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Research Article

Existence and Nonexistence of Global Solutions of the Quasilinear Parabolic Equations with Inhomogeneous Terms

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We consider the quasilinear parabolic equation with inhomogeneous term $u_t = \Delta u^m + \langle x \rangle^\sigma u^p + f(x)$, $u(x,0) = u_0(x)$, where $0 \le f(x)$, $u_0(x) \in C(\mathbb{R}^N)$, $m > 0$, $p > \max\{1, m\}$, and $\sigma > -2$, $\langle x \rangle := (|x|^2 + 1)^{1/2}$. In this paper, we investigate the critical exponents of this equation.

1. Introduction

We consider the quasi-linear parabolic equation with inhomogeneous term

$$\begin{gathered} u_t = \Delta u^m + \langle x \rangle^\sigma u^p + f(x) \quad \left(x \in \mathbb{R}^N,\ t > 0\right), \\ u(x,0) = u_0(x) \quad \left(x \in \mathbb{R}^N\right), \end{gathered} \tag{1.1}$$

where $0 \le f(x)$, $u_0(x) \in C(\mathbb{R}^N)$, $m > 0$, $p > \max\{1, m\}$, and $\sigma > -2$, $\langle x \rangle := (|x|^2 + 1)^{1/2}$.

For the solution $u(x,t)$ of (1.1), let $T^* > 0$ be the maximal existence time, that is,

$$T^* := \sup\left\{ T > 0;\ \sup_{t \in [0,T)} \|u(\cdot,t)\|_\infty < \infty \right\}. \tag{1.2}$$

If $T^* = \infty$, we say that $u(x,t)$ is a global solution; if $T^* < \infty$, we say that $u(x,t)$ blows up in finite time.

For quasi-linear parabolic equations, the authors of [1–5] and so on. study the homogeneous equations (i.e., $f(x) \equiv 0$ in (1.1)). Baras and Kersner [1] proved that (1.1) with $m = 1$ and $f(x) \equiv 0$ has a global solution, two constants c_1 and c_2 depending on N and p exist such that

$$\liminf_{r\to\infty}\left\{ r^{-2/(p-1)} \int_{|x|<r} \frac{dx}{\langle x\rangle^{\sigma/(p-1)}} \right\} \geq c_1 \int u_0 dx,$$
$$\liminf_{|x|\to\infty} \langle x\rangle^{\sigma+2} u_0(x)^{p-1} \leq c_2. \tag{1.3}$$

Mochizuki and Mukai [2] and Qi [4] study the case $m > 0$, $\sigma = 0$, Pinsky [3] studies the case $m = 1$, $\sigma > -2$, and Suzuki [5] studies the case $m \geq 1$, $-\infty < \sigma < \infty$. The following two results are proved by them:

(1) if $p \leq p^*_{m,\sigma}$, then every nontrivial solution $u(x,t)$ of (1.1) blows up in finite time;

(2) if $p > p^*_{m,\sigma}$, then (1.1) has a global solution for some initial value $u_0(x)$,

where $p^*_{m,\sigma} = m + (2+\sigma)/N$ for $N \geq 2$, $\sigma > -2$ and for $N = 1$, $\sigma > -1$, $p^*_{m,\sigma} = m + 1$ for $N = 1$, $\sigma \leq -1$. This $p^*_{m,\sigma}$ is called the critical exponent.

On the other hand, [6–9] and so on. study the inhomogeneous equations (i.e., $f(x) \not\equiv 0$ in (1.1)). Bandle et al. [6] study the case $m = 1$, $\sigma = 0$, and Zeng [8] and Zhang [9] study the case $\sigma = 0$. In this paper, we investigate the critical exponents of (1.1) in the case $f(x) \not\equiv 0$. Our results are as follows.

Theorem 1.1. *Suppose that $N \geq 3$, $\sigma > -2$, $m > (N-2)/(N+\sigma)$, and $p > \max\{1, m\}$. Put*

$$p^*_{m,\sigma} := \frac{m(N+\sigma)}{N-2}. \tag{1.4}$$

(a) *If $p \leq p^*_{m,\sigma}$, then every nontrivial solution $u(x,t)$ of* (1.1) *blows up in finite time.*

(b) *If $p > p^*_{m,\sigma}$, $u_0(x) \leq C_1\langle x\rangle^{-(2+\sigma)/(p-m)}$, and $f(x) \leq C_2\langle x\rangle^{-m(2+\sigma)/(p-m)-4}$, then* (1.1) *has a global solution for some constants C_1 and C_2.*

Theorem 1.2. *Suppose that $N = 1, 2$, $\sigma \geq -2$, $m > 0$, and $p > \max\{1, m\}$. Then every nontrivial solution $u(x,t)$ of* (1.1) *blows up in finite time.*

Remark 1.3. Theorems 1.1 and 1.2 are the extension of the results of [8]. If we put $\sigma = 0$ in these theorems, the same results as Theorem 1 in [8] are obtained.

We will prove Theorem 1.1(a) and (b) in Sections 3 and 4, respectively. The proof of Theorem 1.2 is included in the proof of Theorem 1.1(a).

In the following, R and T are two given positive real numbers greater than 1. C is a positive constant independent of R and T, and its value may change from line to line.

2. Preliminaries

In this section, we first give the definition of a solution for Problem (1.1) and then cite the comparison theorem and a known result.

Definition 2.1. A continuous function $u = u(x,t)$ is called a solution of Problem (1.1) in $Q_T \equiv \mathbb{R}^N \times [0,T)$ if the following holds:

(i) $\nabla_x u^m \in L^2_{\mathrm{loc}}(\mathbb{R}^N)$;

(ii) for any bounded domain $D \subset \mathbb{R}^N$ and for all $\varphi \in C^2(D \times [0,T))$ and vanishing on $\partial D \times [0,T)$,

$$\int_0^\tau \int_D \left(u\partial_t\varphi - \nabla u^m \nabla\varphi + \langle x\rangle^\sigma u^p \varphi + f\varphi\right) dx\, dt = \int_D u(x,\cdot)\varphi(x,\cdot)\big|_0^\tau dx, \tag{2.1}$$

for all $\tau \in [0,T)$.

Lemma 2.2 (the comparison theorem). *Let u, $v \in C(0, T; L^2_{\mathrm{loc}}(\Omega))$, ∇u^m, $\nabla v^m \in L^2(0, T; L^2_{\mathrm{loc}}(\Omega))$, and satisfy*

$$\begin{gathered} u_t - \Delta u^m \le v_t - \Delta v^m, \quad (x,t) \in \Omega_T, \\ u \le v, \quad (x,t) \in \partial\Omega_T. \end{gathered} \tag{2.2}$$

Then $u \le v$ for all $(x,t) \in \Omega_T$, where Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$ or $\Omega = \mathbb{R}^N$ and $\Omega_T = \Omega \times (0,T]$.

Lemma 2.3 (the monotonicity property). *Let $\underline{u}(x)$ be a nonnegative sub-solution to the stationary problems of Problem* (1.1). *Then the positive solution $u(x,t)$ with initial data $\underline{u}(x)$ is monotone increasing to t.*

3. Proof of Theorem 1.1(a)

We first consider the following problem:

$$\begin{gathered} u_t = \Delta u^m + \langle x\rangle^\sigma u^p + f(x) \quad \left(x \in \mathbb{R}^N,\ t > 0\right), \\ u(x,0) = 0 \quad \left(x \in \mathbb{R}^N\right). \end{gathered} \tag{3.1}$$

It is clear that the positive solution of Problem (3.1) is a sub-solution of Problem (1.1). If every positive solution of Problem (3.1) blows up in finite time, then, by Lemma 2.2, every positive solution of Problem (1.1) also blows up in finite time. Therefore, we only need to consider Problem (3.1).

The stationary problem of Problem (3.1) is as follows:

$$\Delta u^m + \langle x\rangle^\sigma u^p + f(x) = 0 \quad \left(x \in \mathbb{R}^N\right). \tag{3.2}$$

It is obvious that 0 is a sub-solution of Problem (3.2) and does not satisfy Problem (3.2). Thus, by making use of Lemmas 2.2 and 2.3, the positive solution of Problem (3.1) is monotone increasing to t.

We argue by contradiction. Assume that Problem (3.1) has a global positive solution for $p \leq p^*_{m,\sigma}$.

Let $\varphi(r)$ and $\eta(t)$ be two functions in $C^\infty([0,\infty))$, and satisfy

(i) $0 \leq \varphi(r) \leq 1$ in $[0,\infty)$; $\varphi(r) \equiv 1$ in $[0,1]$, $\varphi(r) \equiv 0$ in $[2,\infty)$; $-C \leq \varphi'(r) \leq 0$, $|\varphi''(r)| \leq C$;

(ii) $0 \leq \eta(t) \leq 1$ in $[0,\infty)$; $\eta(t) \equiv 1$ in $[0,1]$, $\eta(t) \equiv 0$ in $[2,\infty)$; $-C \leq \eta'(t) \leq 0$.

For $R > 1$ and $T > 1$, define $Q_{R,T} \equiv B_{2R} \times [0,4T]$, and let $\Psi(r,t) = \varphi_R(r)\eta_T(t)$ be a cut-off function, where $\varphi_R(r) = \varphi(r/R)$, $\eta_T(t) = \eta(t/2T)$. It is easy to check that

$$-\frac{C}{R} \leq \frac{d\varphi_R(r)}{dr} \leq 0, \qquad \left|\frac{d^2\varphi_R(r)}{dr^2}\right| \leq \frac{C}{R^2}, \qquad -\frac{C}{2T} \leq \frac{d\eta_T(t)}{dt} \leq 0. \tag{3.3}$$

Let

$$I_R = \int_{Q_{R,T}} \langle x\rangle^\sigma u^p \Psi^s \,dx\,dt, \tag{3.4}$$

where $s > 1$ is a positive number to be determined. Then

$$\begin{aligned}
I_R &= \int_{Q_{R,T}} \left(-u\partial_t\Psi^s + \nabla u^m \nabla\Psi^s - f\Psi^s\right)dx\,dt + \int_{B_{2R}} u(x,\cdot)\Psi(r,\cdot)^s\Big|_0^{4T}dx \\
&= -\int_{Q_{R,T}} u\varphi_R^s \frac{d\eta_T^s}{dt}dx\,dt + \int_{Q_{R,T}} \nabla u^m \eta_T^s \nabla\varphi_R^s dx\,dt - \int_{Q_{R,T}} f\Psi^s dx\,dt \\
&\quad + \int_{B_{2R}} u(x,\cdot)\varphi_R(r)^s\eta_T(\cdot)^s\Big|_0^{4T}dx \\
&= -\int_{Q_{R,T}} u\varphi_R^s \frac{d\eta_T^s}{dt}dx\,dt - \int_{Q_{R,T}} u^m \eta_T^s \Delta\varphi_R^s dx\,dt - \int_{Q_{R,T}} f\Psi^s dx\,dt \\
&\quad + \int_0^{4T}\int_{|x|=2R} u^m\eta_T^s \frac{\partial\varphi_R^s}{\partial\nu}dS\,dt.
\end{aligned} \tag{3.5}$$

Since $\int_{\mathbb{R}^N} f(x)dx > 0$, there exist $\delta > 0$ and $R_0 > 1$ such that $\int_{B_R} f(x)dx \geq \delta$ for $R > R_0$:

$$\int_{Q_{R,T}} f\Psi^s dx\,dt = \int_0^{4T}\eta_T^s\int_{B_{2R}} f\varphi_R^s dx\,dt \geq \int_T^{2T}\int_{B_R} f\,dx\,dt \geq \delta T. \tag{3.6}$$

Hence, by the definition of φ_R and η_T, we have

$$I_R \leq -\int_{2T}^{4T}\int_{B_{2R}} u\varphi_R^s\frac{d\eta_T^s}{dt}dx\,dt - \int_0^{4T}\int_{B_{2R}\setminus\overline{B_R}} u^m\eta_T^s\Delta\varphi_R^s dx\,dt - \delta T. \tag{3.7}$$

Since $\Delta\varphi_R^s = s\varphi_R^{s-1}\Delta\varphi_R + s(s-1)\varphi_R^{s-2}|\nabla\varphi_R|^2$ and

$$\Delta\varphi_R(r) = \frac{d^2\varphi_R(r)}{dr^2} + \frac{N-1}{r}\frac{d\varphi_R(r)}{dr}, \qquad |\nabla\varphi_R|^2 = \left(\frac{d\varphi_R(r)}{dr}\right)^2, \tag{3.8}$$

we obtain from (3.3) that

$$|\Delta\varphi_R^s| \le s\varphi_R^{s-1}\left(\frac{C}{R^2} + \frac{N-1}{R}\cdot\frac{C}{R}\right) + s(s-1)\varphi_R^{s-2}\left(\frac{C}{R}\right)^2 \le \frac{C}{R^2}\varphi_R^{s-2} \tag{3.9}$$

in $B_{2R} \setminus \overline{B_R}$ and

$$\frac{d\eta_T^s}{dt} = s\eta_T^{s-1}\frac{d\eta_T}{dt} \ge -s\eta_T^{s-1}\frac{C}{2T} \ge -\frac{C}{T}\eta_T^{s-1} \tag{3.10}$$

in $[2T, 4T]$. Thus, (3.7) becomes

$$I_R \le \frac{C}{T}\int_{2T}^{4T}\int_{B_{2R}} u\Psi^{s-1}dx\,dt + \frac{C}{R^2}\int_0^{4T}\int_{B_{2R}\setminus\overline{B_R}} u^m\Psi^{s-2}dx\,dt - \delta T. \tag{3.11}$$

Let s be large enough such that $(s-1)p \ge s$ and $(s-2)p/m \ge s$, and let $A_\sigma(R)$ be as follows:

$$A_\sigma(R) = \begin{cases} R^{N-\sigma/(p-1)} & (\sigma < N(p-1)), \\ \log(R+1) & (\sigma \ge N(p-1)). \end{cases} \tag{3.12}$$

Then, by making use of Young's inequality, we have

$$\begin{aligned} &\frac{C}{T}\int_{2T}^{4T}\int_{B_{2R}} u\Psi^{s-1}dx\,dt \\ &\quad\le \int_{2T}^{4T}\int_{B_{2R}}\left(\frac{1}{4^p p}\langle x\rangle^\sigma u^p\Psi^{(s-1)p} + \frac{4^q}{q}\langle x\rangle^{-\sigma q/p}C^qT^{-q}\right)dx\,dt \\ &\quad\le \frac{1}{4}\int_0^{4T}\int_{B_{2R}}\langle x\rangle^\sigma u^p\Psi^s dx\,dt + CT^{-p/(p-1)}\int_{2T}^{4T}\int_{B_{2R}}\langle x\rangle^{-\sigma/(p-1)}dx\,dt \\ &\quad\le \frac{1}{4}I_R + CT^{1-p/(p-1)}A_\sigma(R), \end{aligned} \tag{3.13}$$

where $1/p + 1/q = 1$ and

$$\begin{aligned}
&\frac{C}{R^2}\int_0^{4T}\int_{B_{2R}\setminus\overline{B_R}} u^m\Psi^{s-2}dx\,dt \\
&\quad\le \int_0^{4T}\int_{B_{2R}\setminus\overline{B_R}}\left(\frac{1}{4^{p'}p'}\langle x\rangle^{\sigma}u^{mp'}\Psi^{(s-2)p'} + \frac{4^{q'}}{q'}\langle x\rangle^{-\sigma q'/p'}C^{q'}R^{-2q'}\right)dx\,dt \\
&\quad\le \frac{1}{4}\int_0^{4T}\int_{B_{2R}}\langle x\rangle^{\sigma}u^p\Psi^s dx\,dt + CR^{-2p/(p-m)}\int_0^{4T}\int_{B_{2R}\setminus\overline{B_R}}\langle x\rangle^{-m\sigma/(p-m)}dx\,dt \\
&\quad\le \frac{1}{4}I_R + CTR^{-2p/(p-m)}R^{N-m\sigma/(p-m)},
\end{aligned} \tag{3.14}$$

where $p' = p/m$, $1/p' + 1/q' = 1$. Thus, (3.11) becomes

$$I_R \le \frac{1}{2}I_R + T\left(CT^{-p/(p-1)}A_{\sigma}(R) + CR^{N-(2p+m\sigma)/(p-m)} - \delta\right). \tag{3.15}$$

For $N \ge 3$, since $\sigma > -2$, $1/p' + 1/q' = 1$, and $\max\{1, m\} < p \le m(N+\sigma)/(N-2)$, we have

$$N - \frac{2p+m\sigma}{p-m} = \frac{(N-2)p - (N+\sigma)m}{p-m} \le 0. \tag{3.16}$$

For $N = 2$, since $\sigma \ge -2$, $m > 0$, and $p > \max\{1, m\}$, we have

$$2 - \frac{2p+m\sigma}{p-m} = \frac{-(2+\sigma)m}{p-m} \le 0. \tag{3.17}$$

For $N = 1$, since $\sigma \ge -2$, $m > 0$, and $p > max\{1, m\}$, we have

$$1 - \frac{2p+m\sigma}{p-m} = \frac{-p-(1+\sigma)m}{p-m} < \frac{-(2+\sigma)m}{p-m} \le 0. \tag{3.18}$$

Let $T \ge A_{\sigma}(R)^{(p-1)/p}$ such that $T^{-p/(p-1)}A_{\sigma}(R) \le 1$, then

$$I_R \le CT, \tag{3.19}$$

that is,

$$\int_0^{4T}\int_{B_{2R}}\langle x\rangle^{\sigma}u^p\Psi^s dx\,dt \le CT. \tag{3.20}$$

Thus

$$\int_T^{2T}\int_{B_R}\langle x\rangle^{\sigma}u^p dx\,dt \le CT. \tag{3.21}$$

By the integral mean value theorem, there exists $t_1 \in [T, 2T]$ such that

$$\int_T^{2T} \int_{B_R} \langle x \rangle^\sigma u^p dx\, dt = T \int_{B_R} \langle x \rangle^\sigma u(x, t_1)^p dx \leq CT, \tag{3.22}$$

that is,

$$\int_{B_R} \langle x \rangle^\sigma u(x, t_1)^p dx \leq C. \tag{3.23}$$

Since T is a large positive number and a random selection, and $u(x,t)$ is monotone increasing to t, there exists a positive number $T(R) > 1$ for any fixed $R > R_0$ such that, for all $t > T(R)$,

$$\int_{B_R} \langle x \rangle^\sigma u(x, t)^p dx \leq C. \tag{3.24}$$

By the monotone increasing property of $u(x,t)$, $\int_{B_R} \langle x \rangle^\sigma u(x,t)^p dx$ also is increasing to t. This, combined with (3.24), yields that the limit I_R^∞ exists such that

$$I_R^\infty \equiv \lim_{t \to \infty} \int_{B_R} \langle x \rangle^\sigma u(x, t)^p dx \leq C. \tag{3.25}$$

Since $u(x,t)$ is nonnegative, I_R^∞ is monotone increasing to R. This, combined with (3.25), yields that $\lim_{R\to\infty} I_R^\infty$ exists. Thus, for any small $\varepsilon > 0$, there exists a large positive constant which still is denoted by R_0, such that, for $R > R_0$,

$$\lim_{t \to \infty} \int_{B_{2R} \setminus \overline{B_R}} \langle x \rangle^\sigma u(x, t)^p dx \equiv I_{2R}^\infty - I_R^\infty < \varepsilon. \tag{3.26}$$

Hence, by similar argument as that in (3.24), there exists a large positive number $T(R) > 1$ such that

$$\int_{B_{2R} \setminus \overline{B_R}} \langle x \rangle^\sigma u(x, t)^p dx < \varepsilon, \quad \forall t > T(R). \tag{3.27}$$

On the other hand, we argue as in [6, 10]. Let $\xi(x) \in C^2(\mathbb{R}^N)$ be a positive function satisfying.

(i) $0 \leq \xi(x) \leq 1$ in $\mathbb{R}^N$; $\xi(x) \equiv 1$ in B_1, $\xi(x) \equiv 0$ in B_2^c;

(ii) $\partial\xi/\partial\nu = 0$ on $\partial(B_2 \setminus B_1)$;

(iii) for any $\alpha \in (0,1)$, there exists a positive constant C_α such that $|\Delta\xi| \leq C_\alpha \xi^\alpha$.

Let R and $T(R)$ be as defined in (3.26) and (3.27). Multiplying (3.1) by $\xi_R(x) = \xi(x/R)$ and then integrating by parts in $\mathbb{R}^N$, we have

$$\frac{d}{dt} \int_{\mathbb{R}^N} u\xi_R dx = \int_{B_{2R} \setminus \overline{B_R}} u^m \Delta\xi_R dx + \int_{\mathbb{R}^N} \langle x \rangle^\sigma u^p \xi_R dx + \int_{\mathbb{R}^N} f\xi_R dx. \tag{3.28}$$

By the definition of $\xi_R(x)$, Hölder's inequality, and (3.27), we have

$$\begin{aligned}
\left|\int_{B_{2R}\backslash\overline{B_R}} u^m \Delta\xi_R dx\right| &\le \frac{C_\alpha}{R^2}\int_{B_{2R}\backslash\overline{B_R}} u^m \xi_R^\alpha dx \\
&\le \frac{C_\alpha}{R^2}\left(\int_{B_{2R}\backslash\overline{B_R}} \langle x\rangle^\sigma u^{mp'} dx\right)^{1/p'}\left(\int_{B_{2R}\backslash\overline{B_R}} \langle x\rangle^{-\sigma q'/p'} \xi_R^{\alpha q'} dx\right)^{1/q'} \\
&\le \frac{C_\alpha}{R^2}\left(\int_{B_{2R}\backslash\overline{B_R}} \langle x\rangle^\sigma u^{p} dx\right)^{m/p}\left(\int_{B_{2R}\backslash\overline{B_R}} \langle x\rangle^{-m\sigma/(p-m)} dx\right)^{(p-m)/p} \\
&\le C\varepsilon^{m/p}R^{(N-m\sigma/(p-m))(p-m)/p-2} \le C\varepsilon^{m/p},
\end{aligned} \tag{3.29}$$

where $p' = p/m$, $1/p' + 1/q' = 1$, since

$$\left(N - \frac{m\sigma}{p-m}\right)\frac{p-m}{p} - 2 = \frac{(N-2)p - (N+\sigma)m}{p} \le 0. \tag{3.30}$$

Let $F_R(t) = \int_{\mathbb{R}^N} u\xi_R dx$ and $G_R(t) = \int_{\mathbb{R}^N} \langle x\rangle^\sigma u^p \xi_R dx$. Then, by making use of (3.29) and $\int_{\mathbb{R}^N} f(x)dx \ge \delta$ for $R > R_0$, (3.28) becomes

$$F_R'(t) \ge G_R(t) - C\varepsilon^{m/p} + \delta. \tag{3.31}$$

Thus, let ε be small enough such that $C\varepsilon^{m/p} \le \delta/2$, then $F_R'(t) \ge G_R(t) + \delta/2$.

Let $t_0 > T(R)$. By making use of Hölder's inequality, we obtain that

$$\begin{aligned}
F_R(t) &\le \left(\int_{\mathbb{R}^N} \langle x\rangle^\sigma u^p \xi_R dx\right)^{1/p}\left(\int_{\mathbb{R}^N} \langle x\rangle^{-\sigma q/p} \xi_R dx\right)^{1/q} \\
&\le G_R(t)^{1/p}\left(\int_{B_{2R}} \langle x\rangle^{-\sigma/(p-1)} dx\right)^{(p-1)/p} \\
&\le CG_R(t)^{1/p} A_\sigma(R)^{(p-1)/p},
\end{aligned} \tag{3.32}$$

where $1/p + 1/q = 1$. Thus, we obtain that

$$\begin{aligned}
\int_{t_0}^t F_R(s)^p ds &\le CA_\sigma(R)^{p-1}\int_{t_0}^{\mathrm{t}} G_R(s) ds \le CA_\sigma(R)^{p-1}\int_{t_0}^t F\prime_R(s) ds \\
&\le CA_\sigma(R)^{p-1}(F_R(t) - F_R(t_0)).
\end{aligned} \tag{3.33}$$

Since $F_R(t) \geq 0$ for all $t \geq 0$, we have

$$F_R(t) \geq CA_\sigma(R)^{-p+1} \int_{t_0}^{t} F_R(s)^p ds + F_R(t_0)$$
$$\geq CA_\sigma(R)^{-p+1} \int_{t_0}^{t} F_R(s)^p ds. \tag{3.34}$$

Let $g(t) = \int_{t_0}^{t} F_R(s)^p ds$, then

$$g\prime(t) = F_R(t)^p \geq CA_\sigma(R)^{-p(p-1)} g(t)^p. \tag{3.35}$$

Let $t_1 > t_0$ such that $g(t_1) > 0$. Since $p > 1$, by solving the differential inequality (3.35) in $[t_1, t]$, we have

$$\int_{t_1}^{t} \frac{g\prime(s)}{g(s)^p} ds \geq CA_\sigma(R)^{-p(p-1)} \int_{t_1}^{t} ds,$$
$$g(t)^{1-p} \leq g(t_1)^{1-p} - C(p-1)A_\sigma(R)^{-p(p-1)}(t-t_1), \tag{3.36}$$
$$g(t) \geq \left\{ g(t_1)^{1-p} - C(p-1)A_\sigma(R)^{-p(p-1)}(t-t_1) \right\}^{-1/(p-1)}.$$

Thus, there exists T_1 with $t_1 < T_1 \leq t_1 + C(p-1)^{-1} A_\sigma(R)^{p(p-1)} g(t_1)^{1-p}$, such that $\lim_{t \uparrow T_1} g(t) = +\infty$, which implies that $g(t)$ and then u blow up in finite time. It contradicts our assumption. Therefore, every positive solution of Problem (3.1) blows up in finite time. Hence, every positive solution of Problem (1.1) blows up in finite time.

4. Proof of Theorem 1.1(b)

In this section, we prove that for $p > m(N+\sigma)/(N-2)$, there exist some $f(x)$ and $u_0(x)$, such that Problem (1.1) admits a global positive solution.

We first consider the stationary problem of Problem (1.1) as follows:

$$\Delta u^m + \langle x \rangle^\sigma u^p + f(x) = 0 \quad \left(x \in \mathbb{R}^N \right). \tag{4.1}$$

Let $v(x) = C_1 \langle x \rangle^{-s}$, where $s = (2+\sigma)/(p-m)$ and the positive constant C_1 satisfies

$$C_1^{p-m} = ms(N-ms-2) = \frac{m(2+\sigma)\{(N-2)p-(N+\sigma)m\}}{(p-m)^2} > 0. \tag{4.2}$$

Then, we have

$$
\begin{aligned}
-\Delta v^m &= \frac{ms}{2} C_1^m \left(|x|^2+1\right)^{-ms/2-1} \Delta\left(|x|^2+1\right) \\
&\quad - \frac{ms(ms+2)}{4} C_1^m \left(|x|^2+1\right)^{-ms/2-2} \left|\nabla\left(|x|^2+1\right)\right|^2 \\
&= NmsC_1^m \langle x\rangle^{-ms-2} - ms(ms+2)C_1^m |x|^2 \langle x\rangle^{-ms-4} \\
&= ms(N-ms-2)C_1^m \langle x\rangle^{-ms-2} + ms(ms+2)C_1^m \langle x\rangle^{-ms-4}.
\end{aligned}
\tag{4.3}
$$

Since $C_1^{p-m} = ms(N-ms-2)$ and $-ms-2 = \sigma - ps$, we have

$$
-\Delta v^m = C_1^p \langle x\rangle^{\sigma-ps} + C_2 \langle x\rangle^{-ms-4} = \langle x\rangle^{\sigma} v^p + C_2 \langle x\rangle^{-ms-4}, \tag{4.4}
$$

where $C_2 = ms(ms+2)C_1^m$. Thus, if $f(x) \le C_2\langle x\rangle^{-ms-4}$ and $u_0(x) \le v(x)$, then v is a supersolution of Problem (1.1). It is obvious that 0 is s sub-solution of Problem (1.1). Therefore, by the iterative process and the comparison theorem, Problem (1.1) admits a global positive solution.

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