquad\qquad \left. N\left(T(a^n x) - f(a^n x), a^{2n}t_0\sum_{k=n}^{\infty} a^{-2k}\varphi\left(a^k x, 0\right)\right)\right\} \\ &\quad= \min\{N(f(a^n x) - Q(a^n x), t_0\widetilde{\varphi}(a^n x, 0)), N(T(a^n x) - f(a^n x), t_0\widetilde{\varphi}(a^n x, 0))\} \\ &\quad\geq 1-\varepsilon.\end{aligned} \tag{3.33}$$

It follows that $N(Q(x)-T(x),c) = 1$ for all $c > 0$. Thus $Q(x) = T(x)$ for all $x \in X$. □

Corollary 3.3. *Let $\theta \geq 0$ and let p be a real number with $0 < p < 2$ if $|a| > 1$ and with $p > 2$ if $|a| < 1$. Let $f : X \to Y$ be a mapping with $f(0) = 0$ such that*

$$\lim_{t\to\infty} N\left(f(ax+by) + f(ax-by) - 2a^2 f(x) - 2b^2 f(y), t\theta(\|x\|^p + \|y\|^p)\right) = 1 \tag{3.34}$$

uniformly on $X \times X$. Then $Q(x) := N\text{-}\lim_{n\to\infty}(f(a^n x)/a^{2n})$ exists for each $x \in X$ and defines a quadratic mapping $Q : X \to Y$ such that if for some $\delta > 0, \alpha > 0$

$$N\left(f(ax+by) + f(ax-by) - 2a^2 f(x) - 2b^2 f(y), \delta\theta(\|x\|^p + \|y\|^p)\right) \geq \alpha \tag{3.35}$$

for all $x, y \in X$, then

$$N\left(f(x) - Q(x), \frac{\delta\theta}{a^2 - |a|^p}\|x\|^p\right) \geq \alpha \tag{3.36}$$

for all $x \in X$.

Furthermore, the quadratic mapping $Q : X \to Y$ is a unique mapping such that

$$\lim_{t\to\infty} N\left(f(x) - Q(x), \frac{a^2}{a^2 - |a|^p} t\theta\|x\|^p \right) = 1 \tag{3.37}$$

uniformly on X.

Proof. Define $\varphi(x, y) := \theta(\|x\|^p + \|y\|^p)$ and apply Theorem 3.2 to get the result. □

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