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Research Article

New Approach to q-Euler Numbers and Polynomials

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We give a new construction of the q-extensions of Euler numbers and polynomials. We present new generating functions which are related to the q-Euler numbers and polynomials. We also consider the generalized q-Euler polynomials attached to Dirichlet's character χ and have the generating functions of them. We obtain distribution relations for the q-Euler polynomials and have some identities involving q-Euler numbers and polynomials. Finally, we derive the q-extensions of zeta functions from the Mellin transformation of these generating functions, which interpolate the q-Euler polynomials at negative integers.

1. Introduction

Let $\mathbb{C}$ be the complex number field. We assume that $q \in \mathbb{C}$ with $|q| < 1$ and that the q-number is defined by $[x]_q = (1 - q^x)/(1 - q)$ in this paper.

Recently, many mathematicians have studied for q-Euler and q-Bernoulli polynomials and numbers (see [1–18]). Specially, there are papers for the q-extensions of Euler polynomials and numbers approaching with two kinds of viewpoint among remarkable papers (see [7, 10]). It is known that the Euler polynomials are defined by $(2/(e^t + 1))e^{xt} = \sum_{n=0}^{\infty} E_n(x)(t^n/n!)$, for $|t| < \pi$, and $E_n = E_n(0)$ are called the nth Euler numbers. The recurrence formula for the original Euler numbers E_n is as follows:

$$E_0 = 1, \quad (E + 1)^n + E_n = 0, \quad \text{if } n > 0 \tag{1.1}$$

see [7, 10]. As for the q-extension of the recurrence formula for the Euler numbers, Kim [10] had the following recurrence formula:

$$E^{*}_{0,q} = \frac{[2]_q}{2}, \quad \text{and } \left(qE^{*}+1\right)^{n} + E^{*}_{n,q} = \begin{cases} [2]_q & \text{if } n = 0, \\ 0 & \text{if } n \geq 1, \end{cases} \tag{1.2}$$

with the usual convention of replacing $(E^{*})^{n}$ by $E^{*}_{n,q}$. Many researchers have made a wider and deeper study of the q-number up to recently (see [1–18]). In the field of number theory and mathematical physics, zeta functions and l-functions interpolating these numbers in negative integers have been studied by Cenkci and Can [3], Kim [4–12], and Ozden et al. [16–18].

This research for q-Euler numbers seems to be motivated by Carlitz who had constructed the q-Bernoulli numbers and polynomials for the first time. In [1, 2], Carlitz considered the recurrence formulae for the q-extension of the Bernoulli numbers as follows:

$$B_{0,q} = 1, \qquad \left(qB+1\right)^{k} - B_{k,q} = \begin{cases} 1 & \text{if } k = 1, \\ 0 & \text{if } k > 1, \end{cases} \tag{1.3}$$

with the usual convention of replacing B^{k} by $B_{k,q}$. These numbers diverge when $q = 1$, and so Carlitz modified and constructed them as following:

$$\beta_{0,q} = 1, \qquad q\left(q\beta+1\right)^{k} - \beta_{k,q} = \begin{cases} 1 & \text{if } k = 1, \\ 0 & \text{if } k > 1, \end{cases} \tag{1.4}$$

with the usual convention of replacing β^{k} by $\beta_{k,q}$. From this, it was shown that $\lim_{q\to 1}\beta_{k,q} = B_k$. Here B_k are the Bernoulli numbers.

Lately, Carlitz's q-Bernoulli numbers have been studied actively by many mathematicians in the field of number theory, discrete mathematics, analysis, mathematical physics, and so on (see [3–18]).

The purpose of this paper is to give a new construction of the q-extensions of Euler numbers and polynomials. It is expected that new constructed q-Euler numbers and polynomials in this paper are more useful to be applied to various areas related to number theory. In this paper, we present new generating functions which are related to q-Euler numbers and polynomials. We also consider the generalized q-Euler polynomials attached to Dirichlet's character χ with an odd conductor and have the generating functions of them. We obtain distribution relations for the q-Euler polynomials, and have some identities involving the q-Euler numbers and polynomials. Finally, we derive the q-extensions of zeta functions from the Mellin transformation of these generating functions. Using the Cauchy residue theorem and Laurent series, we show that these q-extensions of zeta functions interpolate the q-Euler polynomials at negative integers.

2. New Approach to q-Euler Numbers and Polynomials

Let $\mathbb{N}$ be the set of natural numbers and $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$. For $q \in \mathbb{C}$ with $|q| < 1$, let us define the q-Euler polynomials $E_{n,q}(x)$ as follows:

$$F_q(t,x) = 2\sum_{m=0}^{\infty}(-1)^m e^{[m+x]_q t} = \sum_{n=0}^{\infty} E_{n,q}(x)\frac{t^n}{n!}. \tag{2.1}$$

Note that

$$\lim_{q\to 1} F_q(t,x) = \frac{2}{e^t+1}e^{xt} = \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!}, \quad \text{for } |t| < \pi, \tag{2.2}$$

where $E_n(x)$ are called the nth Euler polynomials. In the special case $x = 0$, $E_{n,q}(= E_{n,q}(0))$ are called the nth q-Euler numbers. That is,

$$F_q(t) = F_q(t,0) = 2\sum_{m=0}^{\infty}(-1)^m e^{[m]_q t} = \sum_{n=0}^{\infty} E_{n,q}\frac{t^n}{n!}. \tag{2.3}$$

From (2.1) and (2.3), we note that

$$\begin{aligned}
F_q(t,1) + F_q(t) &= e^t F_q(qt) + F_q(t) \\
&= \left(\sum_{l=0}^{\infty}\frac{t^l}{l!}\right)\left(\sum_{m=0}^{\infty} q^m E_{m,q}\frac{t^m}{m!}\right) + \sum_{n=0}^{\infty} E_{n,q}\frac{t^n}{n!} \\
&= \sum_{\substack{n=0 \\ n=l+m}}^{\infty}\left(\sum_{l=0}^{n}\frac{n!q^l E_{l,q}}{l!(n-l)!}\right)\frac{t^n}{n!} + \sum_{n=0}^{\infty} E_{n,q}\frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l} q^l E_{l,q}\right)\frac{t^n}{n!} + \sum_{n=0}^{\infty} E_{n,q}\frac{t^n}{n!}.
\end{aligned} \tag{2.4}$$

From (2.1) and (2.3), we can easily derive the following equation:

$$F_q(t,1) + F_q(t) = 2. \tag{2.5}$$

By (2.4) and (2.5), we see that $E_{0,q} = 1$ and

$$\sum_{l=0}^{n}\binom{n}{l} q^l E_{l,q} + E_{n,q} = \begin{cases} 2 & \text{if } n = 0, \\ 0 & \text{if } n > 0. \end{cases} \tag{2.6}$$

Therefore, we obtain the following theorem.

Theorem 2.1. *For $n \in \mathbb{Z}_+$, one has*

$$E_{0,q} = 1, \qquad (qE+1)^n + E_{n,q} = \begin{cases} 2 & \text{if } n = 0, \\ 0 & \text{if } n > 0, \end{cases} \tag{2.7}$$

with the usual convention of replacing E^i by $E_{i,q}$.

Theorem 2.1 of this paper seems to be more interesting and valuable than the q-Euler numbers which are introduced in [7, 10].

From (2.1), we note that

$$F_q(t,x) = e^{[x]_q t} F_q(q^x t) = \sum_{n=0}^{\infty} \left(\sum_{l=0}^{n} \binom{n}{l} q^{lx} [x]_q^{n-l} E_{l,q} \right) \frac{t^n}{n!}. \tag{2.8}$$

Therefore, we obtain the following theorem.

Theorem 2.2. *For $n \in \mathbb{Z}_+$, one has*

$$E_{n,q}(x) = \sum_{l=0}^{n} \binom{n}{l} [x]_q^{n-l} q^{lx} E_{l,q}. \tag{2.9}$$

By (2.1), we see that

$$\begin{aligned} F_q(t,x) &= \sum_{n=0}^{\infty} \left(2 \sum_{m=0}^{\infty} (-1)^m [m+x]_q^n \right) \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \left(\frac{2}{(1-q)^n} \sum_{l=0}^{n} \binom{n}{l} (-1)^l q^{lx} \frac{1}{1+q^l} \right) \frac{t^n}{n!}. \end{aligned} \tag{2.10}$$

By (2.1) and (2.10), we obtain the following theorem.

Theorem 2.3. *For $n \in \mathbb{Z}_+$, one has*

$$E_{n,q}(x) = \frac{2}{(1-q)^n} \sum_{l=0}^{n} \binom{n}{l} (-1)^l q^{lx} \frac{1}{1+q^l}. \tag{2.11}$$

From (2.1), we can derive that, for $f \in \mathbb{N}$ with $f \equiv 1 \pmod{2}$,

$$F_q(t,x) = \sum_{a=0}^{f-1} (-1)^a F_{q^f}\left(t[f]_q, \frac{x+a}{f} \right). \tag{2.12}$$

By (2.12), we see that, for $f \in \mathbb{N}$ with $f \equiv 1 \pmod 2$,

$$\sum_{n=0}^{\infty} E_{n,q}(x)\frac{t^n}{n!} = \sum_{n=0}^{\infty}\left([f]_q^n \sum_{a=0}^{f-1}(-1)^a E_{n,q^f}\left(\frac{x+a}{f}\right)\right)\frac{t^n}{n!}. \tag{2.13}$$

Therefore, we obtain the following theorem.

Theorem 2.4 (Distribution relation for $E_{n,q}(x)$). *For $n \in \mathbb{Z}_+$, $f \in \mathbb{N}$ with $f \equiv 1 \pmod 2$, one has*

$$E_{n,q}(x) = [f]_q^n \sum_{a=0}^{f-1}(-1)^a E_{n,q^f}\left(\frac{x+a}{f}\right). \tag{2.14}$$

By (2.1), we observe the following equations:

$$\begin{aligned} F_q(t,n) + F_q(t) &= 2\sum_{l=0}^{n-1}(-1)^l e^{[l]_q t} \quad \text{if } n = \text{odd}, \\ F_q(t,n) - F_q(t) &= 2\sum_{l=0}^{n-1}(-1)^{l-1} e^{[l]_q t} \quad \text{if } n = \text{even}. \end{aligned} \tag{2.15}$$

By (2.15), we obtain the following result.

Theorem 2.5. *Let $n \in \mathbb{N}$ with $n \equiv 1 \pmod 2$. Then one has*

$$E_{m,q}(n) + E_{m,q} = 2\sum_{l=0}^{n-1}(-1)^l [l]_q^m, \tag{2.16}$$

where $m \in \mathbb{Z}_+$.

Let χ be Dirichlet's character with an odd conductor $f \in \mathbb{N}$. Then we define the generalized q-Euler polynomials attached to χ as follows:

$$\begin{aligned} F_{q,\chi}(t,x) &= 2\sum_{m=0}^{\infty}\chi(m)(-1)^m e^{[m+x]_q t} \\ &= \sum_{n=0}^{\infty} E_{n,\chi,q}(x)\frac{t^n}{n!}. \end{aligned} \tag{2.17}$$

In the special case $x = 0$, $E_{n,\chi,q}(= E_{n,\chi,q}(0))$ are called the nth generalized q-Euler numbers attached to χ. Thus the generating functions of the generalized q-Euler numbers attached to χ are as follows:

$$\begin{aligned} F_{q,\chi}(t) &= 2\sum_{m=0}^{\infty}\chi(m)(-1)^m e^{[m]_q t} \\ &= \sum_{n=0}^{\infty} E_{n,\chi,q}\frac{t^n}{n!}. \end{aligned} \tag{2.18}$$

By (2.1) and (2.17), we see that

$$\begin{aligned} F_{q,\chi}(t,x) &= \sum_{a=0}^{f-1}(-1)^a\chi(a)F_{q^f}\left(t[f]_q, \frac{x+a}{f}\right) \\ &= \sum_{n=0}^{\infty}\left([f]_q^n\sum_{a=0}^{f-1}(-1)^a\chi(a)E_{n,q^f}\left(\frac{x+a}{f}\right)\right)\frac{t^n}{n!}. \end{aligned} \tag{2.19}$$

Therefore, we obtain the following theorem.

Theorem 2.6. *For $n \in \mathbb{Z}_+$, $f \in \mathbb{N}$ with $f \equiv 1 \pmod 2$, one has*

$$E_{n,\chi,q}(x) = [f]_q^n\sum_{a=0}^{f-1}(-1)^a\chi(a)E_{n,q^f}\left(\frac{x+a}{f}\right). \tag{2.20}$$

By (2.17) and (2.18), we see that

$$F_{q,\chi}(t,x) = e^{[x]_q t}F_{q,\chi}(q^x t) = \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}q^{lx}[x]_q^{n-l}E_{l,\chi,q}\right)\frac{t^n}{n!}. \tag{2.21}$$

Hence

$$E_{n,\chi,q}(x) = \sum_{l=0}^{n}\binom{n}{l}q^{lx}[x]_q^{n-l}E_{l,\chi,q}. \tag{2.22}$$

From (2.17), we note that

$$F_{q,\chi}(t,x) = \sum_{n=0}^{\infty}\left(\frac{2}{(1-q)^n}\sum_{a=0}^{f-1}(-1)^a\chi(a)\sum_{l=0}^{n}\frac{\binom{n}{l}(-1)^l q^{l(x+a)}}{1+q^{lf}}\right)\frac{t^n}{n!}. \tag{2.23}$$

From (2.17) and (2.23), we have

$$\begin{aligned} E_{n,\chi,q}(x) &= \frac{2}{(1-q)^n}\sum_{a=0}^{f-1}(-1)^a\chi(a)\sum_{l=0}^{n}\frac{\binom{n}{l}(-1)^l q^{l(x+a)}}{1+q^{lf}} \\ &= 2\sum_{m=0}^{\infty}\chi(m)(-1)^m[m+x]_q^n. \end{aligned} \tag{2.24}$$

In (2.19), it is easy to show that

$$\lim_{q\to 1}F_{q,\chi}(t,x) = \left(\frac{2\sum_{a=0}^{f-1}(-1)^a\chi(a)e^{at}}{e^{ft}+1}\right)e^{xt} = \sum_{n=0}^{\infty}E_{n,\chi}(x)\frac{t^n}{n!}, \tag{2.25}$$

where $E_{n,\chi}(x)$ are called the nth generalized Euler polynomials attached to χ.

For $s \in \mathbb{C}$, we now consider the Mellin transformation for the generating function of $F_q(t,x)$. That is,

$$\frac{1}{\Gamma(s)}\int_0^{\infty}F_q(-t,x)t^{s-1}dt = 2\sum_{n=0}^{\infty}\frac{(-1)^n}{[n+x]_q^s}, \tag{2.26}$$

for $s \in \mathbb{C}$, and $x \neq 0,-1,-2,\ldots$.

From (2.26), we define the zeta function as follows:

$$\zeta^*(s,x) = \sum_{n=0}^{\infty}\frac{(-1)^n}{[n+x]_q^s}, \quad s \in \mathbb{C},\ x \neq 0,-1,-2,\ldots. \tag{2.27}$$

Note that $\zeta^*(s,x)$ is analytic function in whole complex s-plane. Using the Laurent series and the Cauchy residue theorem, we have

$$\zeta^*(-n,x) = E_{n,q}(x), \quad \text{for } n \in \mathbb{Z}_+. \tag{2.28}$$

By the same method, we can also obtain the following equation:

$$\frac{1}{\Gamma(s)}\int_0^{\infty}F_{q,\chi}(-t,x)t^{s-1}dt = 2\sum_{n=0}^{\infty}\frac{\chi(n)(-1)^n}{[n+x]_q^s}. \tag{2.29}$$

For $s \in \mathbb{C}$, we define Dirichlet type q-l-function as

$$l_q(s,x \mid \chi) = 2\sum_{n=0}^{\infty}\frac{\chi(n)(-1)^n}{[n+x]_q^s}, \tag{2.30}$$

where $x \neq 0, -1, -2, \ldots$. Note that $l_q(s, x \mid \chi)$ is also holomorphic function in whole complex s-plane. From the Laurent series and the Cauchy residue theorem, we can also derive the following equation:

$$l_q(-n, x \mid \chi) = E_{n,\chi,q}(x), \quad \text{for } n \in \mathbb{Z}_+. \tag{2.31}$$

Remark 2.7. It is easy to see that

$$\begin{aligned} E_{n,q}(x) &= \int_{\mathbb{Z}_p} [x+y]_q^n d\mu_{-1}(y), \\ E_{n,\chi,q}(x) &= \int_X [x+y]_q^n \chi(y) d\mu_{-1}(y), \end{aligned} \tag{2.32}$$

see [19, Lemma 1].

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