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Research Article

On Regularly Varying and History-Dependent Convergence Rates of Solutions of a Volterra Equation with Infinite Memory

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We consider the rate of convergence to equilibrium of Volterra integrodifferential equations with infinite memory. We show that if the kernel of Volterra operator is regularly varying at infinity, and the initial history is regularly varying at minus infinity, then the rate of convergence to the equilibrium is regularly varying at infinity, and the exact pointwise rate of convergence can be determined in terms of the rate of decay of the kernel and the rate of growth of the initial history. The result is considered both for a linear Volterra integrodifferential equation as well as for the delay logistic equation from population biology.

1. Introduction

In this paper we consider the asymptotic behaviour of linear and nonlinear Volterra integrodifferential equations with infinite memory, paying particular attention to the connection between the asymptotic behaviour of the initial history as $t \to -\infty$ and the rate of convergence of the solution to a limit. In fact we focus our attention on the cases where the initial history ϕ obeys $\lim_{t\to\infty}|\phi(t)| = \infty$. We do not aim to be comprehensive in our analysis and focus only on scalar equations whose initial histories and kernels are regularly varying functions. However, we note that such history-dependent asymptotic behaviour does not seem to be generic behaviour for equations with a finite memory.

We consider both linear and nonlinear equations. In particular we consider the linear Volterra integrodifferential equation given by

$$x'(t) = -ax(t) - \int_{-\infty}^{t} b(t-s)x(s)ds, \quad t > 0; \; x(t) = \phi(t), \; t \in (-\infty, 0], \tag{1.1}$$

as well as the nonlinear logistic equation with infinite delay given by

$$N'(t) = N(t)\left(r - aN(t) - \int_{-\infty}^{t} b(t-s)N(s)ds\right), \quad t > 0,$$
$$N(t) = \phi(t), \quad t \in (-\infty, 0]. \tag{1.2}$$

In both cases, we presume that b is continuous, positive, and integrable, and that ϕ is continuous on $(-\infty, 0]$. When $t \mapsto |\phi(t)|$ is bounded and $a > \int_0^\infty b(s)ds$, the solution of x of (1.1) obeys $x(t) \to 0$ as $t \to \infty$; moreover, Miller [1] has shown when in addition $r > 0$, the solution N of (1.2) obeys $N(t) \to r/(a + \int_0^\infty b(s)ds) =: K$ as $t \to \infty$.

For definiteness, we concentrate in this introduction on solutions of (1.2). In Appleby et al. [2], an extension of Miller's global asymptotic stability result was given to a class of initial functions Φ, which can include initial histories ϕ which are unbounded in the sense that $\lim_{t\to-\infty}\phi(t) = +\infty$. Furthermore, the rate at which $N(t)$ converges K as $t \to \infty$ is also of interest.

It was shown in [2] that for certain classes of kernels b that the rate of convergence of $N(t)$ to K as $t \to \infty$ depends on the asymptotic behaviour of $\phi(t)$ as $t \to -\infty$. When b is a type of slowly decaying function, called subexponential, it has been shown that when

$$\int_{-\infty}^{0} \left|\phi(t) - K\right| dt < +\infty \tag{1.3}$$

then $N(t) - K$ tends to zero like $b(t)$ as $t \to \infty$. In the case when $\phi(t) - K$ tends to a nonzero limit as $t \to -\infty$, $N(t) - K$ tends to zero like $\int_t^\infty b(s)ds$ as $t \to \infty$. Moreover, this rate of decay to zero is slower than b. It is therefore of interest to consider how this history-dependent decay rate develops in the case that $\phi \in \Phi$ is unbounded.

In our main results, we show that if b decays polynomially (in the sense that b is in the class of integrable and regularly varying functions) and the history $\phi(t)$ grows polynomially as $t \to -\infty$ (in the sense that ϕ is a regularly varying function at $-\infty$), and the rate of growth of ϕ is not too rapid relative to the rate of decay of b, then problems (1.1) and (1.2) are well posed and $x(t) \to 0$ and $N(t) - K \to 0$ as $t \to \infty$ at the rate $tb(t)\phi(-t)$ as $t \to \infty$.

The question of history-dependent asymptotic behaviour is of interest not only in demography and population dynamics, but also in financial mathematics and time series, and this also motivates our study here. It is well known that certain discrete- and continuous-time stochastic processes have autocovariance functions which can be represented as linear difference or delay-differential equations (see, e.g., Küchler and Mensch [3]). In the case of the so-called ARCH (∞) processes which are stationary, the resulting equation for the autocovariance function of the process can be represented as a Volterra summation equation with infinite memory. For details on the stationarity and autocovariance function of such ARCH processes; see, for example, Zaffaroni [4], Giraitis et al. [5], Robinson [6], and Giraitis et al. [7]. In the nonstationary case, the process may be autocorrelated on $-\mathbb{N}$ is a manner which is inconsistent with the autocovariance function in the stationary case (which must be an even function), while the mean and variance of the process still converge. Therefore, the process can have a limiting autocovariance function which may differ from that of the stationary process. This phenomenon is impossible for processes with bounded memory, and the different convergence rates which depend on the asymptotic behaviour of the initial history in this case is an exact analogue to the history-dependent decay rates recorded here. Therefore, this paper also lays the groundwork for analysis of this phenomenon in finance from the perspective of infinite memory Volterra equations. The interest in such so-called long memory stems in part from the presence of inefficiency in financial markets and the applicability of ARCH-type processes in modelling the evolution of market volatility. Some of the fundamental papers in this direction are Comte and Renault [8], Baillie et al. [9], and Bollerslev and Mikkelsen [10]. An up-to-date survey of work on long memory processes is given by Cont [11].

2. Mathematical Preliminaries

We introduce some standard notation. We denote by $\mathbb{R}$ the set of real numbers. If J is an interval in $\mathbb{R}$ and V a finite dimensional normed space, we denote by $C(J,V)$ the family of continuous functions $\phi : J \to V$. The space of Lebesgue integrable functions $\phi : (0,\infty) \to V$ will be denoted by $L^1((0,\infty),V)$. Where V is clear from the context we omit it from the notation.

The convolution of $f : [0,\infty) \to \mathbb{R}$ and $g : [0,\infty) \to \mathbb{R}$ is denoted by $f * g$ and defined to be the function given by

$$(f * g)(t) = \int_0^t f(s)g(t-s)\,ds, \quad t \geq 0. \tag{2.1}$$

If the domain of f contains an interval of the form (T,∞) and $\gamma : [0,\infty) \to (0,\infty)$, then $L_\gamma f$ denotes $\lim_{t\to\infty} f(t)/\gamma(t)$, if it exists.

2.1. Subexponential and Regularly Varying Functions

We make a definition, based on the hypotheses of Theorem 3 of [12].

Definition 2.1. Let $\beta : [0,\infty) \to (0,\infty)$ be a continuous function. Then we say that β is *subexponential* if

$$\int_0^\infty \beta(t)\,dt < \infty, \tag{2.2}$$

$$\lim_{t\to\infty} \frac{1}{\beta(t)} \int_0^t \beta(t-s)\beta(s)\,ds = 2\int_0^\infty \beta(s)\,ds, \tag{2.3}$$

$$\lim_{t\to\infty} \frac{\beta(t-s)}{\beta(t)} = 1 \quad \text{uniformly for } 0 \leq s \leq S,\ \forall S > 0. \tag{2.4}$$

In [13] the terminology *positive subexponential* function was used instead of just subexponential function. Because subexponential functions play the role here of weight functions, it is natural that they have strictly positive values. The nomenclature subexponential is suggested by the fact that (2.4) implies that, for every $\epsilon > 0$, $\beta(t)e^{\epsilon t} \to \infty$ as $t \to \infty$. This is proved, for example, in [14]. It is also true that

$$\lim_{t\to\infty} \beta(t) = 0. \tag{2.5}$$

In Definition 2.1 above, condition (2.3) can be replaced by

$$\lim_{T\to\infty} \limsup_{t\to\infty} \frac{1}{\beta(t)} \int_T^{t-T} \beta(t-s)\beta(s)\,ds = 0, \tag{2.6}$$

and this latter condition often proves to be useful in proofs.

The properties of subexponential functions have been extensively studied, for example, in [12–15]. Simple examples of subexponential functions are $\beta(t) = (1+t)^{-\alpha}$ for $\alpha > 1$, $\beta(t) = e^{-(1+t)^{\alpha}}$ for $0 < \alpha < 1$ and $\beta(t) = e^{-t/\log(t+2)}$. The class of subexponential functions therefore includes a wide variety of functions exhibiting polynomial and slower-than-exponential decay: nor is the slower-than-exponential decay limited to a class of polynomially decaying functions.

In this paper, however, we restrict our attention to an important subclass of subexponential functions. It is noted in [13] that the class of subexponential functions includes all positive, continuous, integrable functions which are regularly varying at infinity. We recall that a function $\gamma : [0,\infty) \to \mathbb{R}$ is said to be regularly varying at infinity with index $\alpha \in \mathbb{R}$ if

$$\lim_{t\to\infty} \frac{\gamma(\lambda t)}{\gamma(t)} = \lambda^{\alpha}, \quad \forall \lambda > 0, \tag{2.7}$$

and we write $\gamma \in \mathrm{RV}_{\infty}(\alpha)$. When $\alpha < -1$, γ is subexponential. A useful property of a continuous function $\gamma \in \mathrm{RV}_{\infty}(\alpha)$ for $\alpha < -1$ is

$$\lim_{t\to\infty} \frac{\int_t^{\infty} \gamma(s)\,ds}{t\gamma(t)} = -\frac{1}{1+\alpha}. \tag{2.8}$$

In this paper, we also find it convenient to consider functions in $\mathrm{RV}_{\infty}(\alpha)$ for $\alpha \geq -1$. We list some of the important properties used here. A characteristic of regularly varying functions of nonzero index is that they exhibit a type of power-law growth or decay as $t \to \infty$. Indeed, if $\gamma \in \mathrm{RV}_{\infty}(\alpha)$, then

$$\lim_{t\to\infty} \frac{\log|\gamma(t)|}{\log t} = \alpha. \tag{2.9}$$

Hence $|\gamma(t)| \to \infty$ as $t \to \infty$ if $\alpha > 0$ and $\gamma(t) \to 0$ as $t \to \infty$ if $\alpha < 0$.

If $\gamma \in \mathrm{RV}_{\infty}(\alpha)$ for $\alpha > 0$ (and $\gamma(t) \to \infty$ as $t \to \infty$), then γ is asymptotic to a continuous function δ, which is also in $\mathrm{RV}_{\infty}(\alpha)$, such that δ is increasing on $(0,\infty)$. Similarly, if $\gamma \in \mathrm{RV}_{\infty}(\alpha)$ for $\alpha < 0$, and γ is ultimately positive, then γ is asymptotic to a continuous function δ, which is also in $\mathrm{RV}_{\infty}(\alpha)$, such that δ is decreasing on $(0,\infty)$.

A function $\gamma : (-\infty, 0] \to \mathbb{R}$ is said to be regularly varying at minus infinity with index α if the function $\gamma_- : [0,\infty) \to \mathbb{R}$ defined for $t \geq 0$ by $\gamma_-(t) = \gamma(-t)$ is in $\mathrm{RV}_\infty(\alpha)$. We denote the class of regularly varying functions at minus infinity with index α by $\mathrm{RV}_{-\infty}(\alpha)$.

For further details on regularly varying functions, consult [16].

3. Existence and Asymptotic Behaviour of Functionals of the Initial History

3.1. Hypotheses on b and ϕ

We make the following standing hypotheses concerning the kernel b and initial history ϕ of (1.1) and (1.2)

$$b : [0,\infty) \longrightarrow (0,\infty) \text{ is continuous and in } L^1(0,\infty), \tag{3.1}$$

$$\phi : (-\infty, 0] \longrightarrow \mathbb{R} \text{ is continuous.} \tag{3.2}$$

We introduce a function $f : [0,\infty) \to \mathbb{R}$ which depends on the continuous function ϕ. Suppose that

$$\text{For every } t \geq 0, \quad f(t;\phi) \text{ defined by } f(t;\phi) := \int_{-\infty}^{0} b(t-s)\phi(s)ds \text{ exists} \tag{3.3a}$$

$$f(\cdot;\phi) \in C([0,\infty),\mathbb{R}) \tag{3.3b}$$

$$\lim_{t\to\infty} f(t;\phi) = 0. \tag{3.3c}$$

Following [2], we define by Φ the space of initial functions ϕ for which such $f(\cdot;\phi)$ exists and has the properties (3.3b) and (3.3c):

$$\Phi = \{\phi \in C((-\infty,0];\mathbb{R}) : f(\cdot;\phi) \text{ obeys (3.3)}\}. \tag{3.4}$$

The importance of f and Φ in this paper is the following. Suppose that we have an infinite memory integrodifferential equation with solution x and initial history ϕ, that is, $x(t) = \phi(t)$ for $t \in (-\infty, 0]$. If the equation involves a convolution term of the form

$$\int_{-\infty}^{t} b(t-s)x(s)ds, \quad t \geq 0 \tag{3.5}$$

on the right-hand side, the infinite memory equation is equivalent to an initial value integrodifferential equation with unbounded memory, provided that ϕ is such that (3.3a) holds. The existence and uniqueness of a solution of the integrodifferential equation is essentially guaranteed by (3.3b), and asymptotic analysis (and in particular stability) is aided by (3.3c). Therefore the class of initial histories Φ helps us to recast questions about the existence, uniqueness, and asymptotic stability of solutions of an infinite memory convolution equation in terms of a perturbed initial value convolution equation, where f, to a certain extent, plays the role of a forcing term or perturbation.

We now impose some additional conditions on b and ϕ which enable us to demonstrate that $\phi \in \Phi$ and which are also central to the asymptotic analysis of solutions of (1.1) and (1.2). To this end suppose that b obeys

$$b \in \mathrm{RV}_{\infty}(\delta) \quad \text{for some } \delta \in (-\infty, -1], \tag{3.6}$$

In addition to (3.2), ϕ also obeys

$$\phi \in \mathrm{RV}_{-\infty}(\eta) \quad \text{for some } \eta > 0 \text{ with } \lim_{t \to -\infty} \phi(t) = \infty. \tag{3.7}$$

Suppose further that

$$\delta + \eta + 1 < 0. \tag{3.8}$$

It is often convenient to work with the function $\phi_{+} : [0, \infty) \to \mathbb{R}$ defined by $\phi_{+}(t) = \phi(-t)$ for $t \geq 0$, rather than with ϕ itself. An important property of ϕ_{+} is that it is in $\mathrm{RV}_{\infty}(\eta)$.

By virtue of the fact that b is regularly varying at infinity with index $\delta < -1$, it follows that there exists a function β such that

$$\beta \in C([0, \infty); (0, \infty)) \text{ is decreasing}, \quad \lim_{t \to \infty} \beta(t) = 0, \quad \lim_{t \to \infty} \frac{b(t)}{\beta(t)} \neq 0 \text{ exists}, \tag{3.9}$$

and β is also forced to satisfy

$$\frac{\beta(t-s)}{\beta(t)} \longrightarrow 1 \quad \text{as } t \longrightarrow \infty \text{ uniformly for } 0 \leq s \leq T, \ \forall T \geq 0. \tag{3.10}$$

Also, because ϕ_{+} is regularly varying at infinity with index $\eta > 0$, there exists a function φ which is increasing and which obeys $\phi_{+}(t)/\varphi(t) \to 1$ as $t \to \infty$.

3.2. Existence and Asymptotic Behaviour of $f(\cdot; \phi)$

Our results in this section demonstrate that, under the hypotheses (3.6) and (3.7), $f(\cdot; \phi)$ has the properties given in (3.3a), (3.3b), and (3.3c). The proofs of the main results are postponed to later in the paper.

Remark 3.1. Condition (3.8) implies that

$$\int_{0}^{\infty} b(s)\phi(-s)ds \text{ is finite.} \tag{3.11}$$

It can be seen that (3.11) is necessary for the existence of $f(t; \phi)$ for $t \geq 0$ (i.e., for the validity of (3.3a)). This is because the integral in (3.11) is $f(0; \phi)$.

Condition (3.11) is close to being sufficient for the existence of $f(\cdot;\phi)$ and indeed is sufficient if b and ϕ are nonnegative. In fact, because b is integrable, we have that $f(t;\phi) \to 0$ as $t \to \infty$.

Proposition 3.2. *Suppose that b obeys* (3.1) *and* (3.6) *and that ϕ obeys* (3.2) *and* (3.7). *If δ and η obey* (3.8), *then b and ϕ obey*

$$\int_0^\infty |b(s)|\left|\phi(-s)\right| ds < +\infty, \tag{3.12}$$

and $f(\cdot;\phi)$ exists for all $t \geq 0$ and therefore obeys (3.3a). *Moreover $f(\cdot;\phi)$ obeys* (3.3c).

It is notable that condition (3.12) does not require that $|\phi|$ be bounded, but merely that it cannot grow too quickly as $t \to -\infty$, relative to the rate of decay of $b(t) \to 0$ as $t \to \infty$. This is the significance of the parameter restriction (3.8). Scrutiny of the proof, which is in Section 7, reveals that the regular variation of b and ϕ is used sparingly. Indeed, if one assumes (3.12), the properties (3.9) and (3.10) suffice to prove the result.

Conditions (3.9) and (3.10) will be used to establish the continuity of $t \mapsto f(t;\phi)$, as well as for later asymptotic analysis of $f(t;\phi)$ as $t \to \infty$.

We notice that by virtue of (3.9) that $b(t) \to 0$ as $t \to \infty$. Since b is also continuous, it follows that it is uniformly continuous. This fact is used at important points in the proof of the following result.

Proposition 3.3. *Suppose that b obeys* (3.1) *and* (3.6) *and that ϕ obeys* (3.2) *and* (3.7). *Since b and ϕ also obey* (3.12), *then $f(t;\phi)$ exists for all $t \geq 0$, $t \mapsto f(t;\phi)$ is continuous and $f(t;\phi) \to 0$ as $t \to \infty$ (i.e., $\phi \in \Phi$).*

A careful reading of the proof (again deferred to Section 7) reveals that it is the properties (3.9) and (3.10), together with (3.12), that are employed, and that the full strength of (3.6) and (3.7) is unnecessary.

Having shown that f obeys all the properties in (3.3a), (3.3b), and (3.3c), including the fact that $f(t;\phi) \to 0$ as $t \to \infty$, our first main result determines the exact rate of decay to zero of $f(t;\phi)$ as $t \to \infty$. In contrast to the other results in this section, the proof of this result employs extensively the regular variation of b and of ϕ.

Theorem 3.4. *Suppose that b is a positive continuous function which obeys* (3.6) *for some $\delta < -1$. Let ϕ be a function which obeys* (3.7) *for some $\eta > 0$, and suppose that $\delta + \eta + 1 < 0$. Then $\phi \in \Phi$ and f defined in* (3.3a), (3.3b), *and* (3.3c) *obeys*

$$\lim_{t\to\infty} \frac{f(t;\phi)}{tb(t)\phi(-t)} = \int_0^\infty x^\eta (1+x)^\delta dx. \tag{3.13}$$

The proof of Theorem 3.4 is postponed to Section 5. We note that the integral on the right-hand side of (3.13) exists because $\eta > 0$ and $\delta + \eta + 1 < 0$. We also notice that as $b \in \mathrm{RV}_\infty(\delta)$ and $\phi_+ \in \mathrm{RV}_\infty(\eta)$, by (3.13), the function $f(\cdot;\phi) \in \mathrm{RV}_\infty(\delta + \eta + 1)$.

4. Statement and Discussion of Main Results

4.1. Linear Equations with Unbounded Initial History

Once Theorem 3.4 has been proven, we are able to determine the rate of decay of the solution of the following linear infinite memory convolution equation

$$\begin{gathered} x'(t) = ax(t) + \int_{-\infty}^{t} b(t-s)x(s)ds, \quad t > 0, \\ v = \phi(t), \quad t \in (-\infty, 0]. \end{gathered} \tag{4.1}$$

If we suppose that b and ϕ obey merely (3.1) and (3.2), and that $\phi \in \Phi$ (where Φ is the space defined by (3.4)), the function f defined in (3.3a), (3.3b), and (3.3c) is well defined and continuous on $[0, \infty)$. Therefore, we see that (4.1) can be written in the equivalent form

$$x'(t) = ax(t) + \int_{0}^{t} b(t-s)x(s)ds + f\left(t;\phi\right), \quad t > 0, \quad x(0) = \phi(0), \tag{4.2}$$

where $x(t) = \phi(t)$ for $t \in (-\infty, 0]$. Since this initial value problem has a unique continuous solution, it follows that there is a unique continuous solution of (4.1). However, as we assume that b and ϕ obey the hypotheses (3.6), (3.7), (3.1), (3.2), and (3.8) throughout, it follows that $\phi \in \Phi$, and therefore (4.1) has a unique continuous solution x.

We now investigate conditions under which $x(t) \to 0$ as $t \to \infty$, and the rate of convergence to zero. To study this asymptotic behaviour, it is conventional to introduce the linear differential resolvent r which is defined to be the unique continuous solution of the integrodifferential equation

$$r'(t) = ar(t) + \int_{0}^{t} b(t-s)r(s)ds, \quad t > 0, \quad r(0) = 1. \tag{4.3}$$

The significance of r is that it enables us to represent the unique continuous solution x of (4.1) in terms of $f(t;\phi)$ (defined in (3.3a)). Using (4.2), the formula for x is given by

$$x(t) = r(t)\phi(0) + \int_{0}^{t} r(t-s)f\left(s;\phi\right)ds, \quad t \geq 0. \tag{4.4}$$

In the case when $a + \int_0^\infty |b(s)|ds < 0$, it is known that $r \in L^1(0,\infty)$ and $r(t) \to 0$ as $t \to \infty$. Therefore, in this case $x(t) \to 0$ as $t \to \infty$. Some recent results on the asymptotic stability of Volterra equations with unbounded delay include [17, 18].

Moreover, as b is in $\mathrm{RV}_\infty(\delta)$ for $\delta < -1$, b is subexponential, and so it is known by results of, for example, [19], that

$$\lim_{t\to\infty} \frac{r(t)}{\beta(t)} \text{ exists and is finite,} \tag{4.5}$$

due to (3.9). Therefore, as we already have good information about the rate of convergence of $f(t;\phi) \to 0$ as $t \to \infty$ from Theorem 3.4, the representation (4.4) together with (4.5) opens the prospect that the rate of convergence of $x(t) \to 0$ as $t \to \infty$ can be obtained. Our main result in this direction is as follows.

Theorem 4.1. *Suppose that b is a continuous and integrable function which obeys* (3.6) *for some $\delta < -1$. Let ϕ be a continuous function which obeys* (3.7) *for some $\eta > 0$ and suppose that $\delta+\eta+1 < 0$. If $a + \int_0^\infty |b(s)|ds < 0$, then x, the unique continuous solution of* (4.1), *obeys*

$$\lim_{t\to\infty} \frac{x(t)}{tb(t)\phi(-t)} = -\int_0^\infty u^\eta (1+u)^\delta \, du \cdot \frac{1}{a+\int_0^\infty b(s)ds}. \tag{4.6}$$

It is worth re-emphasising that the condition $\delta + \eta + 1 < 0$ is not a merely a technical convenience; in the case when $\phi(t) > 0$ for $t \in (-\infty, 0]$ and $\delta + \eta + 1 > 0$, problem (4.1) is not well posed, because $f(0;\phi)$, for example, is not well defined.

The proof of Theorem 4.1 is given in Section 6.1, and uses results from the admissibility theory of linear Volterra operators. These results are stated in Section 6, in advance of the proof of Theorem 4.1.

4.2. Delay Logistic Equation with Unbounded Initial History

In this section, we state and discuss a result similar to Theorem 4.1 for a nonlinear integrodifferential equation with infinite memory. We consider the logistic equation with infinite delay

$$\begin{gathered} N'(t) = N(t)\left(r - aN(t) - \int_{-\infty}^t b(t-s)N(s)ds\right), \quad t > 0, \\ N(t) = \phi(t), \quad t \in (-\infty, 0], \end{gathered} \tag{4.7}$$

where b is continuous and integrable, ϕ is continuous, and a and r are real numbers. This equation, and related equations, have been used to study the population dynamics of a single species, where $N(t)$ stands for the population at time t.

If $f(\cdot;\phi)$ is the function given in (3.3a), it is seen that the existence of a solution of (4.7) is equivalent to the existence of a solution of

$$N'(t) = N(t)\left(r - aN(t) - \int_0^t b(t-s)N(s)ds - f(t;\phi)\right), \quad t > 0, \quad N(0) = \phi(0). \tag{4.8}$$

Therefore, it is necessary that the function f be well defined in order for solutions of (4.7) to exist. In the case that $\phi \in \Phi$, then the function $f(\cdot;\phi)$ given in (3.3a) is well defined and is moreover continuous. Therefore standard results on existence, uniqueness, and continuation of solutions of Volterra integral equations (cf., e.g., Burton [20], Gripenberg et al. [21], Miller [1]) ensure that there is a unique solution of (4.7) (up to a possible explosion time). For the proof of positivity of the solution see, for example, Miller [1]. We state [2, Theorem 1] which concerns on the asymptotic behaviour of solutions of (4.7).

Theorem 4.2. *Let $b \in C([0,\infty),(0,\infty)) \cap L^1([0,\infty),(0,\infty))$, $a > \int_0^\infty b(s)ds$, $r > 0$. Let Φ be defined by* (3.4)*, and suppose that $\phi \in C((-\infty,0],(0,\infty))$ is also in Φ. Then there is a unique continuous positive solution N of* (4.7) *which obeys*

$$\lim_{t\to\infty} N(t) = \frac{r}{a+\int_0^\infty b(s)ds} =: K. \tag{4.9}$$

This theorem extends a result of [1] which deals with the case when ϕ is a bounded continuous function. We remark once more that the condition $\phi \in \Phi$ does not require ϕ to be bounded. Some other recent papers which employ Volterra equations with unbounded delay to model stable population dynamics include [22, 23].

With Theorem 4.2 in hand, we can determine the convergence rate of the solution N of (4.7) to K defined in (4.9).

Theorem 4.3. *Suppose that b is a positive continuous function which obeys* (3.6) *for some $\delta < -1$. Let $a > \int_0^\infty b(s)ds$ and $r > 0$. Let $\phi \in C((-\infty,0],(0,\infty))$ obey* (3.7) *for some $\eta > 0$, and suppose that $\delta + \eta + 1 < 0$. Then N, the unique continuous positive solution of* (4.7)*, obeys*

$$\lim_{t\to\infty} \frac{K-N(t)}{tb(t)\phi(-t)} = \int_0^\infty u^\eta (1+u)^\delta \; du \cdot \frac{1}{a+\int_0^\infty b(s)ds}, \tag{4.10}$$

where K is defined by (4.9).

Once again, the proof appeals to results from the admissibility theory of linear Volterra operators. The proof of Theorem 4.3 is deferred to Section 6.2.

It is interesting to compare this result with those obtained for (4.7) under different conditions on ϕ and subexponential b in [2]. Suppose, as in Theorem 4.3 above, that $a > \int_0^\infty b(s)ds$, $\phi \in \Phi$ is continuous and positive, and b is positive, continuous, and integrable. In the case when b a fortiori obeys (3.6), and there exists $L \neq 0$ such that

$$\lim_{t\to\infty} \{\phi(-t) - K\} = L, \tag{4.11}$$

then by [2, Theorem 2] there exists $c \neq 0$ such that

$$\lim_{t\to\infty} \frac{K-N(t)}{tb(t)} = c. \tag{4.12}$$

On the other hand, by [2, Theorem 2] we have that

$$\int_{-\infty}^0 \left|\phi(t) - K\right| dt < +\infty \tag{4.13}$$

implies

$$\lim_{t\to\infty} \frac{K-N(t)}{b(t)} \text{ exists.} \tag{4.14}$$

In both these cases, there is a history-dependent (i.e., a ϕ-dependent) rate of convergence to the equilibrium; moreover, it appears that the larger the "size" of the history (as measured by the discrepancy of ϕ from K), the slower the rate of convergence to K. In Theorem 4.3 we show that the rate of convergence is slower ($tb(t)\phi(-t)$ as $t \to \infty$) than in both (4.12) ($tb(t)$ as $t \to \infty$) and (4.14) ($b(t)$ as $t \to \infty$). This is consistent with the picture that a "larger" history leads to a slower rate of convergence, as the history in Theorem 4.3 obeys $\lim_{t\to-\infty}\phi(t) = \infty$, in contrast to the "bounded" histories in (4.11) and (4.13). Unbounded histories are studied in [2], but only for equations in which b decays exponentially fast to zero, in the sense that $t \mapsto b(t)e^{\lambda t}$ is subexponential for some $\lambda > 0$, in which case, $\phi(t)$ can grow as $t \to -\infty$ according to

$$\lim_{t\to-\infty} e^{\lambda t}(\phi(t) - K) = L \neq 0 \quad \text{or} \quad \int_{-\infty}^{0} e^{\lambda}\left|\phi(t) - K\right| dt < +\infty, \tag{4.15}$$

and results similar to (4.14) or (4.12) can be established.

An interesting question, which we do not address here, is to determine the rate of convergence to the equilibrium for solutions of (4.7) in the case when $\phi - K \in \mathrm{RV}_{-\infty}(\eta)$ for $\eta \in (-1, 0)$. In this case, $\phi - K$ is not integrable, but $\phi(t)$ tends to K as $t \to -\infty$. Therefore, these cases cover histories ϕ whose discrepancy from K is intermediate between those ϕ covered by conditions (4.13) and (4.11). It might be expected that a similar rate of convergence to zero would be found for solutions of (4.1) in the case when $\phi \in \mathrm{RV}_{-\infty}(\eta)$ for $\eta \in (-1, 0)$. Obviously, the key ingredient to proving such results is an analysis of the rate of convergence of $f(t;\phi) \to 0$ as $t \to \infty$.

5. Proof of Theorem 3.4

Theorem 3.4 follows by a number of lemmas. The first part of this section discusses and presents these results; the rest of the section is devoted to their proofs.

5.1. Discussion of Supporting Lemmas

We suppose that b and ϕ obey (3.1) and (3.2) throughout. In the first lemma supporting Theorem 3.4, we show that the requirement that b and ϕ be nonmonotone can essentially be lifted. The key result is the following.

Lemma 5.1. *Suppose that b obeys* (3.6) *and ϕ obeys* (3.7), *then there exist a decreasing continuous function β such that $b(t)/\beta(t) \to 1$ as $t \to \infty$ and an increasing function φ such that $\phi(-t)/\varphi(t) \to 1$ as $t \to \infty$. If there exists $L > 0$ such that*

$$\lim_{t\to\infty} \frac{\int_0^\infty \varphi(s)\beta(s+t)ds}{t\beta(t)\varphi(t)} = L, \tag{5.1}$$

and $f(\cdot;\phi)$ is the function defined by (3.3a), *then*

$$\lim_{t\to\infty} \frac{f(t;\phi)}{t\beta(t)\varphi(t)} = L. \tag{5.2}$$

The next result shows that, subject to a technical condition, the conclusion of Theorem 3.4 holds for monotone β and φ.

Lemma 5.2. *Suppose that β is a decreasing and continuous function in $RV_\infty(\delta)$ for $\delta < -1$, and that φ is an increasing and continuous function in $RV_\infty(\eta)$ for $\eta > 0$. Let $\delta + \eta + 1 < 0$. If*

$$\lim_{t\to\infty} \frac{1}{\beta(t)\varphi(t)} \sum_{j=0}^{\infty} \varphi((j+1)ht)\beta(jht+t) = \overline{\Lambda}(h) := \sum_{j=0}^{\infty} ((j+1)h)^{\eta}(jh+1)^{\delta}, \quad \text{for each fixed } h > 0, \tag{5.3}$$

$$\lim_{t\to\infty} \frac{1}{\beta(t)\varphi(t)} \sum_{j=1}^{\infty} \varphi(jht)\beta((j+1)ht+t) = \underline{\Lambda}(h) := \sum_{j=1}^{\infty} (jh)^{\eta}((j+1)h+1)^{\delta}, \quad \text{for each fixed } h > 0, \tag{5.4}$$

then

$$\lim_{t\to\infty} \frac{\int_0^\infty \varphi(s)\beta(s+t)ds}{t\beta(t)\varphi(t)} = \int_0^\infty x^{\eta}(1+x)^{\delta} dx. \tag{5.5}$$

To prove Lemma 5.2, we need the following auxiliary result.

Lemma 5.3. *If $\overline{\Lambda}$ is defined by* (5.3) *and $\underline{\Lambda}$ by* (5.4)*, then*

$$\lim_{h\to 0^+} h\overline{\Lambda}(h) = \int_0^\infty x^{\eta}(1+x)^{\delta} dx, \qquad \lim_{h\to 0^+} h\underline{\Lambda}(h) = \int_0^\infty x^{\eta}(1+x)^{\delta} dx. \tag{5.6}$$

Finally, we need to prove the suppositions (5.3) and (5.4).

Lemma 5.4. *If β is a decreasing and continuous function in $RV_\infty(\delta)$ for $\delta < -1$ and $\varphi \in RV_\infty(\eta)$ for $\eta > 0$ and $\delta + \eta + 1 < 0$, then* (5.3) *and* (5.4) *hold.*

The proofs of these lemmas are given in the following subsections. It is readily seen that by taking the results of Lemmas 5.2 and 5.4 together with the result of Lemma 5.1 with $L = \int_0^\infty x^{\eta}(1+x)^{\delta} dx$, Theorem 3.4 is true.

5.2. Proof of Lemma 5.1

Since $b(t)/\beta(t) \to 1$ as $t \to \infty$ and $\phi_+(t)/\varphi(t) \to 1$ as $t \to \infty$ for every $\varepsilon \in (0,1)$ there exists $T(\varepsilon) > 0$ such that $1-\varepsilon < b(t)/\beta(t) < 1+\varepsilon$ for all $t \geq T(\varepsilon)$ and $1-\varepsilon < \phi_+(t)/\varphi(t) < 1+\varepsilon$ for all $t \geq T(\varepsilon)$. Therefore

$$\int_T^\infty \phi_+(s)b(s+t)ds \leq (1+\varepsilon)^2 \int_T^\infty \varphi(s)\beta(s+t)ds, \tag{5.7}$$

$$\int_T^\infty \phi_+(s)b(s+t)ds \geq (1-\varepsilon)^2 \int_T^\infty \varphi(s)\beta(s+t)ds. \tag{5.8}$$

Now

$$\begin{aligned}\lim_{t\to\infty} \frac{1}{\beta(t)} \int_0^T \phi_+(s)b(s+t)ds &= \lim_{t\to\infty} \int_0^T \phi_+(s) \frac{b(s+t)}{\beta(s+t)} \frac{\beta(s+t)}{\beta(t)} ds \\ &= \int_0^T \phi_+(s)ds.\end{aligned} \tag{5.9}$$

Therefore as $t \mapsto t\varphi(t)$ is in $\mathrm{RV}_\infty(\eta+1)$ and $\eta+1>0$, we have $t\varphi(t) \to \infty$ as $t \to \infty$, and so

$$\lim_{t\to\infty} \frac{1}{t\beta(t)\varphi(t)} \int_0^T \phi_+(s)b(s+t)ds = 0. \tag{5.10}$$

Similarly

$$\lim_{t\to\infty} \frac{1}{t\beta(t)\varphi(t)} \int_0^T \varphi(s)\beta(s+t)ds = 0. \tag{5.11}$$

Hence by (5.1) we have

$$\lim_{t\to\infty} \frac{1}{t\beta(t)\varphi(t)} \int_T^\infty \varphi(s)\beta(s+t)ds = L. \tag{5.12}$$

Now

$$\frac{f(t;\phi)}{t\beta(t)\varphi(t)} = \frac{1}{t\beta(t)\varphi(t)} \int_0^T \phi_+(s)b(s+t)ds + \frac{1}{t\beta(t)\varphi(t)} \int_T^\infty \phi_+(s)b(s+t)ds. \tag{5.13}$$

so by (5.7) and (5.9)

$$\limsup_{t\to\infty} \frac{f(t;\phi)}{t\beta(t)\varphi(t)} \leq L(1+\varepsilon)^2. \tag{5.14}$$

Similarly by (5.8) and (5.9) we have

$$\liminf_{t\to\infty}\frac{f(t;\phi)}{t\beta(t)\varphi(t)}\geq L(1-\varepsilon)^2. \tag{5.15}$$

Letting $\varepsilon\to 0^+$ in (5.15) and (5.14) gives (5.2).

5.3. Proof of Lemma 5.2

Fix $h\in(0,1)$. Since β is decreasing and φ is increasing, we have

$$\begin{aligned}\int_0^\infty \varphi(s)\beta(s+t)ds &= \sum_{j=0}^{\infty}\int_{jht}^{(j+1)ht}\varphi(s)\beta(s+t)ds\\ &\leq \sum_{j=0}^{\infty}\int_{jht}^{(j+1)ht}\varphi((j+1)ht)\beta(jh\mathrm{t}+t)ds\\ &\leq \sum_{j=0}^{\infty}ht\varphi((j+1)ht)\beta(jht+t).\end{aligned} \tag{5.16}$$

Similarly

$$\int_0^\infty \varphi(s)\beta(s+t)ds\geq\sum_{j=0}^{\infty}ht\varphi(jht)\beta((j+1)ht+t). \tag{5.17}$$

Suppose we can show that (5.4) holds, then

$$\begin{aligned}&\liminf_{t\to\infty}\frac{1}{t\beta(t)\varphi(t)}\int_0^\infty\varphi(s)\beta(s+t)ds\\ &\quad\geq\liminf_{t\to\infty}\frac{1}{t\beta(t)\varphi(t)}\cdot\sum_{j=0}^{\infty}ht\varphi(jht)\beta((j+1)ht+t)=h\underline{\Lambda}(h).\end{aligned} \tag{5.18}$$

Also by (5.3)

$$\begin{aligned}&\limsup_{t\to\infty}\frac{1}{t\beta(t)\varphi(t)}\int_0^\infty\varphi(s)\beta(s+t)ds\\ &\quad\leq\limsup_{t\to\infty}\frac{1}{t\beta(t)\varphi(t)}\sum_{j=0}^{\infty}ht\varphi((j+1)ht)\beta(jht+t)=h\overline{\Lambda}(h).\end{aligned} \tag{5.19}$$

Therefore

$$\limsup_{t\to\infty}\frac{1}{t\beta(t)\varphi(t)}\int_0^\infty \varphi(s)\beta(s+t)ds \le h\overline{\Lambda}(h),$$
$$\liminf_{t\to\infty}\frac{1}{t\beta(t)\varphi(t)}\int_0^\infty \varphi(s)\beta(s+t)ds \ge h\underline{\Lambda}(h). \tag{5.20}$$

By (5.20), using the facts that $\delta+\eta+1<0$ and $\eta>0$, and by employing (5.6), we have (5.5) as required.

5.4. *Proof of Lemma 5.3*

The required results are

$$\lim_{h\to 0^+}\sum_{j=0}^{\infty} h\big((j+1)h\big)^{\eta}(jh+1)^{\delta} = \int_0^\infty x^{\eta}(1+x)^{\delta}dx, \tag{5.21}$$

$$\lim_{h\to 0^+}\sum_{j=0}^{\infty} h(jh)^{\eta}\big((j+1)h+1\big)^{\delta} = \int_0^\infty x^{\eta}(1+x)^{\delta}dx. \tag{5.22}$$

We pause to remark that the integrals on the right-hand side of both (5.21) and (5.22) are finite. To start, notice that

$$\int_0^\infty x^{\eta}(1+x)^{\delta}dx = \sum_{j=0}^{\infty}\int_{jh}^{(j+1)h} x^{\eta}(1+x)^{\delta}dx \ge \sum_{j=0}^{\infty} h(jh)^{\eta}\big(1+(j+1)h\big)^{\delta}, \tag{5.23}$$

$$\int_0^\infty x^{\eta}(1+x)^{\delta}dx \le \sum_{j=0}^{\infty} h\big((j+1)h\big)^{\eta}(1+jh)^{\delta}. \tag{5.24}$$

Let $\varepsilon\in(0,1)$. Let $J(\varepsilon)\in\mathbb{N}$ be such that $J(\varepsilon)>(1-\varepsilon)/\varepsilon$. Then for $j>J(\varepsilon)$ we have $j>(1-\varepsilon)/\varepsilon$ and so

$$\frac{j}{j+1}>1-\varepsilon, \quad j>J(\varepsilon). \tag{5.25}$$

Also, as $\varepsilon\in(0,1)$, for every $h>0$ we have $(\varepsilon+h)(1-\varepsilon)<\varepsilon+(1-\varepsilon)h$ so because $j>J(\varepsilon)>(1-\varepsilon)/\varepsilon$, we have

$$\frac{1+jh+h}{1+jh}=1+\frac{h}{1+jh}<1+\frac{h}{1+(1-\varepsilon)/\varepsilon\cdot h}<\frac{1}{1-\varepsilon}. \tag{5.26}$$

This implies

$$\frac{1+jh}{1+jh+h}>1-\varepsilon, \quad j>J(\varepsilon). \tag{5.27}$$

By (5.23), we have

$$\int_0^\infty x^\eta (1+x)^\delta dx \geq \sum_{j=0}^{\infty} h((j+1)h)^\eta (1+jh)^\delta \left(\frac{j}{j+1}\right)^\eta \left(\frac{1+jh}{1+(j+1)h}\right)^{-\delta}. \tag{5.28}$$

Hence by (5.25) and (5.27) and the fact that $\eta > 0$ and $-\delta > 0$, we get

$$\sum_{j=J(\varepsilon)+1}^{\infty} h(jh)^\eta (1+(j+1)h)^\delta > (1-\varepsilon)^\eta (1-\varepsilon)^{-\delta} \sum_{j=J(\varepsilon)+1}^{\infty} h((j+1)h)^\eta (1+jh)^\delta. \tag{5.29}$$

Therefore

$$\begin{aligned}\sum_{j=0}^{\infty} h(jh)^\eta (1+(j+1)h)^\delta > &\sum_{j=0}^{J(\varepsilon)} h(jh)^\eta (1+(j+1)h)^\delta \\ &+ (1-\varepsilon)^{\eta-\delta} \sum_{j=J(\varepsilon)+1}^{\infty} h((j+1)h)^\eta (1+jh)^\delta.\end{aligned} \tag{5.30}$$

Hence

$$\begin{aligned}&\liminf_{h\to 0^+} \sum_{j=0}^{\infty} h(jh)^\eta (1+(j+1)h)^\delta \\ &\qquad \geq (1-\varepsilon)^{\eta-\delta} \liminf_{h\to 0^+} \sum_{j=J(\varepsilon)+1}^{\infty} h((j+1)h)^\eta (1+jh)^\delta, \\ &\limsup_{h\to 0^+} \sum_{j=0}^{\infty} h(jh)^\eta (1+(j+1)h)^\delta \\ &\qquad \geq (1-\varepsilon)^{\eta-\delta} \limsup_{h\to 0^+} \sum_{j=J(\varepsilon)+1}^{\infty} h((j+1)h)^\eta (1+jh)^\delta.\end{aligned} \tag{5.31}$$

Now

$$\lim_{h\to 0^+} \sum_{j=0}^{J(\varepsilon)} h((j+1)h)^\eta (1+jh)^\delta = 0, \tag{5.32}$$

so we have

$$\begin{aligned}\liminf_{h\to 0^+} \sum_{j=0}^{\infty} h(jh)^\eta (1+(j+1)h)^\delta &\geq (1-\varepsilon)^{\eta-\delta} \liminf_{h\to 0^+} \sum_{j=0}^{\infty} h((j+1)h)^\eta (1+jh)^\delta, \\ \limsup_{h\to 0^+} \sum_{j=0}^{\infty} h(jh)^\eta (1+(j+1)h)^\delta &\geq (1-\varepsilon)^{\eta-\delta} \limsup_{h\to 0^+} \sum_{j=0}^{\infty} h((j+1)h)^\eta (1+jh)^\delta.\end{aligned} \tag{5.33}$$

Letting $\varepsilon \to 0$ yields

$$\liminf_{h\to 0^+}\sum_{j=0}^{\infty} h(jh)^{\eta}(1+(j+1)h)^{\delta} \geq \liminf_{h\to 0^+}\sum_{j=0}^{\infty} h((j+1)h)^{\eta}(1+jh)^{\delta}, \tag{5.34}$$

$$\limsup_{h\to 0^+}\sum_{j=0}^{\infty} h(jh)^{\eta}(1+(j+1)h)^{\delta} \geq \limsup_{h\to 0^+}\sum_{j=0}^{\infty} h((j+1)h)^{\eta}(1+jh)^{\delta}. \tag{5.35}$$

By (5.23) and (5.35), we have

$$\begin{aligned}\int_0^{\infty} x^{\eta}(1+x)^{\delta}dx &\geq \limsup_{h\to 0^+}\sum_{j=0}^{\infty} h(jh)^{\eta}(1+(j+1)h)^{\delta} \\ &\geq \limsup_{h\to 0^+}\sum_{j=0}^{\infty} h((j+1)h)^{\eta}(1+jh)^{\delta}.\end{aligned} \tag{5.36}$$

Similarly, by (5.24) we have

$$\int_0^{\infty} x^{\eta}(1+x)^{\delta}dx \leq \liminf_{h\to 0^+}\sum_{j=0}^{\infty} h((j+1)h)^{\eta}(1+jh)^{\delta}. \tag{5.37}$$

Combining these inequalities gives (5.21) as required. By (5.34) and (5.21), we get

$$\liminf_{h\to 0^+}\sum_{j=0}^{\infty} h(jh)^{\eta}(1+(j+1)h)^{\delta} \geq \lim_{h\to 0^+}\sum_{j=0}^{\infty} h((j+1)h)^{\eta}(1+jh)^{\delta} = \int_0^{\infty} x^{\eta}(1+x)^{\delta}dx. \tag{5.38}$$

On the other hand, by (5.23) we have

$$\int_0^{\infty} x^{\eta}(1+x)^{\delta}dx \geq \limsup_{h\to 0^+}\sum_{j=0}^{\infty} h(jh)^{\eta}(1+(j+1)h)^{\delta}, \tag{5.39}$$

so by combining these inequalities, we get (5.22) as required.

5.5. Proof of Lemma 5.4

Let $h \in (0,1)$. Since β is decreasing for $j \geq 0$ we have

$$\frac{\beta((1+jh)t)}{\beta(t)}\cdot\frac{\varphi((1+j)ht)}{\varphi(t)} \leq \frac{\beta((1+j)ht)}{\beta(ht)}\cdot\frac{\beta(ht)}{\beta(t)}\cdot\frac{\varphi((1+j)ht)}{\varphi(ht)}\cdot\frac{\varphi(ht)}{\varphi(t)}. \tag{5.40}$$

Since β and φ are continuous and are in $\mathrm{RV}_\infty(\delta)$ and $\mathrm{RV}_\infty(\eta)$, respectively, there exists $K(h) > 0$ such that $\beta(ht)/\beta(t) \le K(h)$ and $\varphi(ht)/\varphi(t) \le K(h)$ for all $t \ge 0$. Hence with $\beta_h(t) = \beta(ht)$ and $\varphi_h(t) = \varphi(ht)$ we have

$$\frac{\beta((1+jh)t)}{\beta(t)} \cdot \frac{\varphi((1+j)ht)}{\varphi(t)} \le K(h)^2 \frac{\beta_h((1+j)t)}{\beta_h(t)} \cdot \frac{\varphi_h((1+j)t)}{\varphi_h(t)}, \quad t \ge 0. \tag{5.41}$$

We see that $\beta_h \in \mathrm{RV}_\infty(\delta)$ and β_h is decreasing, while $\varphi_h \in \mathrm{RV}_\infty(\eta)$ and φ_h is increasing. Therefore, we have

$$\lim_{t\to\infty} \frac{\beta_h(2t)}{\beta_h(t)} = 2^\delta, \qquad \lim_{t\to\infty} \frac{\varphi_h(2t)}{\varphi_h(t)} = 2^\eta. \tag{5.42}$$

Since $\delta + \eta + 1 < 0$, we may choose $\varepsilon > 0$ so small that $A_\varepsilon := (1+\varepsilon)^2 2^{\delta+\eta+1} < 1$. By (5.42), for every $\varepsilon > 0$ sufficiently small, there exists $T(\varepsilon, h) > 1$ such that

$$\frac{\beta_h(2t)}{\beta_h(t)} < (1+\varepsilon)2^\delta, \quad \frac{\varphi_h(2t)}{\varphi_h(t)} < (1+\varepsilon)2^\eta, \quad t > T(\varepsilon, h). \tag{5.43}$$

Let $n \ge 0$ be an integer. Then for $t > T(\varepsilon, h)$ we have

$$\begin{gathered}
\frac{\beta_h(2^n t)}{\beta_h(t)} = \prod_{l=1}^{n} \frac{\beta_h(2^l t)}{\beta_h(2^{l-1} t)} < \left((1+\varepsilon)2^\delta\right)^n, \\
\frac{\varphi_h(2^{n+1} t)}{\varphi_h(t)} = \prod_{l=1}^{n+1} \frac{\varphi_h(2^l t)}{\varphi_h(2^{l-1} t)} < ((1+\varepsilon)2^\eta)^{n+1}.
\end{gathered} \tag{5.44}$$

For every integer $j \ge 0$ there exists a unique integer $n \ge 0$ such that $2^n \le j+1 < 2^{n+1}$. Suppose that $t > T(\varepsilon, h)$. Then as β_h is decreasing, and φ_h is increasing, we have

$$\begin{gathered}
\frac{\beta_h((j+1)t)}{\beta_h(t)} \le \frac{\beta_h(2^n t)}{\beta_h(t)} < \left((1+\varepsilon)2^\delta\right)^n, \\
\frac{\varphi_h((j+1)t)}{\varphi_h(t)} \le \frac{\varphi_h(2^{n+1} t)}{\varphi_h(t)} < ((1+\varepsilon)2^\eta)^{n+1}.
\end{gathered} \tag{5.45}$$

Hence by (5.41) for $2^n \le j+1 < 2^{n+1}$, we have

$$\frac{\beta((1+jh)t)}{\beta(t)} \cdot \frac{\varphi((1+j)ht)}{\varphi(t)} \le K(h)^2 \left((1+\varepsilon)2^\delta\right)^n ((1+\varepsilon)2^\eta)^{n+1}, \quad t > T(\varepsilon, h). \tag{5.46}$$

Define for $n \ge 0$

$$C_n(t) := \sum_{j=2^n-1}^{2^{n+1}-2} \frac{\beta((1+jh)t)}{\beta(t)} \cdot \frac{\varphi((1+j)ht)}{\varphi(t)}. \tag{5.47}$$

Then for $n \geq 0$ and $t > T(\varepsilon, h)$ we have

$$C_n(t) \leq \sum_{j=2^n-1}^{2^{n+1}-2} K(h)^2 \left((1+\varepsilon)^2 2^{\eta+\delta}\right)^n (1+\varepsilon)2^\eta = K(h)^2 (1+\varepsilon) 2^\eta A_\varepsilon^n. \tag{5.48}$$

Since $A_\varepsilon < 1$, the sequence $M_n := K(h)^2(1+\varepsilon)2^\eta A_\varepsilon^n$ is summable. Next, as $\beta \in \mathrm{RV}_\infty(\delta)$ and $\varphi \in \mathrm{RV}_\infty(\eta)$, we have

$$\lim_{t\to\infty} C_n(t) = \sum_{j=2^n-1}^{2^{n+1}-2} \lim_{t\to\infty} \frac{\beta((1+jh)t)}{\beta(t)} \cdot \frac{\varphi((1+j)ht)}{\varphi(t)} = \sum_{j=2^n-1}^{2^{n+1}-2} (1+jh)^\delta ((1+j)h)^\eta. \tag{5.49}$$

Since $0 \leq C_n(t) \leq M_n$ for all $t > T(\varepsilon, h)$, by the summability of $(M_n)_{n\geq 1}$ and (5.49), the Dominated Convergence Theorem gives

$$\lim_{t\to\infty} \sum_{n=0}^{\infty} C_n(t) = \sum_{n=0}^{\infty} \lim_{t\to\infty} C_n(t). \tag{5.50}$$

This is equivalent to

$$\lim_{t\to\infty} \sum_{n=0}^{\infty} \sum_{j=2^n-1}^{2^{n+1}-2} \frac{\beta((1+jh)t)}{\beta(t)} \cdot \frac{\varphi((1+j)ht)}{\varphi(t)} = \sum_{n=0}^{\infty} \sum_{j=2^n-1}^{2^{n+1}-2} (1+jh)^\delta ((1+j)h)^\eta, \tag{5.51}$$

which implies (5.3).

To prove that (5.4) holds, note that as β is decreasing and φ is increasing, we have

$$\frac{\beta((1+(j+1)h)t)}{\beta(t)} \cdot \frac{\varphi(jht)}{\varphi(t)} \leq \frac{\beta((1+jh)t)}{\beta(t)} \cdot \frac{\varphi((1+j)ht)}{\varphi(t)}. \tag{5.52}$$

Define

$$D_n(t) := \sum_{j=2^n-1}^{2^{n+1}-2} \frac{\beta((1+(j+1)h)t)}{\beta(t)} \cdot \frac{\varphi(jht)}{\varphi(t)}. \tag{5.53}$$

Thus, $0 \leq D_n(t) \leq C_n(t) \leq M_n$ for $t > T(\varepsilon, h)$ and all $n \geq 0$. Also as $\beta \in \mathrm{RV}_\infty(\delta)$ and $\varphi \in \mathrm{RV}_\infty(\eta)$, we have

$$\lim_{t\to\infty} D_n(t) = \sum_{j=2^n-1}^{2^{n+1}-2} \lim_{t\to\infty} \frac{\beta((1+(j+1)h)t)}{\beta(t)} \cdot \frac{\varphi(jht)}{\varphi(t)} = \sum_{j=2^n-1}^{2^{n+1}-2} (1+(j+1)h)^\delta (jh)^\eta. \tag{5.54}$$

Now by the summability of $(M_n)_{n\geq 1}$, the last limit and the fact that $0 \leq D_n(t) \leq M_n$, by the Dominated Convergence Theorem we have

$$\lim_{t\to\infty}\sum_{n=0}^{\infty}\sum_{j=2^n-1}^{2^{n+1}-2}\frac{\beta((1+(j+1)h)t)}{\beta(t)}\cdot\frac{\varphi(jht)}{\varphi(t)} = \sum_{n=0}^{\infty}\sum_{j=2^n-1}^{2^{n+1}-2}(1+(j+1)h)^{\delta}(jh)^{\eta}, \tag{5.55}$$

and therefore (5.4) holds, as required.

6. Proofs of Theorems 4.1 and 4.3

The proofs of Theorems 4.1 and 4.3, which concern the asymptotic behaviour of Volterra equations, are greatly facilitated by applying extant results on the admissibility of certain linear Volterra operators. For the convenience of the reader, two results from [2] are restated.

Let $H : \Delta \to \mathbb{R}$ be a continuous function on $\Delta = \{(s,t) \in \mathbb{R}^2 : 0 \leq s \leq t < \infty\}$. Associated with H is the linear operator $\mathscr{H} : C[0,\infty) \to C[0,\infty)$ defined by

$$(\mathscr{H}\xi)(t) = \int_0^t H(t,s)\xi(s)ds, \quad t \geq 0. \tag{6.1}$$

Firstly we restate a theorem, which is a variant of part of a result in Corduneanu [24, page 74].

Theorem 6.1 (see [2, Theorem 3]). *Suppose that for all $T > 0$,*

$$H(t,s) \longrightarrow H_\infty(s) \quad \text{as } t \longrightarrow \infty \text{ uniformly with respect to } s \in [0,T]. \tag{6.2}$$

Further assume that

$$W := \lim_{T\to\infty}\limsup_{t\to\infty}\int_T^t |H(t,s)|ds < \infty, \tag{6.3}$$

$$\lim_{T\to\infty}\limsup_{t\to\infty}\left|\int_T^t H(t,s)ds - V\right| = 0 \quad \text{for some } V \in \mathbb{R}, \tag{6.4}$$

then $\lim_{t\to\infty}(\mathscr{H}\xi)(t)$ exists for all ξ for which $\lim_{t\to\infty}\xi(t) =: \xi(\infty)$ exists, and

$$\lim_{t\to\infty}(\mathscr{H}\xi)(t) = V\xi(\infty) + \int_0^\infty H_\infty(s)\xi(s)ds. \tag{6.5}$$

The next result is [2, Theorem 4]. It extends Appleby et al. [19, Theorem 5] to nonconvolution integral equations (cf. [19, Theorem A.1]); it is also the counterpart of Appleby et al. [25, Theorems 3.1 and 5.1], and Győri and Horváth [26, Theorem 3.1] which concerns linear nonconvolution difference equations.

Theorem 6.2 (see [2, Theorem 4]). *Suppose that* (6.2) *and* (6.4) *hold, and that* (6.3) *holds with*

$$W < 1. \tag{6.6}$$

Assume that ξ *is in* $C[0,\infty)$ *and that* $\lim_{t\to\infty}\xi(t) =: \xi(\infty)$ *exists. If* $\eta : [0,\infty) \to \mathbb{R}^n$ *is the continuous solution of*

$$\eta(t) = \xi(t) + \int_0^t H(t,s)\eta(s)ds, \quad t \geq 0, \tag{6.7}$$

then $\lim_{t\to\infty}\eta(t) =: \eta(\infty)$ *exists and satisfies the limit formula*

$$\eta(\infty) = (I-V)^{-1}\left[\xi(\infty) + \int_0^\infty H_\infty(s)\eta(s)ds\right]. \tag{6.8}$$

6.1. Proof of Theorem 4.1

The method of [2] is now used to prove Theorems 4.1 and 4.3.

Let $\beta \in \mathrm{RV}_\infty(\delta)$ be the positive function defined in (3.9), which is decreasing on $[\Theta_1,\infty)$ for some $\Theta_1 > 0$. As remarked the solution r of (4.3) is in $r \in L^1(0,\infty)$; it also obeys

$$\int_0^\infty r(s)ds = -\frac{1}{a+\int_0^\infty b(s)ds}. \tag{6.9}$$

If

$$\lim_{t\to\infty}\frac{(r * f(\cdot;\phi))(t)}{tb(t)\phi(-t)} = -\int_0^\infty u^\eta(1+u)^\delta\, du \cdot \frac{1}{a+\int_0^\infty b(s)ds} \tag{6.10}$$

holds, then by (4.4), (4.5), and the fact that $t\phi(-t) \to \infty$ as $t \to \infty$, we have

$$\begin{aligned}\lim_{t\to\infty}\frac{x(t)}{tb(t)\phi(-t)} &= \phi(0)\lim_{t\to\infty}\frac{r(t)}{b(t)}\cdot\frac{1}{t\phi(-t)} + \lim_{t\to\infty}\frac{(r * f(\cdot;\phi))(t)}{tb(t)\phi(-t)} \\ &= -\int_0^\infty u^\eta(1+u)^\delta\, du\cdot\frac{1}{a+\int_0^\infty b(s)ds},\end{aligned} \tag{6.11}$$

which is nothing other than (4.6).

It therefore remains to establish (6.10). Since $\phi_+ \in \mathrm{RV}_\infty(\eta)$ for some $\eta > 0$ there exists φ such that φ is increasing on $[\Theta_2,\infty)$ for some $\Theta_2 > 0$, positive, differentiable and obeys $\phi_+(t)/\varphi(t) \to 1$ as $t \to \infty$. Define $\gamma(t) = t\beta(t)\varphi(t)$. Then by Theorem 3.4, we have

$$L_\gamma f(\cdot;\phi) := \lim_{t\to\infty}\frac{f(t;\phi)}{\gamma(t)} = \lim_{t\to\infty}\frac{f(t;\phi)}{tb(t)\phi_+(t)}\cdot\frac{b(t)\phi_+(t)}{\beta(t)\varphi(t)} = \int_0^\infty u^\eta(1+u)^\delta\, du. \tag{6.12}$$

Note also that as $t\varphi(t) \longrightarrow \infty$ as $t \longrightarrow \infty$, we have

$$L_\gamma \beta = \lim_{t\to\infty} \frac{\beta(t)}{\gamma(t)} = \lim_{t\to\infty} \frac{1}{t\varphi(t)} = 0. \tag{6.13}$$

Our strategy here is to use Theorem 6.1 to show that $\lim_{t\to\infty}(r * f(\cdot;\phi))(t)/\gamma(t)$ exists and to determine it. To this end write

$$\frac{1}{\gamma(t)}\int_0^t r(t-s) f(s;\phi)ds = \int_0^t \frac{r(t-s)\gamma(s)}{\gamma(t)} \cdot \frac{f(s;\phi)}{\gamma(s)} ds = \int_0^t H_1(t,s)\xi_1(s)ds, \tag{6.14}$$

where we identify

$$H_1(t,s) = \frac{r(t-s)\gamma(s)}{\gamma(t)}, \qquad \xi_1(s) = \frac{f(s;\phi)}{\gamma(s)}. \tag{6.15}$$

In the notation of Theorem 6.1, H_1 here plays the role of H and ξ_1 the role of ξ. Evidently H_1 and ξ_1 are continuous. By (6.12), as $t \longrightarrow \infty$, it follows that $\xi_1(t) \longrightarrow \int_0^\infty u^\eta(1+u)^\delta du =: \xi_1(\infty)$. By (6.13) and (4.5), we have $L_\gamma r = L_\beta r \cdot L_\gamma \beta = 0$. Using this and the fact that $\gamma(t-s)/\gamma(t) \longrightarrow 1$ as $t \longrightarrow \infty$ uniformly on compact intervals (by (2.4)), we obtain

$$|H_1(t,s)| = \frac{|r(t-s)|}{\gamma(t-s)} \cdot \frac{\gamma(t-s)}{\gamma(t)} \cdot \gamma(s) \longrightarrow 0, \quad \text{as } t \longrightarrow \infty, \tag{6.16}$$

where the convergence is uniform for $s \in [0,T]$, for any $T > 0$.

Next let $T > \max(\Theta_1, \Theta_2)$. For $t > 2T$, we have the identity

$$\begin{aligned}\int_T^t H_1(t,s)ds - \int_0^\infty r(s)ds = {}& \int_T^{t-T} \frac{r(t-s)\gamma(s)}{\gamma(t)} ds \\ &+ \int_0^T r(s)\left(\frac{\gamma(t-s)}{\gamma(t)} - 1\right)ds - \int_T^\infty r(s)ds.\end{aligned} \tag{6.17}$$

Since γ obeys (2.4), the second term on the right-hand side has zero limit as $t \to \infty$. As for the first term, by the continuity of $|r|/\beta$ and (4.5), the facts that $\gamma(t) = t\beta(t)\varphi(t)$ and that φ is increasing, we deduce the estimate

$$\begin{aligned}\int_T^{t-T} \frac{|r(t-s)|\gamma(s)}{\gamma(t)} ds &= \int_T^{t-T} \frac{|r(t-s)|}{\beta(t-s)} \frac{\beta(t-s)\gamma(s)}{\gamma(t)} ds \\ &\le \sup_{u\ge 0} \frac{|r(u)|}{\beta(u)} \int_T^{t-T} \frac{\beta(t-s)\gamma(s)}{\gamma(t)} ds \\ &= \sup_{u\ge 0} \frac{|r(u)|}{\beta(u)} \int_T^{t-T} \frac{\beta(t-s)s\beta(s)\varphi(s)}{t\beta(t)\varphi(t)} ds \\ &\le \sup_{u\ge 0} \frac{|r(u)|}{\beta(u)} \int_T^{t-T} \frac{\beta(t-s)\beta(s)}{\beta(t)} ds.\end{aligned} \tag{6.18}$$

Since β is subexponential, and therefore obeys (2.3), we have

$$\lim_{t\to\infty} \frac{1}{\beta(t)} \int_T^{t-T} \beta(t-u)\beta(u) du = 2\int_T^{\infty} \beta(s) ds, \tag{6.19}$$

so it follows that

$$\limsup_{t\to\infty} \int_T^{t-T} \frac{|r(t-s)|\gamma(s)}{\gamma(t)} ds \le 2\sup_{u\ge 0} \frac{|r(u)|}{\beta(u)} \int_T^{\infty} \beta(s) ds. \tag{6.20}$$

Therefore by (6.17) and (6.20), we arrive at the estimate

$$\limsup_{t\to\infty} \left| \int_T^t H_1(t,s) ds - \int_0^{\infty} r(s) ds \right| \le 2\sup_{u\ge 0} \frac{|r(u)|}{\beta(u)} \int_T^{\infty} \beta(s) ds + \int_T^{\infty} |r(s)| ds. \tag{6.21}$$

Since $\beta, r \in L^1(0,\infty)$, we have

$$\lim_{T\to\infty} \limsup_{t\to\infty} \left| \int_T^t H_1(t,s) ds - \int_0^{\infty} r(s) ds \right| = 0. \tag{6.22}$$

Returning to (6.17) we get

$$\int_T^t |H_1(t,s)| ds \le \int_T^{t-T} \frac{|r(t-s)|\gamma(s)}{\gamma(t)} ds + \int_0^T |r(s)| \left| \frac{\gamma(t-s)}{\gamma(t)} - 1 \right| ds + \int_0^T |r(s)| ds. \tag{6.23}$$

Hence by using (6.20), we obtain

$$\limsup_{t\to\infty} \int_T^t |H_1(t,s)| ds \le 2\sup_{u\ge 0} \frac{|r(u)|}{\beta(u)} \int_T^{\infty} \beta(s) ds + \int_0^T |r(s)| ds. \tag{6.24}$$

Finally, letting $T \to \infty$ yields

$$\lim_{T\to\infty}\limsup_{t\to\infty}\int_T^t |H_1(t,s)|ds \le \int_0^\infty |r(s)|ds. \tag{6.25}$$

Since all of the hypotheses of Theorem 6.1 are satisfied with $V = \int_0^\infty r(s)ds$ we have, by (6.12) and (6.9),

$$\begin{aligned}\lim_{t\to\infty}\frac{1}{\gamma(t)}\int_0^t r(t-s)f(s;\phi)ds &= \lim_{t\to\infty}\int_0^t H_1(t,s)\xi_1(s)ds\\ &= \int_0^\infty r(s)ds\cdot L_\gamma f(\cdot;\phi)\\ &= -\frac{1}{a+\int_0^\infty b(s)ds}\int_0^\infty u^\eta(1+u)^\delta\, du\end{aligned} \tag{6.26}$$

which is nothing but (6.10), and so the theorem is proven.

6.2. Proof of Theorem 4.3

First, define $x(t) := N(t) - K$ for $t \ge 0$, where K is given by (4.9). Then by Theorem 4.2, $x(t) \to 0$ as $t \to \infty$. Our strategy, as in [2], is to show that x satisfies a linear integral equation (where nonlinearities are subsumed into the kernel and forcing function). Once this is done, we scale the resulting integral equation appropriately and apply Theorem 6.2 to determine the asymptotic behaviour of x.

Although the same derivation of the integral equation for x is given in [2], we give it afresh here, partly to make the exposition self contained, and partly because it enables us to define and analyse a number of auxiliary functions that will be important in the proof.

Substitution of $N = x + K$ into (4.7) with $\tilde{f}$ defined by

$$\tilde{f}(t;\phi) = \int_{-\infty}^0 b(t-s)(\phi(s)-K)ds, \quad t\ge 0, \tag{6.27}$$

leads to the initial-value problem

$$x'(t) = (K+x(t))\left(-ax(t)-\int_0^t b(t-s)x(s)ds-\tilde{f}(t;\phi)\right), \quad t\ge 0, \tag{6.28}$$

where $x(0) = \phi(0) - K$. We note by (3.3a) that the function $\tilde{f}(\cdot;\phi)$ is well defined, continuous and obeys $\tilde{f}(t;\phi) \to 0$ as $t \to \infty$. Define

$$c(t;\phi) = -ax(t)-\int_0^t b(t-s)x(s)ds-\tilde{f}(t;\phi), \quad t\ge 0. \tag{6.29}$$

Then $t \mapsto c(t;\phi)$ is continuous and $c(t;\phi) \to 0$ as $t \to \infty$, and x obeys

$$x'(t) = -aKx(t) - K\int_0^t b(t-s)x(s)ds - K\tilde{f}(t;\phi) + c(t;\phi)x(t), \quad t \geq 0. \tag{6.30}$$

Define the differential resolvent r by

$$r'(t) = -aKr(t) - K\int_0^t b(t-s)r(s)ds, \quad t \geq 0, \quad r(0) = 1. \tag{6.31}$$

Therefore by the variation of constants formula, we have

$$x(t) = h(t) + \int_0^t r(t-s)c(s)x(s)ds, \quad t \geq 0, \tag{6.32}$$

where

$$h(t) = r(t)x(0) - K\int_0^t r(t-s)\tilde{f}(s;\phi)ds, \quad t \geq 0, \tag{6.33}$$

and we have suppressed the ϕ-dependence in c and h. Since $a > \int_0^\infty b(s)ds$, it follows that $r(t) \to 0$ as $t \to \infty$ and $r \in L^1(0,\infty)$ with

$$\int_0^\infty r(s)ds = \frac{1}{K\left(a + \int_0^\infty b(s)ds\right)}. \tag{6.34}$$

Moreover we have that $\lim_{t\to\infty} r(t)/\beta(t)$ exists and is finite. Define $\gamma(t) = t\beta(t)\varphi(t)$ as in the proof of Theorem 4.1. Dividing (6.32) by γ, and defining $\xi(t) = h(t)/\gamma(t)$ and $\eta(t) = x(t)/\gamma(t)$ for $t \geq 0$ we arrive at

$$\eta(t) = \xi(t) + \int_0^t H(t,s)\eta(s)ds, \quad t \geq 0, \tag{6.35}$$

where H is given by

$$H(t,s) = \frac{r(t-s)\gamma(s)c(s)}{\gamma(t)}, \quad 0 \leq s \leq t. \tag{6.36}$$

Clearly η, ξ, and H are continuous. Our strategy now is to apply Theorem 6.2 to determine the integral equation (6.35), showing that $\eta(t) = x(t)/\gamma(t)$ tends to a finite limit as $t \to \infty$. From this fact, we will be able to deduce the speed of convergence of $N(t)$ to K as $t \to \infty$.

First we show that $\lim_{t\to\infty}\xi(t)$ exists and is finite. Since $L_\beta r$ is finite, and $L_\gamma\beta = 0$, we have $L_\gamma r = 0$. Also with f defined by (3.3a), $\tilde{f}(t;\phi) = f(t,\phi) - K\int_t^\infty b(s)ds$ for $t \geq 0$. Since $b \in \mathrm{RV}_\infty(\delta)$ for $\delta < -1$, it obeys

$$\lim_{t\to\infty}\frac{\int_t^\infty b(s)ds}{tb(t)} = -\frac{1}{\delta+1}. \tag{6.37}$$

Therefore $\lim_{t\to\infty}\int_t^\infty b(s)ds/\gamma(t) = 0$, and so we get

$$\lim_{t\to\infty}\frac{\tilde{f}(t;\phi)}{\gamma(t)} = \lim_{t\to\infty}\frac{f(t;\phi)}{\gamma(t)} = \int_0^\infty x^\eta(1+x)^\delta dx, \tag{6.38}$$

by Theorem 3.4. Therefore, by also using the fact that $L_\gamma r = 0$, we see that

$$\lim_{t\to\infty}\xi(t) = -K\lim_{t\to\infty}\frac{1}{\gamma(t)}\int_0^t r(t-s)\tilde{f}(s;\phi)ds = -K\int_0^\infty r(s)ds\cdot L_\gamma f(\cdot;\phi), \tag{6.39}$$

by using the argument used to prove (6.26). Hence by (6.34), ξ obeys

$$\xi(\infty) := \lim_{t\to\infty}\xi(t) = -\frac{1}{a+\int_0^\infty b(s)ds}\int_0^\infty x^\eta(1+x)^\delta dx. \tag{6.40}$$

With H defined by (6.36), we use the facts that $L_\gamma r = 0$ and the fact that $\gamma(t-s)/\gamma(t) \to 1$ as $t \to \infty$ uniformly on compact intervals to establish that

$$|H(t,s)| = \frac{|r(t-s)|}{\gamma(t-s)}\cdot\frac{\gamma(t-s)}{\gamma(t)}\cdot\gamma(s)|c(s)| \longrightarrow 0 \quad \text{as } t \longrightarrow \infty \tag{6.41}$$

for all $s \in [0,T]$ and any $T > 0$. Next, as c is uniformly bounded on $[0,\infty)$, for any $T > 0$, we have

$$\int_T^t |H(t,s)|ds = \int_T^t \frac{|r(t-s)|\gamma(s)}{\gamma(t)}|c(s)|ds \leq \sup_{s\geq T}|c(s)|\int_T^t \frac{|r(t-s)|\gamma(s)}{\gamma(t)}ds. \tag{6.42}$$

Therefore by (6.20), we deduce that

$$\limsup_{t\to\infty}\int_T^t |H(t,s)|ds \leq \sup_{s\geq T}|c(s)|\cdot 2\sup_{u\geq 0}\frac{|r(u)|}{\beta(u)}\cdot\int_T^\infty \beta(s)ds. \tag{6.43}$$

Since c is bounded and $\beta \in L^1(0,\infty)$, it follows that H obeys

$$\lim_{T\to\infty}\limsup_{t\to\infty}\int_T^t |H(t,s)|ds = 0. \tag{6.44}$$

Hence we also have

$$\lim_{T\to\infty}\limsup_{t\to\infty}\left|\int_T^t H(t,s)ds\right| = 0, \tag{6.45}$$

and therefore all the conditions of Theorem 6.2 hold, with $V = W = 0 < 1$. Hence the solution η of (6.35) obeys $\lim_{t\to\infty}\eta(t) = \xi(\infty)$. Therefore by (6.40) we have

$$\lim_{t\to\infty}\frac{N(t)-K}{\gamma(t)} = \lim_{t\to\infty}\frac{x(t)}{\gamma(t)} = \lim_{t\to\infty}\eta(t) = -\frac{1}{a+\int_0^\infty b(s)ds}\int_0^\infty x^{\eta}(1+x)^{\delta}dx, \tag{6.46}$$

which implies (4.10).

7. Proof of Propositions 3.2 and 3.3

7.1. Proof of Proposition 3.2

By (3.9), (3.12), I is finite, where

$$I := \int_0^\infty \beta(s)\left|\phi(-s)\right|ds < +\infty. \tag{7.1}$$

By hypothesis

$$\left|f(t;\phi)\right| \le \int_0^\infty |b(t+s)|\left|\phi(-s)\right|ds = \int_0^\infty \frac{|b(t+s)|}{\beta(t+s)}\beta(t+s)\left|\phi(-s)\right|ds. \tag{7.2}$$

Hence as β is nonincreasing, we have

$$\left|f(t;\phi)\right| \le \sup_{u\ge t}\frac{|b(u)|}{\beta(u)}\int_0^\infty \beta(t+s)\left|\phi(-s)\right|ds \le \sup_{u\ge t}\frac{|b(u)|}{\beta(u)}\int_0^\infty \beta(s)\left|\phi(-s)\right|ds, \tag{7.3}$$

so $|f(t;\phi)| \le I\sup_{u\ge t}|b(u)|/\beta(u) \le IB$, where we define

$$B := \sup_{u\ge 0}\frac{|b(u)|}{\beta(u)}. \tag{7.4}$$

Hence $t\mapsto |f(t;\phi)|$ is uniformly bounded.

Since $L_\beta|b| \ne 0$, β is nonincreasing, and $b\in L^1(0,\infty)$ we have that $\beta(t)\to 0$ as $t\to\infty$. Returning to (7.3) with B given by (7.4), we get

$$\left|f(t;\phi)\right| \le B\int_0^\infty \beta(t+s)\left|\phi(-s)\right|ds, \quad t\ge 0. \tag{7.5}$$

Let $T > 0$. Then as β is nonincreasing, we have

$$\begin{aligned} \left|f(t;\phi)\right| &\leq B\int_0^T \beta(t+s)\left|\phi(-s)\right|ds + B\int_T^\infty \frac{\beta(t+s)}{\beta(s)}\beta(s)\left|\phi(-s)\right|ds \\ &\leq B\beta(t)\int_0^T \left|\phi(-s)\right|ds + B\int_T^\infty \beta(s)\left|\phi(-s)\right|ds. \end{aligned} \tag{7.6}$$

The second integral on the right-hand side is finite, and bounded above by I. Since ϕ is continuous, $\int_0^T|\phi(-s)|ds$ is finite. Hence, as $\beta(t) \to 0$ as $t \to \infty$, we have

$$\limsup_{t\to\infty}\left|f(t;\phi)\right| \leq B\int_T^\infty \beta(s)\left|\phi(-s)\right|ds. \tag{7.7}$$

Since T is arbitrary, and $\beta|\phi(-\cdot)| \in L^1(0,\infty)$ we have $\lim_{T\to\infty}\int_T^\infty \beta(s)|\phi(-s)|ds = 0$. Hence $\limsup_{t\to\infty}|f(t;\phi)| = 0$, as claimed.

7.2. Proof of Proposition 3.3

Fix $t_0 \geq 0$ and let $0 \leq t \neq t_0$. We will show for every $\varepsilon > 0$ that there exists $\delta(\varepsilon) > 0$ such that $|t - t_0| < \delta(\varepsilon)$ implies

$$\sup_{s\geq 0}\left|\frac{b(t+s) - b(t_0+s)}{\beta(s)}\right| < \frac{\varepsilon}{I}, \tag{7.8}$$

Assume temporarily that this holds. Then for $|t - t_0| < \delta(\varepsilon)$, we have

$$\begin{aligned} \left|f(t;\phi) - f(t_0;\phi)\right| &\leq \int_0^\infty |b(t+s) - b(t_0+s)|\left|\phi(-s)\right|ds \\ &= \int_0^\infty \frac{|b(t+s) - b(t_0+s)|}{\beta(s)}\beta(s)\left|\phi(-s)\right|ds \\ &\leq \sup_{s\geq 0}\frac{|b(t+s) - b(t_0+s)|}{\beta(s)}\int_0^\infty \beta(s)\left|\phi(-s)\right|ds < \frac{\varepsilon}{I}\cdot I = \varepsilon, \end{aligned} \tag{7.9}$$

where we used (7.8) and (7.1) at the last step. This establishes the continuity of $t \mapsto f(t;\phi)$. It remains to prove (7.8). The identity

$$\frac{b(\tau+s)}{\beta(s)} = L_\beta b + \frac{b(\tau+s)}{\beta(\tau+s)}\left(\frac{\beta(\tau+s)}{\beta(s)} - 1\right) + \left(\frac{b(\tau+s)}{\beta(\tau+s)} - L_\beta b\right) \tag{7.10}$$

applied with $\tau = t$ and $\tau = t_0$ gives

$$\begin{aligned}\frac{b(t+s)}{\beta(s)} - \frac{b(t_0+s)}{\beta(s)} &= \frac{b(t+s)}{\beta(t+s)}\left(\frac{\beta(t+s)}{\beta(s)} - 1\right) + \left(\frac{b(t+s)}{\beta(t+s)} - L_\beta b\right)\\ &\quad - \frac{b(t_0+s)}{\beta(t_0+s)}\left(\frac{\beta(t_0+s)}{\beta(s)} - 1\right) - \left(\frac{b(t_0+s)}{\beta(t_0+s)} - L_\beta b\right).\end{aligned} \tag{7.11}$$

For every $\varepsilon > 0$ there exists $S_1(\varepsilon) > 0$ such that

$$\left|\frac{b(t)}{\beta(t)} - L_\beta b\right| < \frac{\varepsilon}{8I}, \quad t > S_1(\varepsilon), \tag{7.12}$$

where $I > 0$ is given by (7.1). Suppose also that $|t - t_0| < 1$ so that $t < t_0 + 1$. Since β obeys (3.10), we have

$$\lim_{s\to\infty} \sup_{0\le u\le t_0+1}\left|\frac{\beta(u+s)}{\beta(s)} - 1\right| = 0. \tag{7.13}$$

Hence for every $\varepsilon > 0$ there exists $S_2(\varepsilon, t_0) > 0$ such that

$$\sup_{0\le u\le t_0+1}\left|\frac{\beta(u+s)}{\beta(s)} - 1\right| < \frac{\varepsilon}{8BI}, \quad s > S_2(\varepsilon, t_0), \tag{7.14}$$

where B is given by (7.4). Hence as $0 \le t < t_0 + 1$ we have

$$\left|\frac{\beta(t_0+s)}{\beta(s)} - 1\right| < \frac{\varepsilon}{8BI}, \quad \left|\frac{\beta(t+s)}{\beta(s)} - 1\right| < \frac{\varepsilon}{8BI}, \quad s > S_2(\varepsilon, t_0). \tag{7.15}$$

Let $S_3(\varepsilon) = \max(S_1(\varepsilon), S_2(\varepsilon, t_0))$. Then as t and t_0 are nonnegative, using (7.4), (7.12), and (7.15) in (7.11), for $|t - t_0| < 1$ we have

$$\sup_{s\ge S_3(\varepsilon,t_0)}\left|\frac{b(t+s) - b(t_0+s)}{\beta(s)}\right| \le B\frac{\varepsilon}{8BI} + \frac{\varepsilon}{8I} + B\frac{\varepsilon}{8BI} + \frac{\varepsilon}{8I} \le \frac{3\varepsilon}{4I}. \tag{7.16}$$

Set $C_1(\varepsilon, t_0) := 1/\min_{0\le u\le S_3(\varepsilon,t_0)}\beta(u)$. Since b is uniformly continuous on $[0,\infty)$, for every $\varepsilon > 0$ there is a $\delta_1(\varepsilon) > 0$ such that $|t - t_0| < \delta_1(\varepsilon) < 1$ implies

$$|b(t - t_0 + u) - b(u)| < \frac{\varepsilon}{4C_1(\varepsilon, t_0)I}, \quad u \ge 0. \tag{7.17}$$

The t_0-dependence here is permissible as t_0 is fixed. Thus, for $|t-t_0|<\delta_1(\varepsilon)$,

$$\begin{aligned}
\sup_{0\le s\le S_3(\varepsilon,t_0)}\left|\frac{b(t+s)-b(t_0+s)}{\beta(s)}\right| &\le C_1(\varepsilon,t_0)\sup_{0\le s\le S_3(\varepsilon,t_0)}|b(t+s)-b(t_0+s)| \\
&= C_1(\varepsilon,t_0)\sup_{t_0\le u\le t_0+S_3(\varepsilon,t_0)}|b(t-t_0+u)-b(u)| \\
&< C_1(\varepsilon,t_0)\cdot\frac{\varepsilon}{4C_1(\varepsilon,t_0)I}=\frac{\varepsilon}{4I}.
\end{aligned} \tag{7.18}$$

Let $\delta(\varepsilon)=\min(\delta_1(\varepsilon),1)$. By the last inequality and (7.16), for $|t-t_0|<\delta(\varepsilon)$ we get

$$\sup_{s\ge0}\left|\frac{b(t+s)-b(t_0+s)}{\beta(s)}\right|\le\sup_{0\le s\le S_3(\varepsilon,t_0)}\left|\frac{b(t+s)-b(t_0+s)}{\beta(s)}\right|+\sup_{s\ge S_3(\varepsilon,t_0)}\left|\frac{b(t+s)-b(t_0+s)}{\beta(s)}\right|<\frac{\varepsilon}{I}, \tag{7.19}$$

which is nothing other than (7.8), proving the result.

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