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Research Article

Attractor for a Viscous Coupled Camassa-Holm Equation

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The global existence of solution to a viscous coupled Camassa-Holm equation with the periodic boundary condition is investigated. We obtain the compact and bounded absorbing set and the existence of the global attractor for the viscous coupled Camassa-Holm equation in H^2 by uniform prior estimate.

1. Introduction

The Camassa-Holm equation

$$u_t - u_{xxt} + 3uu_x - 2u_x u_{xx} - uu_{xxx} = 0, \quad x \in R, \ t > 0, \tag{1.1}$$

has been paid considerable attention due to its rich phenomenology all the time. Its abstract derivation was first discovered by Fuchssteiner and Fokas [1], while in the physical derivation of Camassa and Holm (see [2, 3]), the equation models unidirectional propagation of shallow water waves, and $u(x,t)$ represents the fluid velocity in the x direction, equivalently the height of the fluid's free surface above a flat bottom. They also found that the solitary waves interact like solitons. Unlike the Korteweg-de Vries equation (which is an approximation to the equations of motion), this model is obtained by approximating directly in the Hamiltonian for Euler's equations in the shallow water regime (see [3, 4]). Equation (1.1) retains higher-order terms in a small amplitude expansion of incompressible Euler's equations for unidirectional motion of waves at the free surface under the influence of gravity. Dropping these terms leads to the BBM equation, or at the same order, the KdV equation. The Camassa-Holm has quite a few interesting features: it admits solitary waves called "peakons" with the form of $u = ce^{-|x-ct|}$, $x \in R$, $c > 0$. The peakons of (1.1) are orbitally stable [5]—that is, their shape is stable under small perturbations and therefore these waves are recognized physically. For waves that approximate the peakons in a special way, a stability result was proved by a variation method [6]. This is in sharp contrast to the Korteweg-de Vries equation, where solitary waves are generally smooth. The peaked traveling waves of the Camassa-Holm equation replicate a future that is characteristic for waves of great height–waves of the largest amplitude that are exact solutions of the governing equations for water waves (see [7, 8]). A breaking wave is a solution which remains bounded but whose slope becomes unbounded in finite time, and, in contrast to the KdV equation, the Camassa-Holm equation models breaking waves [9], as well as a breaking rod (see [4, 10]), since the equation models the propagation of axisymmetric waves in hyperelastic rods. After breaking, the solution can be continued either as a global conservative weak solution or as a global dissipative solution (see [11–13]). Peakons interact "elastically" in the manner typical of all solitons, and their wave dynamics are now wellunderstood (see [3, 14, 15]). Some authors have even argued recently that the Camassa-Holm equation might be relevant to the modeling of tsunamis (see [16, 17]). Moreover, the equation has a bi-Hamiltonian structure [2]. As the Camassa-Holm is completely integrable, it has many conserved qualities. Especially for smooth solutions, the qualities

$$\int_R u\,dx, \qquad \int_R \left(u^2 + u_x^2\right)dx, \quad \int_R \left(u^3 + uu_x^2\right)dx \tag{1.2}$$

are all time independent [18].

Up to now, great efforts have been already devoted to the Camassa-Holm equation. A. Constantin (see [19–21]) considered the Cauchy problem, inverse spectral problem, and inverse scatting transform for Camassa-Holm equation, proving that the corresponding solution to (1.1) does not exist globally for smooth initial data. Rui et al. (see [22, 23]) employed both bifurcation method and numerical simulation to investigate bounded traveling waves of (1.1) in a general compressible hyperelastic rod. Lenells [24] used the inverse scattering transform to show that a solution of the Camassa-Holm equation is identically zero whenever it vanishes on two horizontal half-lines in the $x - t$ space. In particular, a solution that has compact support at two different times vanishes everywhere, proving that the Camassa-Holm equation has infinite propagation speed. Cohen et al. [25] presented two new multisymplectic formulations for the Camassa-Holm equation, and the associated local conservation laws were shown to correspond to certain well-known Hamiltonian functionals. The multisymplectic discretisation of each formulation was exemplified by means of the Euler box scheme. Yiping Meng and Lixin Tian [26] investigated the boundary control of the viscous generalized Camassa-Holm equation on $[0,1]$. Long et al. [27] obtained the loop soliton solution and periodic loop soliton solution [28], solitary wave solution and solitary cusp wave solution and smooth periodic wave solution and nonsmooth periodic wave solution of (1.1) and also discussed their dynamic characters and relations by the integral bifurcation method. Moreover, Ding and Tian (see [29, 30]) considered the existence of the global solution to dissipative Camassa-Holm equation and the global attractor of semigroup of solutions of dissipative Camassa-Holm equation in H^2. Olson [31] showed that the Cauchy problem for a higher-order modification of (1.1) is locally well posed for initial data in $H^s(R)$ for $s > s'$, where $1/4 \le s' < 1/2$ and the value of s' depends on the order of equation, proved the existence and uniqueness of solutions of (1.1) by a contraction mapping argument. Moreover, Zhou and Tian [32] investigated the initial boundary value problem of a generalized Camassa-Holm equation with dissipation and established local well-posedness of this closed-loop system by using Kato's theorem for abstract quasilinear evolution equation of hyperbolic type. Then they obtained a conservation law that enables us to present a blowup result by using multiplier technique. Lixin Tian et. al. [33] discussed optimal control of the viscous Camassa-Holm equation; they deduce that the norm of solution is related to the control item and initial value in the special Hilbert space according to variational method, optimal control theories and distributed parameter system control theories, The optimal control of the viscous Camassa-Holm equation under boundary condition was given, and the existence of optimal solution of the viscous Camassa-Holm equation was proved. Well-posedness problem and scattering problem for DGH equation were also discussed in [34].

On the basis of deformation of bi-Hamiltonian structure of the hydrodynamic type, Chen et al.[35] obtained the following two-component generalization of (1.1):

$$\begin{gathered} u_t - u_{xxt} + 3uu_x - 2u_x u_{xx} - uu_{xxx} + \rho\rho_x = 0, \\ \rho_t + (\rho u)_x = 0. \end{gathered} \tag{1.3}$$

Equation (1.3) is one of many multicomponent generalizations which are integrable (see [35–37]). It has a Lax pair, and it is bi-Hamiltonian. Constantin and Ivanov [36] showed how (1.3) arises in shallow water theory, and it was derived from the Green-Naghdi equations by using expansions in terms of physical parameters. Recently, the infinite propagation speed property for (1.3) was proved in [38]. Escher et al. [39] probed into well-posedness and blowup phenomena of the two-component Camassa-Holm equation in details. Chen et al. [35] obtained solutions of (1.3) by a reciprocal transformation between (1.3) and the first negative flow of the AKNS hierarchy and stated some examples of peakon and multikink solutions of (1.3). Guan and Yin [40] presented a new global existence result and several new blowup results of strong solutions to (1.3) as $\rho = \overline{\rho} - 1$, improving considerably earlier results. Jibin Li and Yishen Li [41] obtained the existence of solitary wave solutions, kink and antikink wave solutions, uncountable infinite many breaking wave solutions, and smooth and nonsmooth periodic wave solutions with the method of dynamical systems to the two-component generalization of the Camassa-Holm equation. Yujuan Wang et. al.[42] showed that the two-component Camassa-Holm equation possesses a global continuous semigroup of weak conservative solutions for initial data. In [43] a link between central extensions of superconformal algebra and a supersymmetric two-component generalization of the Camassa-Holm equation was concerned. Deformations of superconformal algebra give rise to two compatible bracket structures. For $\rho \equiv 0$, the system (1.3) particularizes to the Camassa-Holm equation which is a re-expression of geodesic flow on the diffeomorphism group of the circle (see [44, 45]).

We know that it is of great use to construct an interacting system of equations [37]:

$$\begin{gathered} m_t = -3m(2u_x + v_x) - m_x(2u + v), \quad m = u - u_{xx}, \\ n_t = -2n(2u_x + v_x) - n_x(2u + v), \quad n = v - v_{xx}, \end{gathered} \tag{1.4}$$

as $n = v = 0$ and $m = u = 0$, it, respectively, leads to the Degasperis-Procesi equation and Camassa-Holm equation. Three independent conserved quantities have been obtained as follows:

$$\int (m+n)dx, \qquad \int n^{\lambda} m^{-(1-2\lambda)/3}\, dx,$$
$$\int \left(-9n_x^2 n^{\lambda-2} m^{-(1+2\lambda)/3} + 12 n_x m_x n^{\lambda-1} m^{-(4+2\lambda)/3} - 4m_x^2 n^{\lambda} m^{-(7+2\lambda)/3}\right) dx, \tag{1.5}$$

here λ is an arbitrary constant.

Ying Fu and Changzheng Qu [46] considered the following coupled Camassa-Holm equation:

$$m_t = 2mu_x + m_x u + (mv)_x + nv_x, \quad m = u - u_{xx},$$
$$n_t = 2nv_x + n_x v + (nu)_x + mu_x, \quad n = v - v_{xx}, \tag{1.6}$$

which has peakon solitons in the form of a superposition of multipeakons. It has the following conserved qualities:

$$G_1(u) = \int_R u\, dx, \qquad G_2(v) = \int_R v\, dx, \qquad G_3(u) = \int_R m\, dx,$$
$$G_4(v) = \int_R n\, dx, \qquad G_5(u,v) = \int_R \left(u^2 + u_x^2 + v^2 + v_x^2\right) dx. \tag{1.7}$$

They investigated local well-posedness and blowup solutions of (1.6) by means of Kato's semigroup approach to nonlinear hyperbolic evolution equation and obtained a criterion and condition on the initial data guaranteeing the development of singularities in finite time for strong solutions of (1.6) by energy estimates; moreover, an existence result for a class of local weak solutions was also given. They also showed that the solution of (1.6) is

$$\frac{d}{dt}\int_R \left(u^2 + \alpha u_x^2 + \beta v^2 + \gamma v_x^2\right) dx = 0, \tag{1.8}$$

for some positive constants α, β, γ.

In the field of infinite-dimensional dynamical systems, one of the most important issues is to obtain the existence of global attractors for the semigroups of solutions associated with some concrete partial differential equations. For instance, Yongsheng Li and Xingyu Yan [47] studied the existence and regularity of the global attractor for a weakly damped forced shallow water equation in $H^1(R)$. Tian et al.[48] studied the global attractor for the viscous weakly damped forced Korteweg-de Vries equations in $H^1(R)$. Yanhong Zhang and Chengkui Zhong [49] investigated the existence of global attractors for a nonlinear wave equation. Lixin Tian and Ruihua Tian [50] studied the attractor for the two-dimensional weakly damped KdV equation in belt field. Ying Xu and Lixin Tian [51] investigated attractor for a coupled nonhomogeneous Camassa-Holm equation with periodic boundary condition. Lixin Tian and Jinglin Fan [52] discussed the global attractors for the viscous Degasperis-Procesi equation in H^2. Lixin Tian and Ying Gao [53] obtained global attractors for the viscous Fornberg-Whitham equation [54] in H^2. Here we investigate the existence of global attractor for a viscous coupled Camassa-Holm equation with the periodic boundary condition in H^2

as follows:

$$m_t - \varepsilon m_{xx} = 2mu_x + m_x u + (mv)_x + nv_x, \qquad m = u - u_{xx}, \tag{1.9}$$

$$n_t - \varepsilon n_{xx} = 2nv_x + n_x v + (nu)_x + mu_x, \qquad n = v - v_{xx}, \tag{1.10}$$

$$u(x,0) = u_0, \qquad v(x,0) = v_0, \tag{1.11}$$

$$u(0,t) = u(D,t), \qquad u'(0,t) = u'(D,t), \qquad u''(0,t) = u''(D,t), \tag{1.12}$$

$$v(0,t) = v(D,t), \qquad v'(0,t) = v'(D,t), \qquad v''(0,t) = v''(D,t), \tag{1.13}$$

where $t > 0$, $x \in \Omega$, $\Omega = [0,D]$, $D > 0$, u, $v \in H = L^2(\Omega)$. To the authors' knowledge, the problem of global attractor for (1.9)–(1.13) has not been discussed in previous publications.

Our paper is organized as follows. In Section 2, we give the main definitions and Lemmas. In Section 3, main results are presented, as the core of the paper, and the proofs of the main theorems are completed. Firstly, we prove that (1.9)–(1.13) has a unique solution in infinite time interval then obtain the existence of global solution of (1.9)–(1.13) in $H^2(\Omega)$ by prior estimates. Meanwhile we obtain that the semigroup of the solution operator has an absorbing set. Finally, we demonstrate the long-time behavior of solution of (1.9)–(1.13) that is described by global attractor. In brief, we obtain the existence of the global attractor for (1.9)–(1.13) in $H^2(\Omega)$.

2. Preliminaries

Definition 2.1. Let $(,)$ *stand for the* L^2 *inner product and* $\|\cdot\|$ *the corresponding* L^2 *norm. one also denotes*

$$\begin{gathered}\|u\|_{L^2(\Omega)} \triangleq |u|, \|Du\|_{L^2(\Omega)} \triangleq \|u\|, \\ \|D^m u\|_{L^2(\Omega)} \triangleq |D^m u|, \|u\|_{L^\infty(\Omega)} \triangleq \operatorname*{esssup}_{x\in\Omega}|u(x)|,\end{gathered} \tag{2.1}$$

and $A = -\Delta$, where Δ is Laplace operators and A is a self-adjoint positive operators with compact inverse. The eigenvalue of A is λ_k satisfying $0 < \lambda_1 \le \lambda_2 \le \cdots \lambda_k, \lambda_k \to \infty$ as $k \to \infty$, $A\omega_k = \lambda_k \omega_k$, where ω_k is the corresponding eigenvector of A. For simplicity we will give the following inequalities and only refer to their names wherever necessary.

Lemma 2.2 (consistent Gronwall inequality). *Assume that* $g(t)$, $y(t)$, *and* $h(t)$ *are three positive locally integrable functions defined on* $[t_0, +\infty]$, $y'(t)$ *is a locally integrable function over* $[t_0, +\infty]$, *satisfying*

$$\begin{gathered}y'(t) \le g(t)y(t) + h(t), \quad \forall t \ge t_0, \\ \int_t^{t+r} g(s)\,ds \le k_1, \qquad \int_t^{t+r} h(s)\,ds \le k_2, \qquad \int_t^{t+r} y(s)\,ds \le k_3, \quad \forall t \ge t_0,\end{gathered} \tag{2.2}$$

where $r, k_1, k_2,$ *and* k_3 *are positive constants. one can get*

$$y(t+r) \le \left(\frac{k_3}{r} + k_2\right)\exp(k_1), \quad \forall \ge t_0. \tag{2.3}$$

Lemma 2.3 (Sobolev inequality). *Suppose that* $u \in L_q(\Omega) \cap W_0^{m,r}(\Omega), \Omega \subset R^k, 1 \leq q, r \leq \infty, 0 \leq j \leq m, j/m \leq \alpha < 1,$ *and* $1 \leq p < \infty$, *and there exists a constant c, such that*

$$\left\| D^j u \right\|_{L^p} \leq c \|D^m u\|_{L^r}^{\alpha} \|u\|_{L^q}^{1-\alpha}, \quad \frac{1}{p} = \frac{i}{k} + \left(\frac{1}{r} - \frac{m}{k}\right) + (1-\alpha)\frac{1}{q}. \tag{2.4}$$

Lemma 2.4 (Young inequality). $ab \leq (\varepsilon a^p/p) + (\varepsilon^{-q/p} b^q/q) \leq \varepsilon a^p + \varepsilon^{-q/p} b^q$, *where* $1 < p < \infty, (1/p) + (1/q) = 1$. *As* $p = q = 2$, *one has* $ab \leq (\varepsilon a^2/2) + (b^2/2\varepsilon) \leq (\varepsilon a^2) + (1/\varepsilon b^2)$.

3. Main Results and the Proof of the Theorems

Based on Galerkin procedure, we will show the existence of global solution of (1.9)–(1.13). Suppose that $\{\omega_k\}_{k=1}^{\infty}$ is an orthonormal basis in the space H consisting of eigenfunctions of the operator A. $H_m = \operatorname{span}\{\omega_1, \omega_2, \cdots \omega_m\}$, and P_m is orthogonal projection from H to H_m. By Galerkin procedure [55], (1.9)–(1.13) can be reduced to ordinary differential system

$$m_{mt} - m_{mxx} = 2m_m\, u_{mx} + m_{mx}\, u_m + (m_m v_m)_x + n_m\, v_{mx}, \quad m_m = u_m - u_{mxx}, \tag{3.1}$$

$$n_{mt} - n_{mxx} = 2n_m v_{mx} + n_{mx}\, v_m + (n_m u_m)_x + m_m\, u_{mx}, \quad n_m = v_m - v_{mxx}, \tag{3.2}$$

$$u_m(x,0) = P_m\, u_0, \qquad v_m(x,0) = P_m\, v_0. \tag{3.3}$$

By means of existence theory of solution to ordinary differential equations, we know that local smooth solution of (3.1)–(3.3) exists. Now we can establish consistent integral estimate on approximate solution with respect to m by Galerkin method.

Theorem 3.1. *If* $u_0, \rho_0 \in H^l(R), l \geq 2$, *then* (1.9)–(1.13) *has a global solution in* $H^2(\Omega)$.

Proof. Taking the inner product of (3.1), and (3.2), respectively, with u_m, v_m in Ω and noting that

$$(u_{mt} - u_{mxxt} - \varepsilon(u_m - u_{mxx})_{xx}, u_m) = \frac{1}{2}\frac{d}{dt}\left(|u_m|^2 + \|u_m\|^2\right) + \varepsilon\left(\|u_m\|^2 + |Au_m|^2\right), \tag{3.4}$$

$$(v_{mt} - v_{mxxt} - \varepsilon(v_m - v_{mxx})_{xx}, v_m) = \frac{1}{2}\frac{d}{dt}\left(|v_m|^2 + \|v_m\|^2\right) + \varepsilon\left(\|v_m\|^2 + |Av_m|^2\right), \tag{3.5}$$

$$\begin{aligned}
&P_m[-2(u_{mx}u_{mxx}, u_m) - (u_{mxxx}u_m, u_m)] \\
&\quad = P_m\left[\int_\Omega (-2u_{mx}u_m u_{mxx})dx - \int_\Omega u_m^2\, du_{mxx}\right] \\
&\quad = P_m\left[\int_\Omega (-2u_m u_{mx} u_{mxx})dx + \int_\Omega u_{mxx}\left(u_m^2\right)_x dx\right] = 0, \\
&P_m[(u_{mx}u_m, v_m) + (u_{mx}v_m + u_m v_{mx}, u_m)] \\
&\quad = P_m\left(2\int_\Omega u_m u_{mx} v_m\, dx + \int_\Omega u_m^2\, dv_m\right) \\
&\quad = P_m\left(2\int_\Omega u_m u_{mx} v_m\, dx - \int_\Omega 2v_m u_m u_{mx}\, dx\right) = 0,
\end{aligned} \tag{3.6}$$

with the same reason, we obtain that

$$P_m[-2(v_{mx}v_{mxx}, v_m) - (v_{mxxx}v_m, v_m)] = 0, \qquad P_m[(v_{mx}v_m, u_m) + (v_{mx}u_m + v_m u_{mx}, v_m)] = 0. \tag{3.7}$$

By integrating by parts we get

$$\begin{aligned}
&P_m[(-u_{mxxx}v_m - u_{mxx}v_{mx} - v_{mxx}v_{mx}, u_m) + (-v_{mxxx}u_m - v_{mxx}u_{mx} - u_{mxx}u_{mx}, v_m)] \\
&\quad = P_m\bigg(\int_\Omega -u_m v_m du_{mxx} - \int_\Omega u_m u_{mxx} v_{mx}\, dx - \int_\Omega u_m v_{mxx} v_{mx}\, dx \\
&\qquad\quad - \int_\Omega u_m v_m dv_{mxx} - \int_\Omega u_{mx} v_{mxx} v_m\, dx - \int_\Omega u_{mxx} v_m u_{mx}\, dx\bigg) \\
&\quad = P_m\bigg(\int_\Omega u_{mxx} u_{mx} v_m\, dx + \int_\Omega u_{mxx} u_m v_{mx}\, dx - \int_\Omega u_m u_{mxx} v_{mx}\, dx \\
&\qquad\quad - \int_\Omega u_m v_{mxx} v_{mx}\, dx + \int_\Omega v_{mxx} u_{mx} v_m\, dx + \int_\Omega v_{mxx} u_m v_{mx}\, dx \\
&\qquad\quad - \int_\Omega u_{mx} v_{mxx} v_m dx - \int_\Omega u_{mxx} v_m u_{mx} dx\bigg) = 0.
\end{aligned} \tag{3.8}$$

From (3.4) and (3.5) we obtain that

$$\frac{1}{2}\frac{d}{dt}\Big(|u_m|^2 + \|u_m\|^2 + |v_m|^2 + \|v_m\|^2\Big) + \varepsilon\Big(\|u_m\|^2 + |Au_m|^2 + \|v_m\|^2 + |Av_m|^2\Big) = 0. \tag{3.9}$$

Applying Poincaré inequality, we get

$$\|u_m\|^2 > \lambda_1 |u_m|^2, \qquad |Au_m|^2 > \lambda_1 \|u_m\|^2, \qquad \|v_m\|^2 > \lambda_1 |v_m|^2, \qquad |Av_m|^2 > \lambda_1 \|v_m\|^2. \tag{3.10}$$

Equality (3.9) implies that

$$\frac{d}{dt}\Big(|u_m|^2 + \|u_m\|^2 + |v_m|^2 + \|v_m\|^2\Big) + 2\varepsilon\lambda_1\Big(|u_m|^2 + \|u_m\|^2 + |v_m|^2 + \|v_m\|^2\Big) \le 0, \tag{3.11}$$

$$\begin{aligned}
&|u_m| + \|u_m\|^2 + |v_m|^2 + \|v_m\|^2 \\
&\quad \le \Big(|u_m(0)|^2 + \|u_m(0)\|^2 + |v_m(0)|^2 + \|v_m(0)\|^2\Big)\exp\{-2\varepsilon\lambda_1 t\} \\
&\quad \le |u_m(0)|^2 + \|u_m(0)\|^2 + |v_m(0)|^2 + \|v_m(0)\ \|^2 \triangleq r_1,
\end{aligned} \tag{3.12}$$

where r_1 is nonnegative constant.

Integrating (3.9) over $[t, t+r]$ yields

$$\varepsilon\int_t^{t+r}\Big(\|u_m(s)\|^2 + |Au_m(s)|^2 + \|v_m(s)\|^2 + |Av_m(s)|^2\Big)ds \le r_1. \tag{3.13}$$

Taking the inner product of (3.1) and (3.2), respectively, with $-u_{mxx}, -v_{mxx}$ in Ω, we get

$$(u_{mt} - u_{mxxt} - \varepsilon(u_m - u_{mxx})_{xx}, -u_{mxx}) = \frac{1}{2}\frac{d}{dt}\left(\|u_m\|^2 + |Au_m|^2\right) + \varepsilon\left(|Au_m|^2 + |\nabla Au_m|^2\right), \tag{3.14}$$

$$(v_{mt} - v_{mxxt} - \varepsilon(v_m - v_{mxx})_{xx}, -v_{mxx}) = \frac{1}{2}\frac{d}{dt}\left(\|v_m\|^2 + |Av_m|^2\right) + \varepsilon\left(|Av_m|^2 + |\nabla Av_m|^2\right) \tag{3.15}$$

since

$$\begin{aligned}
&|P_m(3u_m u_{mx}, -u_{mxx})| \le \frac{3}{2}\|\nabla u_m\|_{L^\infty(\Omega)}\|u_m\|^2,\\
&|P_m(3v_m v_{mx}, -v_{mxx})| \le \frac{3}{2}\|\nabla v_m\|_{L^\infty(\Omega)}\|v_m\|^2,\\
&|P_m[(-2u_{mx}u_{mxx}, -u_{mxx}) - (u_{mxxx}u_m, -u_{mxx})]|\\
&\quad = \left|P_m\left[\int_\Omega\left(2u_{mx}u_{mxx}^2\right)dx + \int_\Omega u_m u_{mxx}\ du_{mxx}\right]\right|\\
&\quad = \left|P_m\int_\Omega \frac{3}{2}u_{mx}u_{mxx}^2\ dx\right| \le \frac{3}{2}\|\nabla u_m\|_{L^\infty(\Omega)}|Au_m|^2,\\
&|P_m[(v_{mx}u_m + v_m u_{mx}, -v_{mxx}) + (v_{mx}v_m, -u_{mxx})]|\\
&\quad = \left|P_m\left(\frac{3}{2}\int_\Omega v_{mx}^2 u_{mx}\,dx\right)\right| \le \frac{3}{2}\|\nabla u_m\|_{L^\infty(\Omega)}\|v_m\|^2,\\
&|P_m(-2v_{mx}v_{mxx}, -v_{mxx}) - (v_{mxxx}v_m, -v_{mxx})| \le \frac{3}{2}\|v_{mx}\|_{L^\infty(\Omega)}|Av_m|^2.
\end{aligned} \tag{3.16}$$

By means of integrating by parts frequently, we obtain that

$$\begin{aligned}
&|P_m[(u_{mx}v_m + u_m v_{mx}, -u_{mxx}) + (u_{mx}u_m, -v_{mxx})]|\\
&\quad = \left|P_m\left(\int_\Omega -u_{mx}v_m du_{mx} - \int_\Omega u_{mxx}u_m v_{mx}\,dx - \int_\Omega u_m u_{mx} dv_{mx}\right)\right|\\
&\quad = \left|P_m\left(\int_\Omega \frac{u_{mx}^2}{2}v_{mx}\,dx - \int_\Omega u_m u_{mxx}\,dx + \int_\Omega u_{mx}^2 v_{mx}\,dx + \int_\Omega u_m u_{mxx}v_{mx}\,dx\right)\right|\\
&\quad = \left|P_m\left(\frac{3}{2}\int_\Omega u_{mx}^2 v_{mx}\,dx\right)\right| \le \frac{3}{2}\|\nabla v_m\|_{L^\infty(\Omega)}\|u_m\|^2,\\
&|P_m[(-u_{mxxx}v_m - u_{mxx}v_{mx} - v_{mxx}v_{mx}, -u_{mxx})\\
&\qquad +(-v_{mxxx}u_m - v_{mxx}u_{mx} - u_{mxx}u_{mx}, -v_{mxx})]|
\end{aligned}$$

$$\begin{aligned}
&= \left| P_m\left(\int_\Omega v_m u_{mxx} du_{mxx} + \int_\Omega v_{mx} u_{mxx}^2\, dx + \int_\Omega u_{mxx} v_{mxx} v_{mx}\, dx \right.\right. \\
&\qquad \left.\left. + \int_\Omega u_m v_{mxx} dv_{mxx} + \int_\Omega u_{mx} v_{mxx}^2\, dx + \int_\Omega u_{mxx} u_{mx} v_{mxx}\, dx \right) \right| \\
&= \left| P_m\left(\int_\Omega \frac{1}{2} u_{xx}^2 v_x\, dx + \int_\Omega \frac{1}{2} v_{xx}^2 u_x\, \mathrm{d}x \right) \right| + \left| P_m\left[\int_\Omega \frac{1}{2}\left(u_{mxx}^2 + v_{mxx}^2 \right) v_{mx}\, dx \right] \right| \\
&\quad + \left| P_m\left[\int_\Omega \frac{1}{2}\left(u_{mxx}^2 + v_{mxx}^2 \right) u_{mx}\, dx \right] \right| \\
&= \left| P_m \int_\Omega u_{mxx}^2 v_{mx}\, dx \right| + \left| P_m \int_\Omega v_{mxx}^2 u_{mx}\, dx \right| \\
&\quad + \left| P_m \int_\Omega \frac{1}{2} u_{mxx}^2 u_{mx}\, dx \right| + \left| P_m \int_\Omega \frac{1}{2} v_{mxx}^2 v_{\mathrm{m}x}\, dx \right| \\
&\le \|\nabla v_m\|_{L^\infty(\Omega)} |Au_m|^2 + \|\nabla u_m\|_{L^\infty(\Omega)} |Av_m|^2 \\
&\quad + \frac{1}{2} \|\nabla u_m\|_{L^\infty(\Omega)} |Au_m|^2 + \frac{1}{2} \|\nabla v_m\|_{L^\infty(\Omega)} |Av_m|^2.
\end{aligned} \tag{3.17}$$

Associating all the above inequalities leads to

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|u_m\|^2 + |Au_m|^2 + \|v_m\|^2 + |Av_m|^2 \right) \\
&\qquad + \varepsilon\left(|Au_m|^2 + |\nabla Au_m|^2 + |Av_m|^2 + |\nabla Av_m|^2 \right) \\
&\quad \le \frac{3}{2} \|\nabla u_m\|_{L^\infty(\Omega)} \|u_m\|^2 + \frac{3}{2} \|\nabla u_m\|_{L^\infty(\Omega)} |Au_m|^2 \\
&\qquad + \frac{3}{2} \|\nabla v_m\|_{L^\infty(\Omega)} \|v_m\|^2 + \frac{3}{2} \|\nabla v_m\|_{L^\infty(\Omega)} |Av_m|^2 \\
&\qquad + \frac{3}{2} \|\nabla v_m\|_{L^\infty(\Omega)} \|u_m\|^2 + \frac{3}{2} \|\nabla u_m\|_{L^\infty(\Omega)} \|v_m\|^2 \\
&\qquad + \|\nabla u_m\|_{L^\infty(\Omega)} |Av_m|^2 + \|\nabla v_m\|_{L^\infty(\Omega)} |Au_m|^2 \\
&\qquad + \frac{1}{2} \|\nabla u_m\|_{L^\infty(\Omega)} |Au_m|^2 + \frac{1}{2} \|\nabla v_m\|_{L^\infty(\Omega)} |Av_m|^2 \\
&\quad \le 2\|\nabla u_{\mathrm{m}}\|_{L^\infty(\Omega)} \left(\|u_m\|^2 + |Au_m|^2 + \|v_m\|^2 + |Av_m|^2 \right) \\
&\qquad + 2\|\nabla v_m\|_{L^\infty(\Omega)} \left(\|u_m\|^2 + |Au_m|^2 + \|v_m\|^2 + |Av_m|^2 \right) \\
&\quad \le 2c_1 \|u_m\|^{1/2} |Au_m|^{1/2} \left(\|u_m\|^2 + |Au_m|^2 + \|v_m\|^2 + |Av_m|^2 \right) \\
&\qquad + 2c_2 \|v_m\|^{1/2} |Av_m|^{1/2} \left(\|u_m\|^2 + |Au_m|^2 + \|v_m\|^2 + |Av_m|^2 \right).
\end{aligned} \tag{3.18}$$

Simplifying the above inequality and employing Young inequality, it follows that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|u_m\|^2+|Au_m|^2+\|v_m\|^2+|Av_m|^2\right)\\
&\quad+\varepsilon\left(|Au_m|^2+|\nabla Au_m|^2+|Av_m|^2+|\nabla Av_m|^2\right)\\
&\leq \varepsilon\lambda_1\left(\|u_m\|^2+|Au_m|^2+\|v_m\|^2+|Av_m|^2\right)\\
&\quad+c_3\|u_m\||Au_m|\left(\|u_m\|^2+|Au_m|^2+\|v_m\|^2+|Av_m|^2\right)\\
&\quad+c_4\|v_m\||Av_m|\left(\|u_m\|^2+|Au_m|^2+\|v_m\|^2+|Av_m|^2\right),
\end{aligned}\tag{3.19}$$

where $c_3=2c_1^2/\varepsilon\lambda_1,\ c_4=2c_2^2/\varepsilon\lambda_1$. By means of Poincaré inequality, we obtain that

$$\begin{aligned}
&\frac{d}{dt}\left(\|u_m\|^2+|Au_m|^2+\|v_m\|^2+|Av_m|^2\right)\\
&\quad\leq 2(c_3\|u_m\||Au_m|+c_4\|v_m\||Av_m|)\left(\|u_m\|^2+|Au_m|^2+\|v_m\|^2+|Av_m|^2\right)\\
&\quad\leq \left[c_3\left(\|u_m\|^2+|Au_m|^2\right)+c_4\left(\|v_m\|^2+|Av_m|^2\right)\right]\left(\|u_m\|^2+|Au_m|^2+\|v_m\|^2+|Av_m|^2\right)\\
&\quad\leq (c_3+c_4)\left(\|u_m\|^2+|Au_m|^2+\|v_m\|^2+|Av_m|^2\right)^2.
\end{aligned}\tag{3.20}$$

Let

$$\begin{gathered}
y=\|u_m\|^2+|Au_m|^2+\|v_m\|^2+|Av_m|^2,\\
g=(c_3+c_4)\left(\|u_m\|^2+|Au_m|^2+\|v_m\|^2+|Av_m|^2\right);
\end{gathered}\tag{3.21}$$

from (3.13) we get

$$\int_t^{t+r}y(s)\ ds\leq\frac{r_1}{\varepsilon},\qquad \int_t^{t+r}g(s)\ ds\leq(c_3+c_4)\frac{r_1}{\varepsilon}.\tag{3.22}$$

Finally we obtain that

$$\|u_m\|^2+|Au_m|^2+\|v_m\|^2+|Av_m|^2\leq\left(\frac{r_1}{\varepsilon r}\right)\exp\left\{(c_3+c_4)\frac{r_1}{\varepsilon}\right\}\triangleq r_2.\tag{3.23}$$

Integrating (3.19) over $[t, t+r]$ yields

$$\begin{aligned}
&\int_t^{t+r} \varepsilon\left(|Au_m(s)|^2 + |\nabla Au_m(s)|^2 + |Av_m(s)|^2 + |\nabla Av_m(s)|^2\right)ds \\
&\quad\le \int_t^{t+r} \varepsilon\lambda_1\left(\|u_m(s)\|^2 + |Au_m(s)|^2 + \|v_m(s)\|^2 + |Av_m(s)|^2\right)ds \\
&\qquad + \int_t^{t+r} (c_3 + c_4)\left(\|u_m(s)\|^2 + |Au_m(s)|^2 + \|v_m(s)\|^2 + |Av_m(s)|^2\right)^2 ds \\
&\qquad + \left(\|u_m\|^2 + |Au_m|^2 + \|v_m\|^2 + |Av_m|^2\right) \\
&\quad\le \varepsilon\lambda_1 r_2 r + (c_3 + c_4) r_2^2 r + r_2 \triangleq r_3,
\end{aligned} \tag{3.24}$$

where $c_1, c_2, c_3, c_4, r_2,$ and r_3 are positive constants.

Taking the inner product of (3.1) and (3.2), respectively, with u_{mxxxx}, v_{mxxxx}, in Ω, we obtain that

$$\begin{aligned}
(u_{mt} - u_{mxxt} - \varepsilon(u_m - u_{mxx})_{xx}, u_{mxxxx}) &= \frac{1}{2}\frac{d}{dt}\left(|Au_m|^2 + |\nabla Au_m|^2\right) + \varepsilon\left(|\nabla Au_m|^2 + \left|A^2 u_m\right|^2\right), \\
(v_{mt} - v_{mxxt} - \varepsilon(v_m - v_{mxx})_{xx}, v_{mxxxx}) &= \frac{1}{2}\frac{d}{dt}\left(|Av_m|^2 + |\nabla Av_m|^2\right) + \varepsilon\left(|\nabla Av_m|^2 + \left|A^2 v_m\right|^2\right), \\
|P_m(3u_m u_{mx} - 2u_{mx}u_{mxx} - u_m u_{mxxx}, u_{mxxxx})| &\le 3\|\nabla u_m\|_{L^\infty(\Omega)}|Au_m|^2 + \frac{5}{2}\|\nabla u_m\|_{L^\infty(\Omega)}|\nabla Au_m|^2, \\
|P_m(3v_m v_{mx} - 2v_{mx}v_{mxx} - v_m v_{mxxx}, v_{mxxxx})| &\le 3\|\nabla v_m\|_{L^\infty(\Omega)}|Av_m|^2 + \frac{5}{2}\|\nabla v_m\|_{L^\infty(\Omega)}|\nabla Av_m|^2.
\end{aligned} \tag{3.25}$$

By integrating by parts and applying Sobolev inequality, we obtain that

$$\begin{aligned}
&|P_m[(u_{mx}v_m + u_m v_{mx}, u_{mxxxx}) + (u_m u_{mx}, v_{mxxxx})]| \\
&\quad = \left|P_m\left[-(u_{mxxx}, (u_{mxx}v_m + 2u_{mx}v_{mx} + u_m v_{mxx}) - \left(v_{mxxx}, u_{mx}^2 + u_m u_{mxx}\right)\right]\right| \\
&\quad \le \|u_{mxxx}\|\|u_{mxx}v_m + 2u_{mx}v_{mx} + u_m v_{mxx}\| + \|v_{mxxx}\|\left\|u_{mx}^2 + u_m u_{mxx}\right\| \\
&\quad \le \frac{\varepsilon}{4}|\nabla Au_m|^2 + \frac{\varepsilon}{4}|\nabla Av_m|^2 + \frac{C_1}{2}, \\
&|P_m[(v_{mx}u_m + v_m u_{mx}, v_{mxxxx}) + (v_m v_{mx}, u_{mxxxx})]| \\
&\quad = \left|P_m\left[-(v_{mxxx}, (v_{mxx}u_m + 2v_{mx}u_{mx} + v_m u_{mxx}) - \left(u_{mxxx}, v_{mx}^2 + v_m v_{mxx}\right)\right]\right| \\
&\quad \le \|v_{mxxx}\|\|v_{mxx}u_m + 2v_{mx}u_{mx} + v_m u_{mxx}\| + \|u_{mxxx}\|\left\|v_{mx}^2 + v_m v_{mxx}\right\| \\
&\quad \le \frac{\varepsilon}{4}|\nabla Au_m|^2 + \frac{\varepsilon}{4}|\nabla Av_m|^2 + \frac{C_1}{2},
\end{aligned} \tag{3.26}$$

where C_1 is a constant depending on $|u_m|, \|u_m\|, |v_m|,$ and$\|v_m\|$. As well as

$$\begin{aligned}
&|P_m[(-u_{mxxx}v_m - u_{mxx}v_{mx} - v_{mxx}v_{mx}, u_{mxxxx})+(-v_{mxxx}u_m - v_{mxx}u_{mx} - u_{mxx}u_{mx}, v_{mxxxx})]| \\
&\quad = \left|P_m\left(\frac{3}{2}\int_\Omega u_{mxxx}^2 v_{mx}\,dx + \frac{3}{2}\int_\Omega v_{mxxx}^2 u_{mx}\,dx + \frac{1}{2}\int_\Omega u_{mxxx}v_{mxx}^2\,dx\right.\right. \\
&\qquad \left.\left. + \frac{1}{2}\int_\Omega v_{mxxx}u_{mxx}^2\,dx + \int_\Omega u_{mxxx}v_{mxxx}u_{mx}\,dx + \int_\Omega u_{mxxx}v_{mxxx}v_{mx}\,dx\right)\right| \\
&\quad \le \left|P_m\left(2\int_\Omega u_{mxxx}^2 v_{mx}\,dx + 2\int_\Omega v_{mxxx}^2 u_{mx}\,dx + \frac{1}{2}\int_\Omega u_{mxxx}v_{mxx}^2\,dx\right.\right. \\
&\qquad \left.\left. + \frac{1}{2}\int_\Omega v_{mxxx}u_{mxx}^2 dx\right)\right| + \left|P_m\left(\frac{1}{2}\int_\Omega v_{mxxx}^2 v_{mx}\,dx\right)\right| + \left|P_m\left(\frac{1}{2}\int_\Omega u_{mxxx}^2 u_{mx}\,dx\right)\right| \\
&\quad \le 2\|\nabla v_m\|_{L^\infty(\Omega)}|\nabla A u_m|^2 + 2\|\nabla u_m\|_{L^\infty(\Omega)}|\nabla A v_m|^2 + \frac{\varepsilon}{4}\left(|\nabla A u_m|^2 + |\nabla A v_m|^2\right) \\
&\qquad + C_2 + \frac{1}{2}\|\nabla v_m\|_{L^\infty(\Omega)}|\nabla A v_m|^2 + \frac{1}{2}\|\nabla u_m\|_{L^\infty(\Omega)}|\nabla A u_m|^2,
\end{aligned} \tag{3.27}$$

where C_2 is a constant depending on $|u_m|, \|u_m\|, |v_m|, \|v_m\|, |Au_m|,$ and$|Av_m|$. Combining all the above inequalities, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(|Au_m|^2 + |\nabla A u_m|^2 + |Av_m|^2 + |\nabla A v_m|^2\right) \\
&\qquad + \frac{\varepsilon}{4}|\nabla A u_m|^2 + \varepsilon\left|A^2 u_m\right|^2 + \frac{\varepsilon}{4}|\nabla A v_m|^2 + \varepsilon\left|A^2 v_m\right|^2 \\
&\quad \le 3\|\nabla u_m\|_{L^\infty(\Omega)}|Au_m|^2 + 3\|\nabla u_m\|_{L^\infty(\Omega)}|\nabla A u_m|^2 \\
&\qquad + 2\|\nabla u_m\|_{L^\infty(\Omega)}|\nabla A v_m|^2 + 2\|\nabla v_m\|_{L^\infty(\Omega)}|\nabla A u_m|^2 \\
&\qquad + 3\|\nabla v_m\|_{L^\infty(\Omega)}|Av_m|^2 + 3\|\nabla v_m\|_{L^\infty(\Omega)}|\nabla A v_m|^2 + C_1 + C_2.
\end{aligned} \tag{3.28}$$

By employing Young inequality, it follows that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(|Au_m|^2 + |\nabla A u_m|^2 + |Av_m|^2 + |\nabla A v_m|^2\right) + \frac{\varepsilon}{4}\left(|\nabla A u_m|^2 + \left|A^2 u_m\right|^2 + |\nabla A v_m|^2 + \left|A^2 v_m\right|^2\right) \\
&\quad \le 3c_5\|u_m\|^{1/2}|Au_m|^{1/2}\left(|Au_m|^2 + |\nabla A u_m|^2 + |Av_m|^2 + |\nabla A v_m|^2\right) \\
&\qquad + 3c_6\|v_m\|^{1/2}|Av_m|^{1/2}\left(|Av_m|^2 + |\nabla A v_m|^2 + |Au_m|^2 + |\nabla A u_m|^2\right) + C_1 + C_2
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{1}{4}\varepsilon\lambda_1\Big(|Au_m|^2 + |\nabla Au_m|^2 + |\nabla Av_m|^2 + |Av_m|^2\Big) \\
&\quad + c_7\|u_m\||Au_m|\Big(|Au_m|^2 + |\nabla Au_m|^2 + |\nabla Av_m|^2 + |Av_m|^2\Big) \\
&\quad + c_8\|v_m\||Av_m|\Big(|Au_m|^2 + |\nabla Au_m|^2 + |\nabla Av_m|^2 + |Av_m|^2\Big) + C_1 + C_2,
\end{aligned}
\tag{3.29}
$$

where $c_7 = 18c_5^2/\varepsilon\lambda_1,\ c_8 = 18c_6^2/\varepsilon\lambda_1$. Based on Poincaré inequality we obtain that

$$
\begin{aligned}
&\frac{d}{dt}\Big(|Au_m|^2 + |\nabla Au_m|^2 + |Av_m|^2 + |\nabla Av_m|^2\Big) \\
&\quad \leq c_7\Big(\|u_m\|^2 + |Au_m|^2\Big)\Big(|Au_m|^2 + |\nabla Au_m|^2 + |Av_m|^2 + |\nabla Av_m|^2\Big) \\
&\qquad + c_8\Big(\|v_m\|^2 + |Av_m|^2\Big)\Big(|Au_m|^2 + |\nabla Au_m|^2 + |Av_m|^2 + |\nabla Av_m|^2\Big) + 2C_2 + 2C_1 \\
&\quad \leq 2(C_1 + C_2) + (c_7 + c_8)\Big(\|u_m\|^2 + |Au_m|^2 + \|v_m\|^2 + |Av_m|^2\Big) \\
&\qquad \times \Big(|Au_m|^2 + |\nabla Au_m|^2 + |Av_m|^2 + |\nabla Av_m|^2\Big).
\end{aligned}
\tag{3.30}
$$

Let

$$
\begin{gathered}
y = |Au_m|^2 + |\nabla Au_m|^2 + |Av_m|^2 + |\nabla Av_m|^2, \\
g = (c_7 + c_8)\Big(\|u_m\|^2 + |Au_m|^2 + \|v_m\|^2 + |Av_m|^2\Big), \\
h = 2(C_1 + C_2);
\end{gathered}
\tag{3.31}
$$

from (3.12) and (3.23), we conclude that h is bounded, so we suppose that $\int_t^{t+r} h(s)ds \leq r_4$; by (3.24), we have

$$
\int_t^{t+r} y(s)\ ds \leq \frac{r_3}{\varepsilon}, \qquad \int_t^{t+r} g(s)\ ds \leq (c_7 + c_8)r_2 r.
\tag{3.32}
$$

Using Gronwall inequality, we obtain that

$$
|Au_m|^2 + |\nabla Au_m|^2 + |Av_m|^2 + |\nabla Av_m|^2 \leq \left(\frac{r_3}{\varepsilon r} + r_4\right)\exp\{(c_7 + c_8)r_2 r\} \triangleq r_5.
\tag{3.33}
$$

Integrating (3.29) over $[t, t+r]$, we obtain that

$$\begin{aligned}
&\int_t^{t+r} \frac{\varepsilon}{4}\left(|\nabla A u_m(s)|^2 + \left|A^2 u_m(s)\right|^2 + |\nabla A v_m(s)|^2 + \left|A^2 v_m(s)\right|^2\right) ds \\
&\quad \le \int_t^{t+r} \frac{\varepsilon}{4}\lambda_1\left(|A u_m(s)|^2 + |\nabla A u_m(s)|^2 + |A v_m(s)|^2 + |\nabla A v_m(s)|^2\right) ds \\
&\qquad + \int_t^{t+r} [C_1 + C_2] ds + \left(|A u_m|^2 + |\nabla A u_m|^2 + |A v_m|^2 + |\nabla A v_m|^2\right) \\
&\qquad + \int_t^{t+r} c_7 \|u_m(s)\| |A u_m(s)| \left(|A u_m(s)|^2 + |\nabla A u_m(s)|^2 + |A v_m(s)|^2 + |\nabla A v_m(s)|^2\right) ds \\
&\qquad + \int_t^{t+r} c_8 \|v_m(s)\| |A v_m(s)| \left(|A v_m(s)|^2 + |\nabla A v_m(s)|^2 + |A u_m(s)|^2 + |\nabla A u_m(s)|^2 \right) ds \\
&\quad \le \frac{\varepsilon}{4}\lambda_1 r_5 r + (c_7 + c_8) r_5 r_2 r + r_4 + r_5 \triangleq r_6,
\end{aligned} \tag{3.34}$$

where $c_5, c_6, c_7, c_8, r_4, r_5,$ and r_6 are nonnegative constants.

Respectively, taking the inner product of (3.1) and (3.2) with $A^3 u_m, A^3 v_m$ in Ω, we can also get $|\nabla A u_m|^2 + |A^2 u_m|^2 + |\nabla A v_m|^2 + |A^2 v_m|^2 \le r_7$. Connecting (3.12) and (3.23) with (3.33), we can get that

$$\begin{aligned}
&|u_m|^2 \le r_1, \quad \|u_m\|^2 \le r_1, \quad |A u_m|^2 \le r_2, \quad |\nabla A u_m|^2 \le r_5, \quad \left|A^2 u_m\right|^2 \le r_7, \\
&|v_m|^2 \le r_1, \quad \|v_m\|^2 \le r_1, \quad |A v_m|^2 \le r_2, \quad |\nabla A v_m|^2 \le r_5, \quad \left|A^2 v_m\right|^2 \le r_7,
\end{aligned} \tag{3.35}$$

so $|m_m|, \|m_m\|, |A m_m|, |n_m|, \|n_m\|,$ and $|A n_m|$ are bounded.

Then we get that $du_m/dt,\ dv_m/dt, dm_m/dt, dn_m/dt$ are bounded. Considering Aubin's compactness theorem, we conclude that there is a subsequence $u'_m,\ v'_m,\ m'_m,\ n'_m,$ so that $u'_m \to u,\ v'_m \to v,\ m'_m \to m,$ and $n'_m \to n$. Now we replace $u_{m'},\ v_{m'},\ m_{m'},$ and $n_{m'}$ with $u_m,\ v_m,\ m_m,$ and n_m. We will prove that $u,\ v,\ m,$ and n satisfy (1.9)–(1.10). That is to say, approximate solution of (3.1)–(3.2) is convergent to solution of (1.9)–(1.10).

Let $\omega \in D(A), |\omega|$ is finite from the above discussion, and by ordinary differential equation (3.1), we have

$$\begin{aligned}
&(m_m(t), \omega) + \varepsilon \int_{t_0}^t (m_m(s), P_m A\omega) ds \\
&\quad = 2\int_{t_0}^t (m_m(s)\nabla u_m(s), P_m\omega) ds \\
&\qquad + \int_{t_0}^t (\nabla m_m(s) u_m(s), P_m\omega) ds + \int_{t_0}^t (\nabla m_m(s) v_m(s), P_m\omega) ds \\
&\qquad + \int_{t_0}^t (m_m(s)\nabla v_m(s), P_m\omega)\ ds + \int_{t_0}^t (n_m(s)\nabla v_m(s), P_m\omega) ds + (m_m(t_0), \omega).
\end{aligned} \tag{3.36}$$

Now it is clear that

$$\lim_{m\to\infty}\int_{t_0}^{t}(m_m(s),A\omega)ds=\int_{t_0}^{t}(m(s),A\omega)ds,$$
$$\lim_{m\to\infty}|P_m\omega-\omega|=0,\qquad \lim_{m\to\infty}|P_mA\omega-A\omega|=0,\tag{3.37}$$
$$(m_m(t),\omega)\longrightarrow(m(t),\omega),\qquad \int_{t_0}^{t}(m_m(s),P_mA\omega)ds=\int_{t_0}^{t}(m(s),A\omega)ds,\quad m\longrightarrow\infty.$$

Note that

$$\begin{aligned}
&\left|\int_{t_0}^{t}(m_m(s)\nabla u_m(s),P_m\omega)ds-\int_{t_0}^{t}(m(s)\nabla u(s),\omega)ds\right|\\
&\quad=\left|\int_{t_0}^{t}(m_m(s)\nabla u_m(s),P_m\omega-\omega)ds+\int_{t_0}^{t}(m_m(s)\nabla u_m(s),\omega)ds-\int_{t_0}^{t}(m(s)\nabla u(s),\omega)ds\right|\\
&\quad=\left|\int_{t_0}^{t}(m_m(s)\nabla u_m(s),P_m\omega-\omega)ds+\int_{t_0}^{t}(m_m(s)-m(s))\nabla u_m(s),\omega)ds\right.\\
&\qquad\left.+\int_{t_0}^{t}(m(\mathrm{s})(\nabla u_m(s)-\nabla u(s)),\omega)ds\right|\\
&\quad\le\left|\int_{t_0}^{t}(m_m(s)\nabla u_m(s),P_m\omega-\omega)ds\right|+\left|\int_{t_0}^{t}(m_m(s)-m(s))\nabla u_m(s),\omega)ds\right|\\
&\qquad+\left|\int_{t_0}^{t}(m(s)(\nabla u_m(s)-\nabla u(s)),\omega)ds\right|\\
&\quad\le\int_{t_0}^{t}|m_m(s)|\|u_m(s)\||P_m\omega-\omega|ds+\int_{t_0}^{t}|m_m(s)-m(s)|\|u_m(s)\||\omega|ds\\
&\qquad+\int_{t_0}^{t}|m(s)|\|u_m(s)-u(s)\||\omega|ds\longrightarrow 0.
\end{aligned}\tag{3.38}$$

From the above statements we get

$$\lim_{m\to\infty}\int_{t_0}^{t}(m_m(s)\nabla u_m(s),P_m\omega)ds=\int_{t_0}^{t}(m(s)\nabla u(s),\omega)ds.\tag{3.39}$$

Similarly, we have that

$$\begin{aligned}
&\lim_{m\to\infty}\int_{t_0}^{t}(u_m(s)\nabla m_m(s),P_m\omega)ds=\int_{t_0}^{t}(u(s)\nabla m(s),\omega)ds,\\
&\lim_{m\to\infty}\int_{t_0}^{t}(v_m(s)\nabla m_m(s),P_m\omega)ds=\int_{t_0}^{t}(v(s)\nabla m(s),\omega)ds,\\
&\lim_{m\to\infty}\int_{t_0}^{t}(m_m(s)\nabla v_m(s),P_m\omega)ds=\int_{t_0}^{t}(m(s)\nabla v(s),\omega)ds,\\
&\lim_{m\to\infty}\int_{t_0}^{t}(n_m(s)\nabla v_m(s),P_m\omega)ds=\int_{t_0}^{t}(n(s)\nabla v(s),\omega)ds.
\end{aligned}\tag{3.40}$$

For all $\omega\in D(A)$, we obtain that

$$\begin{aligned}
(m(t),\omega)+\varepsilon\int_{t_0}^{t}(m(s),A\omega)ds&=2\int_{t_0}^{t}(m(s)\nabla u(s),\omega)ds+\int_{t_0}^{t}(\nabla m(s)u(s),\omega)ds\\
&\quad+\int_{t_0}^{t}(\nabla m(s)v(s),\omega)ds+\int_{t_0}^{t}(m(s)\nabla v(s),\omega)ds\\
&\quad+\int_{t_0}^{t}(n(s)\nabla v(s),\omega)ds+(m(t_0),\omega).
\end{aligned}\tag{3.41}$$

From ordinary differential equation (3.2), we know that

$$\begin{aligned}
&(n_m(t),\omega)+\varepsilon\int_{t_0}^{t}(n_m(s),P_mA\omega)ds\\
&\quad=2\int_{t_0}^{t}(n_m(s)\nabla v_m(s),P_m\omega)ds+\int_{t_0}^{t}(\nabla n_m(s)v_m(s),P_m\omega)ds\\
&\qquad+\int_{t_0}^{t}(\nabla n_m(s)u_m(s),P_m\omega)ds+\int_{t_0}^{t}(n_m(s)\nabla u_m(s),P_m\omega)ds\\
&\qquad+\int_{t_0}^{t}(m_m(s)\nabla u_m(s),P_m\omega)ds+(n_m(t_0),\omega),
\end{aligned}\tag{3.42}$$

as we know that

$$
\begin{aligned}
&(n_m(t),\omega) \longrightarrow (n(t),\omega), \qquad \int_{t_0}^{t}(n_m(s),P_mA\omega)ds = \int_{t_0}^{t}(n(s),A\omega)ds, \quad m\longrightarrow\infty,\\
&\left|\int_{t_0}^{t}(n_m(s)\nabla v_m(s),P_m\omega)ds - \int_{t_0}^{t}(n(s)\nabla v(s),\omega)ds\right|\\
&\quad = \left|\int_{t_0}^{t}(n_m(s)\nabla v_m(s),P_m\omega-\omega)ds + \int_{t_0}^{t}(n_m(s)\nabla v_m(s),\omega)ds - \int_{t_0}^{t}(n(s)\nabla v(s),\omega)ds\right|\\
&\quad = \left|\int_{t_0}^{t}(n_m(s)\nabla v_m(s),P_m\omega-\omega)ds + \int_{t_0}^{t}(n_m(s)-n(s))\nabla v_m(s),\omega)ds\right.\\
&\qquad \left.+\int_{t_0}^{t}(n(s)(\nabla v_m(s)-\nabla v(s)),\omega)ds\right|\\
&\quad \le \left|\int_{t_0}^{t}(n_m(s)\nabla v_m(s),P_m\omega-\omega)ds\right| + \left|\int_{t_0}^{t}(n_m(s)-n(s))\nabla v_m(s),\omega)ds\right|\\
&\qquad + \left|\int_{t_0}^{t}(n(s)(\nabla u_m(s)-\nabla v(s)),\omega)ds\right|\\
&\quad \le \int_{t_0}^{t}|n_m(s)|\|v_m(s)\||P_m\omega-\omega|ds + \int_{t_0}^{t}|n_m(s)-n(s)|\|v_m(s)\||\omega|ds\\
&\qquad + \int_{t_0}^{t}|n(s)|\|v_m(s)-v(s)\||\omega|ds \longrightarrow 0.
\end{aligned}
\tag{3.43}
$$

From the above discussions we get

$$
\lim_{m\to\infty}\int_{t_0}^{t}(n_m(s)\nabla v_m(s),P_m\omega)ds = \int_{t_0}^{t}(n(s)\nabla v(s),\omega)ds. \tag{3.44}
$$

Simultaneously, we have that

$$
\begin{aligned}
\lim_{m\to\infty}\int_{t_0}^{t}(v_m(s)\nabla n_m(s),P_m\omega)ds &= \int_{t_0}^{t}(v(s)\nabla n(s),\omega)ds,\\
\lim_{m\to\infty}\int_{t_0}^{t}(u_m(s)\nabla n_m(s),P_m\omega)ds &= \int_{t_0}^{t}(u(s)\nabla n(s),\omega)ds,\\
\lim_{m\to\infty}\int_{t_0}^{t}(n_m(s)\nabla u_m(s),P_m\omega)ds &= \int_{t_0}^{t}(n(s)\nabla u(s),\omega)ds,\\
\lim_{m\to\infty}\int_{t_0}^{t}(m_m(s)\nabla u_m(s),P_m\omega)ds &= \int_{t_0}^{t}(m(s)\nabla u(s),\omega)ds.
\end{aligned}
\tag{3.45}
$$

For all $\omega \in D(A)$, we obtain

$$\begin{aligned}(n(t),\omega)+\varepsilon\int_{t_0}^{t}(n(s),A\omega)ds&=2\int_{t_0}^{t}(n(s)\nabla v(s),\omega)ds+\int_{t_0}^{t}(\nabla n(s)v(s),\omega)ds\\&\quad+\int_{t_0}^{t}(\nabla n(s)u(s),\omega)ds+\int_{t_0}^{t}(n(s)\nabla u(s),\omega)ds\\&\quad+\int_{t_0}^{t}(m(s)\nabla u(s),\omega)ds+(n(t_0),\omega).\end{aligned}\tag{3.46}$$

All the above analysis shows that the global solution to (1.9)–(1.13) exists in $H^2(\Omega)$. □

Theorem 3.2. *Denote $S(t)$ as the semigroup of the solution operator to* (1.9)–(1.13)*, $S(t)$: $H^2(\Omega) \to H^2(\Omega)$, $u(t) = S(t)u_0$, $\rho(t) = S(t)\rho_0$. Then $S(t)$ has an absorbing set in $H^2(\Omega)$.*

Proof. Taking the inner product of (1.9) and (1.10),respectively, with u, v in Ω, noting that

$$\begin{aligned}&(u_t-u_{xxt}-\varepsilon(u-u_{xx})_{xx},u)=\frac{1}{2}\frac{d}{dt}\left(|u|^2+\|u\|^2\right)+\varepsilon\left(\|u\|^2+|Au|^2\right),\\&(v_t-v_{xxt}-\varepsilon(v-v_{xx})_{xx},v)=\frac{1}{2}\frac{d}{dt}\left(|v|^2+\|v\|^2\right)+\varepsilon\left(\|v\|^2+|Av|^2\right),\\&-2(u_xu_{xx},u)-(u_{xxx}u,u)=\int_\Omega(-2uu_xu_{xx})dx+\int_\Omega u_{xx}\left(u^2\right)_x dx=0,\\&(u_xu,v)+(u_xv+uv_x,u)=2\int_\Omega uu_xv\,dx-\int_\Omega 2vuu_x\,dx=0,\\&(v_xv,u)+(uv_x+u_xv,v)=0,-2(v_xv_{xx},v)-(v_{xxx}v,v)=0,\\&(-u_{xxx}v-u_{xx}v_x-v_{xx}v_x,u)+(-v_{xxx}u-v_{xx}u_x-u_{xx}u_x,v)=0,\end{aligned}\tag{3.47}$$

and associating with all the above statements, we obtain

$$\frac{1}{2}\frac{d}{dt}\left(|u|^2+\|u\|^2+|v|^2+\|v\|^2\right)+\varepsilon\left(\|u\|^2+|Au|^2+\|v\|^2+|Av|^2\right)=0.\tag{3.48}$$

According to the Poincaré inequality, we obtain that

$$\begin{aligned}&\frac{d}{dt}\left(|u|^2+\|u\|^2+|v|^2+\|v\|^2\right)+2\varepsilon\lambda_1\left(|u|^2+\|u\|^2+|v|^2+\|v\|^2\right)\le 0,\\&|u|+\|u\|^2+|v|^2+\|v\|^2\\&\quad\le\left(|u(0)|^2+\|u(0)\|^2+|v(0)|^2+\|v(0)\|^2\right)\exp\{-2\varepsilon\lambda_1t\}\\&\quad\le|u(0)|^2+\|u(0)\|^2+|v(0)|^2+\|v(0)\|^2\triangleq r_8,\end{aligned}\tag{3.49}$$

where r_8 is nonnegative constant.

Integrating (3.48) over $[t, t+r]$, we obtain that

$$\varepsilon \int_t^{t+r} \left(\|v(x,s)\|^2 + |Av(x,s)|^2 + \|u(x,s)\|^2 + |Au(x,s)|^2 \right) ds \leq r_8. \tag{3.50}$$

We will obtain the uniform estimate of (1.9)–(1.13) in $H^2(\Omega)$ as follows.

Taking the inner product of (1.9) and (1.10), respectively, with $-u_{xx}, -v_{xx}$, we have

$$\begin{aligned}
&(u_t - u_{xxt} - \varepsilon(u - u_{xx})_{xx}, -u_{xx}) = \frac{1}{2}\frac{d}{dt}\left(\|u\|^2 + |Au|^2\right) + \varepsilon\left(|Au|^2 + |\nabla Au|^2\right),\\
&(v_t - v_{xxt} - \varepsilon(v - v_{xx})_{xx}, -v_{xx}) = \frac{1}{2}\frac{d}{dt}\left(\|v\|^2 + |Av|^2\right) + \varepsilon\left(|Av|^2 + |\nabla Av|^2\right),\\
&|(3uu_x, -u_{xx})| \leq \frac{3}{2}\|\nabla u\|_{L^\infty(\Omega)}\|u\|^2, \qquad |(3vv_x, -v_{xx})| \leq \frac{3}{2}\|\nabla v\|_{L^\infty(\Omega)}\|v\|^2,\\
&|(-2u_x u_{xx}, -u_{xx}) - (u_{xxx}u, -u_{xx})| = \left|\int_\Omega \left(2u_x u_{xx}^2\right) dx + \int_\Omega u u_{xx}\, du_{xx}\right|\\
&\quad = \left|\int_\Omega \frac{3}{2} u_x u_{xx}^2\, dx\right| \leq \frac{3}{2}\|\nabla u\|_{L^\infty(\Omega)}|Au|^2,\\
&|(-2v_x v_{xx}, -v_{xx}) - (v_{xxx}v, -v_{xx})| \leq \frac{3}{2}\|\nabla v\|_{L^\infty(\Omega)}|Av|^2,\\
&|(u_x v + u v_x, -u_{xx}) + (u_x u, -v_{xx})|\\
&\quad = \left|\int_\Omega \frac{u_x^2}{2} v_x\, dx - \int_\Omega u u_{xx} v_x\, dx + \int_\Omega u_x^2 v_x\, dx + \int_\Omega u u_{xx} v_x\, dx\right|\\
&\quad = \left|\frac{3}{2}\int_\Omega u_x^2 v_x\, dx\right| \leq \frac{3}{2}\|\nabla v\|_{L^\infty(\Omega)}\|u\|^2,\\
&|(v_x u + v u_x, -v_{xx}) + (v_x v, -u_{xx})| = \left|\frac{3}{2}\int_\Omega v_x^2 u_x\, dx\right| \leq \frac{3}{2}\|\nabla u\|_{L^\infty(\Omega)}\|v\|^2;
\end{aligned} \tag{3.51}$$

we also obtain

$$\begin{aligned}
&|(-u_{xxx}v - u_{xx}v_x - v_{xx}v_x, -u_{xx}) + (-v_{xxx}u - v_{xx}u_x - u_{xx}u_x, -v_{xx})|\\
&\quad \leq \left|\int_\Omega u_{xx}^2 v_x\, dx\right| + \left|\int_\Omega v_{xx}^2 u_x\, dx\right| + \left|\int_\Omega \frac{1}{2}u_{xx}^2 u_x\, dx\right| + \left|\int_\Omega \frac{1}{2}v_{xx}^2 v_x\, dx\right|\\
&\quad \leq \|\nabla v\|_{L^\infty(\Omega)}|Au|^2 + \|\nabla u\|_{L^\infty(\Omega)}|Av|^2 + \frac{1}{2}\|\nabla u\|_{L^\infty(\Omega)}|Au|^2 + \frac{1}{2}\|\nabla v\|_{L^\infty(\Omega)}|Av|^2.
\end{aligned} \tag{3.52}$$

From all the previous statements we obtain that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|u\|^2+|Au|^2+\|v\|^2+|Av|^2\right)+\varepsilon\left(|Au|^2+|\nabla Au|^2+|Av|^2+|\nabla Av|^2\right)\\
&\quad\le 2\|\nabla u\|_{L^\infty(\Omega)}\left(\|u\|^2+|Au|^2+\|v\|^2+|Av|^2\right)\\
&\qquad+2\|\nabla v\|_{L^\infty(\Omega)}\left(\|u\|^2+|Au|^2+\|v\|^2+|Av|^2\right)\\
&\quad\le 2c_9\|u\|^{1/2}|Au|^{1/2}\left(\|u\|^2+|Au|^2+\|v\|^2+|Av|^2\right)\\
&\qquad+2c_{10}\|v\|^{1/2}|Av|^{1/2}\left(\|u\|^2+|Au|^2+\|v\|^2+|Av|^2\right)\\
&\quad\le \frac{\varepsilon\lambda_1}{2}\left(\|u\|^2+|Au|^2+\|v\|^2+|Av|^2\right)+c_{11}\|u\||Au|\left(\|u\|^2+|Au|^2+\|v\|^2+|Av|^2\right)\\
&\qquad+\frac{\varepsilon\lambda_1}{2}\left(\|u\|^2+|Au|^2+\|v\|^2+|Av|^2\right)+c_{12}\|v\||Av|\left(\|u\|^2+|Au|^2+\|v\|^2+|Av|^2\right),
\end{aligned}\tag{3.53}$$

where $c_{11}=2c_9^2/\varepsilon\lambda_1$, $c_{12}=2c_{10}^2/\varepsilon\lambda_1$. From Poincaré inequality, we get

$$\begin{aligned}
&\frac{d}{dt}\left(\|u\|^2+|Au|^2+\|v\|^2+|Av|^2\right)\\
&\quad\le\left[c_{11}\left(\|u\|^2+|Au|^2\right)+c_{12}\left(\|v\|^2+|Av|^2\right)\right]\left(\|u\|^2+|Au|^2+\|v\|^2+|Av|^2\right)\\
&\quad\le(c_{11}+c_{12})\left(\|u\|^2+|Au|^2+\|v\|^2+|Av|^2\right)^2.
\end{aligned}\tag{3.54}$$

Let

$$y=\|u\|^2+|Au|^2+\|v\|^2+|Av|^2,\qquad g=(c_{11}+c_{12})\left(\|u\|^2+|Au|^2+\|v\|^2+|Av|^2\right)\tag{3.55}$$

From (3.50) we have that

$$\begin{gathered}
\int_t^{t+r}y(s)\,ds\le\frac{r_8}{\varepsilon},\qquad\int_t^{t+r}g(s)\,ds\le(c_{11}+c_{12})\frac{r_8}{\varepsilon},\\
\|u\|^2+|Au|^2+\|v\|^2+|Av|^2\le\left(\frac{r_8}{\varepsilon r}\right)\exp\left\{(c_{11}+c_{12})\frac{r_8}{\varepsilon}\right\}\triangleq r_9.
\end{gathered}\tag{3.56}$$

Integrating (3.53) over $[t,t+r]$, it follows that

$$\begin{aligned}
&\int_t^{t+r}\varepsilon\left(|Au(x,s)|^2+|\nabla Au(x,s)|^2+|Av(x,s)|^2+|\nabla Av(x,s)|^2\right)ds\\
&\quad\le\ \varepsilon\lambda_1r_9r+(c_{11}+c_{12})r_9^2r+r_9\triangleq r_{10},
\end{aligned}\tag{3.57}$$

where $c_9, c_{10}, c_{11}, c_{12}, r_8, r_9,$ and r_{10} are nonnegative constants. Taking the inner product of (1.9) and (1.10), respectively, with u_{xxxx}, v_{xxxx}, we have that

$$\begin{aligned}
(u_t - u_{xxt} - \varepsilon(u-u_{xx})_{xx}, u_{xxxx}) &= \frac{1}{2}\frac{d}{dt}\left(|Au|^2 + |\nabla Au|^2\right) + \varepsilon\left(|\nabla Au|^2 + \left|A^2 u\right|^2\right),\\
(v_t - v_{xxt} - \varepsilon(v-v_{xx})_{xx}, v_{xxxx}) &= \frac{1}{2}\frac{d}{dt}\left(|Av|^2 + |\nabla Av|^2\right) + \varepsilon\left(|\nabla Av|^2 + \left|A^2 v\right|^2\right),\\
|(3uu_x - 2u_xu_{xx} - uu_{xxx}, u_{xxxx})| &\le 3\|\nabla u\|_{L^\infty(\Omega)}|Au|^2 + \frac{5}{2}\|\nabla u\|_{L^\infty(\Omega)}|\nabla Au|^2,\\
|(3vv_x - 2v_xv_{xx} - vv_{xxx}, v_{xxxx})| &\le 3\|\nabla v\|_{L^\infty(\Omega)}|Av|^2 + \frac{5}{2}\|\nabla v\|_{L^\infty(\Omega)}|\nabla Av|^2.
\end{aligned} \tag{3.58}$$

Integrating by parts and employing Sobolev inequality, we get

$$\begin{aligned}
|(u_xv + uv_x, u_{xxxx}) + (uu_x, v_{xxxx})| &= \left|-(u_{xxx}, (u_xv+uv_x)_x - \left(v_{xxx}, u_x^2 + uu_{xx}\right)\right|\\
&\le \frac{\varepsilon}{4}|\nabla Au|^2 + \frac{\varepsilon}{4}|\nabla Av|^2 + \frac{C_3}{2},\\
|(v_xu + vu_x, v_{xxxx}) + (vv_x, u_{xxxx})| &= \left|-(v_{xxx}, (v_xu+vu_x)_x - \left(u_{xxx}, v_x^2 + vv_{xx}\right)\right|\\
&\le \frac{\varepsilon}{4}|\nabla Au|^2 + \frac{\varepsilon}{4}|\nabla Av|^2 + \frac{C_3}{2},
\end{aligned} \tag{3.59}$$

where C_3 is a constant depending on $|u|, \|u\|, |v|$, and $\|v\|$.

$$\begin{aligned}
&|(-u_{xxx}v - u_{xx}v_x - v_{xx}v_x, u_{xxxx}) + (-v_{xxx}u - \mathrm{v}_{xx}u_x - u_{xx}u_x, v_{xxxx})|\\
&\quad= \left|\frac{3}{2}\int_\Omega u_{xxx}^2 v_x\,dx + \frac{3}{2}\int_\Omega v_{xxx}^2 u_x\,dx + \frac{1}{2}\int_\Omega u_{xxx}v_{xx}^2\,dx\right.\\
&\qquad\left.+\frac{1}{2}\int_\Omega v_{xxx}u_{xx}^2\,dx + \int_\Omega u_{xxx}v_{xxx}u_x\,dx + \int_\Omega u_{xxx}v_{xxx}v_x\,dx\right|\\
&\quad\le \left|2\int_\Omega u_{xxx}^2 v_x\,dx + 2\int_\Omega v_{xxx}^2 u_x\,dx + \frac{1}{2}\int_\Omega u_{xxx}v_{xx}^2\,dx + \frac{1}{2}\int_\Omega v_{xxx}u_{xx}^2\,dx\right|\\
&\qquad+ \left|\frac{1}{2}\int_\Omega v_{xxx}^2 v_x\,dx\right| + \left|\frac{1}{2}\int_\Omega u_{xxx}^2 u_x\,dx\right|\\
&\quad\le 2\|\nabla v\|_{L^\infty(\Omega)}|\nabla Au|^2 + 2\|\nabla u\|_{L^\infty(\Omega)}|\nabla Av|^2 + \frac{\varepsilon}{4}\left(|\nabla Au|^2 + |\nabla Av|^2\right)\\
&\qquad+ C_4 + \frac{1}{2}\|\nabla v\|_{L^\infty(\Omega)}|\nabla Av|^2 + \frac{1}{2}\|\nabla u\|_{L^\infty(\Omega)}|\nabla Au|^2,
\end{aligned} \tag{3.60}$$

where C_4 is a constant depending on $|u|, \|u\|, |v|, \|v_m\|, |Au|$, and $|Av|$. From all the above inequalities we obtain that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(|Au|^2+|\nabla Au|^2+|Av|^2+|\nabla Av|^2\right)+\varepsilon\left(|\nabla Au|^2+\left|A^2u\right|^2+|\nabla A\mathrm{v}|^2+\left|A^2v\right|^2\right)\\
&\quad\le 3\|\nabla u\|_{L^\infty(\Omega)}|Au|^2+\frac{5}{2}\|\nabla u\|_{L^\infty(\Omega)}|\nabla Au|^2+3\|\nabla v\|_{L^\infty(\Omega)}|Av|^2\\
&\qquad+\frac{5}{2}\|\nabla v\|_{L^\infty(\Omega)}|\nabla Av|^2+\frac{\varepsilon}{2}|\nabla Au|^2+\frac{\varepsilon}{2}|\nabla Av|^2+C_3\\
&\qquad+2\|\nabla v\|_{L^\infty(\Omega)}|\nabla Au|^2+2\|\nabla u\|_{L^\infty(\Omega)}|\nabla Av|^2+\frac{\varepsilon}{4}\left(|\nabla Au|^2+|\nabla Av|^2\right)\\
&\qquad+C_4+\frac{1}{2}\|\nabla v\|_{L^\infty(\Omega)}|\nabla Av|^2+\frac{1}{2}\|\nabla u\|_{L^\infty(\Omega)}|\nabla Au|^2.
\end{aligned}\tag{3.61}$$

Then it follows that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(|Au|^2+|\nabla Au|^2+|Av|^2+|\nabla Av|^2\right)+\frac{\varepsilon}{4}\left(|\nabla Au|^2+\left|A^2u\right|^2+|\nabla Av|^2+\left|A^2v\right|^2\right)\\
&\quad\le 3c_{13}\|u\|^{1/2}|Au|^{1/2}\left(|Au|^2+|\nabla Au|^2+|\nabla Av|^2+|Av|^2\right)+C_3\\
&\qquad+3c_{14}\|v\|^{1/2}|Av|^{1/2}\left(|Av|^2+|\nabla Av|^2+|\nabla Au|^2+|Au|^2\right)+C_4\\
&\quad\le \frac{\varepsilon}{8}\lambda_1\left(|Au|^2+|\nabla Au|^2+|Av|^2+|\nabla \mathrm{A}v|^2\right)\\
&\qquad+c_{15}\|u\||Au|\left(|Au|^2+|\nabla Au|^2+|Av|^2+|\nabla Av|^2\right)+C_3\\
&\qquad+\frac{\varepsilon}{8}\lambda_1\left(|Au|^2+|\nabla Au|^2+|Av|^2+|\nabla Av|^2\right)\\
&\qquad+c_{16}\|v\||Av|\left(|Au|^2+|\nabla Au|^2+|Av|^2+|\nabla Av|^2\right)+C_4,
\end{aligned}\tag{3.62}$$

where $c_{15}=18c_{13}^2/\varepsilon\lambda_1, c_{16}=18c_{14}^2/\varepsilon\lambda_1$. By means of Poincaré inequality, we obtain that

$$\begin{aligned}
&\frac{d}{dt}\left(|Au|^2+|\nabla Au|^2+|Av|^2+|\nabla Av|^2\right)\\
&\quad\le c_{15}\left(\|u\|^2+|Au|^2\right)\left(|Au|^2+|\nabla Au|^2+|Av|^2+|\nabla Av|^2\right)\\
&\qquad+c_{16}\left(\|v\|^2+|Av|^2\right)\left(|Au|^2+|\nabla Au|^2+|Av|^2+|\nabla Av|^2\right)+C_3+C_4\\
&\quad\le 2(C_3+C_4)+(c_{15}+c_{16})\left(\|u\|^2+|Au|^2+\|v\|^2+|Av|^2\right)\left(|Au|^2+|\nabla Au|^2+|Av|^2+|\nabla Av|^2\right).
\end{aligned}\tag{3.63}$$

Let

$$y = |Au|^2 + |\nabla Au|^2 + |Av|^2 + |\nabla Av|^2,$$
$$g = (c_{15} + c_{16})\Big(\|u\|^2 + |Au|^2 + \|v\|^2 + |Av|^2\Big), \qquad h = 2(C_3 + C_4); \tag{3.64}$$

from (3.49) and (3.56), we conclude that h is bounded; assume that $\int_t^{t+r} h(s)\, ds \le r_{11}$. From (3.57), we have

$$\int_t^{t+r} y(s)ds \le \frac{r_{10}}{\varepsilon}, \qquad \int_t^{t+r} g(s)ds \le (c_{15} + c_{16})r_9 r. \tag{3.65}$$

Then through Gronwall inequality, we obtain

$$|Au|^2 + |\nabla Au|^2 + |Av|^2 + |\nabla Av|^2 \le \left(\frac{r_{10}}{\varepsilon r} + r_{11}\right)\exp\{(c_{15} + c_{16})r_9 r\} \triangleq \rho_1. \tag{3.66}$$

Integrating (3.62) on $[t, t+r]$, we have that

$$\int_t^{t+r} \frac{\varepsilon}{4}\left(|\nabla Au(x,s)|^2 + \left|A^2u(x,s)\right|^2 + |\nabla Av(x,s)|^2 + \left|A^2v(x,s)\right|^2\right)ds$$
$$\le \frac{\varepsilon}{4}\lambda_1\rho_1 r + (c_{15} + c_{16})r_9\rho_1 r + r_{11} + \rho_1 \triangleq r_{12}, \tag{3.67}$$

where $c_{13},\ c_{14},\ c_{15},\ c_{16},\ r_{11},\ r_{12}$, and r_{13} are nonnegative constants. Then from (3.66), we can get $|Au|^2 \le \rho_1,\ \ |\mathrm{A}v|^2 \le \rho_1$. In other words, open ball $B(0, \rho_1)$ is the attracting set of $S(t)$ in $H^2(\Omega)$. □

Theorem 3.3. *Suppose that $u_0, \rho_0 \in H^l(R),\ \ l \ge 2$, then the semigroup of the solution operator $S(t)$ to* (1.9)–(1.13) *has a global attractor in $H^2(\Omega)$.*

Proof. To obtain the existence of the global attractor, we will prove that it is a compact operator. Taking inner product of (1.9) and (1.10) with $t^2\Delta^3 u,\ t^2\Delta^3 v$ in Ω, we obtain

$$\Big(u_t - u_{xxt} - \varepsilon(u - u_{xx})_{xx}, t^2 u_{xxxxxx}\Big)$$
$$= -\frac{1}{2}\frac{d}{dt}\left(|t\nabla Au|^2 + \left|tA^2u\right|^2\right) + t\left(|\nabla Au|^2 + \left|A^2u\right|^2\right) - \varepsilon\left(|t\Delta Au|^2 + \left|\nabla A^2u\right|^2\right), \tag{3.68}$$

$$\Big(v_t - v_{xxt} - \varepsilon(v - v_{xx})_{xx}, t^2 v_{xxxxxx}\Big)$$
$$= -\frac{1}{2}\frac{d}{dt}\left(|t\nabla Av|^2 + \left|tA^2v\right|^2\right) + t\left(|\nabla Av|^2 + \left|A^2v\right|^2\right) - \varepsilon\left(|t\Delta Av|^2 + \left|\nabla A^2v\right|^2\right). \tag{3.69}$$

By integrating by parts, we obtain that

$$\begin{aligned}
&\left|\left(3uu_x-2u_xu_{xx}-uu_{xxx},t^2u_{xxxxxx}\right)\right|\\
&\quad=t^2\left|-3\int_\Omega u_x^2u_{xxxxx}\,dx-3\int_\Omega uu_{xx}u_{xxxxx}\,dx+2\int_\Omega u_{xx}^2u_{xxxxx}\,dx\right.\\
&\qquad\left.+2\int_\Omega u_xu_{xxx}u_{xxxxx}\,dx+\int_\Omega u_xu_{xxx}u_{xxxxx}\,dx+\int_\Omega uu_{xxxx}u_{xxxxx}\,dx\right|\\
&\quad=t^2\left|\int_\Omega 6u_xu_{xx}u_{xxxx}\,dx+3\int_\Omega u_xu_{xx}u_{xxxx}\,dx+3\int_\Omega uu_{xxx}u_{xxxx}\,dx\right.\\
&\qquad-4\int_\Omega u_{xx}u_{xxx}u_{xxxx}\,dx-2\int_\Omega u_{xx}u_{xxx}u_{xxxx}\,dx-2\int_\Omega u_xu_{xxxx}^2\,dx\\
&\qquad\left.-\int_\Omega u_{xx}u_{xxx}u_{xxxx}\,dx-\int_\Omega u_xu_{xxxx}^2\,dx-\frac{1}{2}\int_\Omega u_xu_{xxxx}^2\,dx\right|\\
&\quad=t^2\left|\frac{21}{2}\int_\Omega u_xu_{xxx}^2\,dx+\frac{7}{2}\int_\Omega u_{xxx}^3\,dx+\frac{7}{2}\int_\Omega u_xu_{xxxx}^2\,dx\right|\\
&\quad\le\frac{21}{2}\|\nabla u\|_{L^\infty(\Omega)}|t\nabla Au|^2+\frac{7}{2}\left(\|\nabla Au\|_{L^\infty(\Omega)}|t\nabla Au|^2+\|\nabla u\|_{L^\infty(\Omega)}\left|tA^2u\right|^2\right).
\end{aligned}\tag{3.70}$$

Similarly, we obtain

$$\begin{aligned}
&\left|\left(3vv_x-2v_xv_{xx}-vv_{xxx},t^2v_{xxxxxx}\right)\right|\\
&\quad\le\frac{21}{2}\|\nabla v\|_{L^\infty(\Omega)}|t\nabla Av|^2+\frac{7}{2}\left(\|\nabla Av\|_{L^\infty(\Omega)}|t\nabla Av|^2+\|\nabla v\|_{L^\infty(\Omega)}\left|tA^2\mathrm{v}\right|^2\right),\\
&\left|\left(u_xv+uv_x,t^2u_{xxxxxx}\right)+\left(uu_x,t^2v_{xxxxxx}\right)\right|\\
&\quad=\left|\left(t^2u_{xxxx},u_{xxx}v+3u_{xx}v_x+3u_xv_{xx}+uv_{xxx}\right)+\left(t^2v_{xxxx},3u_xu_{xx}+uu_{xxx}\right)\right|\\
&\quad\le\ \frac{\varepsilon}{4}\left|tA^2u\right|^2+\frac{\varepsilon}{4}\left|tA^2v\right|^2+\frac{1}{3}C_5,\\
&\left|\left(v_xu+vu_x,t^2v_{xxxxxx}\right)+\left(vv_x,t^2u_{xxxxxx}\right)\right|\\
&\quad=\left|\left(t^2v_{xxxx},v_{xxx}u+3v_{xx}u_x+3v_xu_{xx}+vu_{xxx}\right)+\left(t^2u_{xxxx},3v_xv_{xx}+vv_{xxx}\right)\right|\\
&\quad\le\ \frac{\varepsilon}{4}\left|tA^2u\right|^2+\frac{\varepsilon}{4}\left|tA^2v\right|^2+\frac{1}{3}C_5,
\end{aligned}\tag{3.71}$$

where C_5 is a constant depending on $|u|, \|u\|, |v|, \|v_m\|, |Au|, |Av|,$ and t. Integrating by parts frequently, we obtain

$$\begin{aligned}
&\left|\left(-u_{xxx}v - u_{xx}v_x - v_{xx}v_x, t^2 u_{xxxxxx}\right) + \left(-v_{xxx}u - v_{xx}u_x - u_{xx}u_x, t^2 v_{xxxxxx}\right)\right| \\
&\quad \leq t^2\left[\left|3\int_\Omega v_{xxxx}^2 v_x\, dx\right| + \left|3\int_\Omega u_{xxxx}^2 u_x\, dx\right| + \frac{1}{2}\int_\Omega u_{xxxx}^2 v_x\, dx + \frac{1}{2}\int_\Omega v_{xxxx}^2 u_x\, dx\right] \\
&\qquad + \frac{\varepsilon}{4}\left|tA^2u\right|^2 + \frac{\varepsilon}{4}\left|tA^2v\right|^2 + \frac{C_5}{3}, \\
&\quad \leq 3\|v_x\|_{L^\infty(\Omega)}\left(\left|tA^2v\right|^2 + \left|tA^2u\right|^2\right) + 3\|u_x\|_{L^\infty(\Omega)}\left(\left|tA^2v\right|^2 + \left|tA^2u\right|^2\right) \\
&\qquad + \frac{\varepsilon}{4}\left|tA^2u\right|^2 + \frac{\varepsilon}{4}\left|tA^2v\right|^2 + \frac{C_5}{3}.
\end{aligned} \tag{3.72}$$

From (3.68) and (3.69), we know that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(|t\nabla Au|^2 + \left|tA^2u\right|^2 + |t\nabla Av|^2 + \left|tA^2v\right|^2\right) \\
&\qquad + \frac{\varepsilon}{4}|t\Delta Au|^2 + \varepsilon\left|\nabla A^2u\right|^2 + \frac{\varepsilon}{4}|t\Delta Av|^2 + \varepsilon\left|\nabla A^2v\right|^2 \\
&\quad \leq \frac{21}{2}\|\nabla u\|_{L^\infty(\Omega)}|t\nabla Au|^2 + \frac{7}{2}\left(\|\nabla Au\|_{L^\infty(\Omega)}|t\nabla Au|^2 + \|\nabla u\|_{L^\infty(\Omega)}\left|tA^2u\right|^2\right) \\
&\qquad + \frac{21}{2}\|\nabla v\|_{L^\infty(\Omega)}|t\nabla Av|^2 + \frac{7}{2}\left(\|\nabla Av\|_{L^\infty(\Omega)}|t\nabla Av|^2 + \|\nabla v\|_{L^\infty\Omega}\left|tA^2v\right|^2\right) \\
&\qquad + 3\|\nabla v\|_{L^\infty(\Omega)}\left(\left|tA^2v\right|^2 + \left|tA^2u\right|^2\right) + 3\|\nabla u\|_{L^\infty(\Omega)}\left(\left|tA^2v\right|^2 + \left|tA^2u\right|^2\right) \\
&\qquad + t\left(|\nabla Au|^2 + \left|A^2u\right|^2 + |\nabla Av|^2 + \left|A^2v\right|^2\right) + C_5 \\
&\quad \leq \left(11\|\nabla u\|_{L^\infty(\Omega)} + 11\|\nabla v\|_{L^\infty(\Omega)}\right)\left(|t\nabla Au|^2 + \left|tA^2u\right|^2 + |t\nabla Av|^2 + \left|tA^2v\right|^2\right) \\
&\qquad + \frac{7}{2}\|\nabla Au\|_{L^\infty(\Omega)}|t\nabla Au|^2 + \frac{7}{2}\|\nabla Av\|_{L^\infty(\Omega)}\, |t\nabla Av|^2 \\
&\qquad + t\left(|\nabla Au|^2 + \left|A^2u\right|^2 + |\nabla Av|^2 + \left|A^2v\right|^2\right) + C_5 \\
&\quad \leq \left(11c_{17}\|u\|^{1/2}|Au|^{1/2} + 11c_{18}\|v\|^{1/2}|Av|^{1/2}\right)\left(|t\nabla Au|^2 + \left|tA^2u\right|^2 + |t\nabla Av|^2 + \left|tA^2v\right|^2\right) \\
&\qquad + \frac{7}{2}c_{19}|\nabla Au|^{1/2}\left|A^2u\right|^{1/2}|t\nabla Au|^2 + \frac{7}{2}c_{20}|\nabla Av|^{1/2}\left|A^2v\right|^{1/2}|t\nabla Av|^2 \\
&\qquad + t\left(|\nabla Au|^2 + \left|A^2u\right|^2 + |\nabla Av|^2 + \left|A^2v\right|^2\right) + C_5
\end{aligned}$$

$$\leq 11c_{21}\left(\|u\|^{1/2}|Au|^{1/2}+\|v\|^{1/2}|Av|^{1/2}+|\nabla Au|^{1/2}\left|A^2u\right|^{1/2}+|\nabla Av|^{1/2}\left|A^2v\right|^{1/2}\right)$$
$$\times\left(|t\nabla Au|^2+\left|tA^2u\right|^2+|t\nabla Av|^2+\left|tA^2v\right|^2\right)$$
$$+t\left(|\nabla Au|^2+\left|A^2u\right|^2+|\nabla Av|^2+\left|A^2v\right|^2\right)+C_5, \tag{3.73}$$

where $c_{21}=\max\{4c_{17},4c_{18},14c_{19},14c_{20}\}$. Employing Young inequality, we obtain that

$$\frac{1}{2}\frac{d}{dt}\left(|t\nabla Au|^2+\left|tA^2u\right|^2+|t\nabla Av|^2+\left|tA^2v\right|^2\right)$$
$$+\frac{\varepsilon}{4}\left(|t\Delta Au|^2+\left|\nabla A^2u\right|^2+|t\Delta Av|^2+\left|\nabla A^2v\right|^2\right)$$
$$\leq\frac{\varepsilon}{4}\lambda_1\left(|t\nabla Au|^2+\left|tA^2u\right|^2+|t\nabla Av|^2+\left|tA^2v\right|^2\right)$$
$$+c_{22}\left(\|u\||Au|+\|v\||Av|+|\nabla Au|\left|A^2u\right||\nabla Av|\left|A^2v\right|\right) \tag{3.74}$$
$$\times\left(|t\nabla Au|^2+\left|tA^2u\right|^2+|t\nabla Av|^2+\left|tA^2v\right|^2\right)$$
$$+c_{23}\left(|\nabla Au|^2+\left|A^2u\right|^2+|\nabla Av|^2+\left|A^2v\right|^2\right)+C_5,$$

where $c_{22}=242c_{21}^2/\varepsilon\lambda_1$, $\quad c_{23}=2/\varepsilon\lambda_1$. From Poincaré inequality, we obtain that

$$\frac{d}{dt}\left(|t\nabla Au|^2+\left|tA^2u\right|^2+|t\nabla Av|^2+\left|tA^2v\right|^2\right)$$
$$\leq 2c_{22}\left(\|u\||Au|+\|v\||Av|+|\nabla Au|\left|A^2u\right|+|\nabla Av|\left|A^2v\right|\right)$$
$$\times\left(|\nabla Au|^2+\left|A^2u\right|^2+|\nabla Av|^2+\left|A^2v\right|^2\right) \tag{3.75}$$
$$+2c_{23}\left(|\nabla Au|^2+\left|A^2u\right|^2+|\nabla Av|^2+\left|^2v\right|^2\right)+2C_5.$$

Let

$$y=|t\nabla Au|^2+\left|tA^2u\right|^2+|t\nabla Av|^2+\left|tA^2v\right|^2,$$
$$g=2c_{22}\left(\|u\||Au|+\|v\||Av|+|\nabla Au|\left|A^2u\right|+|\nabla Av|\left|A^2v\right|\right), \tag{3.76}$$
$$h=2c_{23}\left(|\nabla Au|^2+\left|A^2u\right|^2+|\nabla Av|^2+\left|A^2v\right|^2\right)+2C_5.$$

Through (3.67), we have

$$\int_t^{t+r}\left(|\nabla Au(x,s)|^2+\left|A^2u(x,s)\right|^2+|\nabla Av(x,s)|^2+\left|A^2v(x,s)\right|^2\right)ds\le\frac{4r_{12}}{\varepsilon}. \tag{3.77}$$

We also get

$$\begin{aligned}
&\int_t^{t+r}\left(|t\nabla Au(x,s)|^2+\left|tA^2u(x,s)\right|^2+|t\nabla Av(x,s)|^2+\left|tA^2v(x,s)\right|^2\right)ds\le(t+r)^2\frac{4r_{12}}{\varepsilon},\\
&\int_t^{t+r}2c_{22}\left(\|u\||Au|+\|v\||Av|+|\nabla Au|\left|A^2u\right|+|\nabla Av|\left|A^2v\right|\right)ds\\
&\quad\le\int_t^{t+r}2c_{22}\left(\|u\|^2+|Au|^2+\|v\|^2+|Av|^2+|\nabla Au|^2+\left|A^2u\right|^2+|\nabla Av|^2+\left|A^2v\right|^2\right)ds\\
&\quad\le 2c_{22}\left(r_9r+\frac{4r_{12}}{\varepsilon}\right).
\end{aligned} \tag{3.78}$$

Through (3.49), (3.56), and (3.66), we assume that

$$\int_t^{t+r}C_5\,ds\le r_{13}. \tag{3.79}$$

Then we have

$$\int_t^{t+r}\left[2c_{23}\left(|\nabla Au|^2+\left|A^2u\right|^2+|\nabla Av|^2+\left|A^2v\right|^2\right)+2C_5\right]ds\le 2c_{23}\frac{4r_{12}}{\varepsilon}+2r_{13}, \tag{3.80}$$

and all the analysis indicates that

$$\begin{aligned}
&|t\nabla Au|^2+\left|tA^2u\right|^2+|t\nabla Av|^2+\left|tA^2v\right|^2\\
&\quad\le\left[(t+r)^2\frac{4r_{12}}{\varepsilon r}+2c_{23}\frac{4r_{12}}{\varepsilon}+2r_{13}\right]\exp\left\{2c_{22}\left(r_9r+\frac{4r_{12}}{\varepsilon}\right)\right\}\\
&\quad\triangleq E^2(\lambda_1,\rho_1,\varepsilon,t).
\end{aligned} \tag{3.81}$$

Finally we obtain that

$$|\nabla Au|\le\frac{E(\lambda_1,\rho_1,\varepsilon,t)}{t},\qquad|\nabla Av|\le\frac{E(\lambda_1,\rho_1,\varepsilon,t)}{t}. \tag{3.82}$$

We know that the injection of $H^3(\Omega)$ into $H^2(\Omega)$ is compact, then we can conclude that $S(t)$ is equi-continuity. From Ascoli-Arzela's theorem, we know that $S(t)$ has the global attractor in $H^2(\Omega)$. □

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References

[1] B. Fuchssteiner and A. S. Fokas, "Symplectic structures, their Bäcklund transformations and hereditary symmetries," *Physica D*, vol. 4, no. 1, pp. 47–66, 1981.
[2] R. Camassa and D. D. Holm, "An integrable shallow water equation with peaked solitons," *Physical Review Letters*, vol. 71, no. 11, pp. 1661–1664, 1993.
[3] R. Camassa, D. D. Holm, and J. M. Hyman, "A new integrable shallow water equation," *Advances in Applied Mechanics*, vol. 31, pp. 1–33, 1994.
[4] G. B. Whitham, *Linear and Nonlinear Waves*, Pure and Applied Mathematic, John Wiley & Sons, New York, NY, USA, 1974.
[5] A. Constantin and W. A. Strauss, "Stability of peakons," *Communications on Pure and Applied Mathematics*, vol. 53, no. 5, pp. 603–610, 2000.
[6] A. Constantin and L. Molinet, "Orbital stability of solitary waves for a shallow water equation," *Physica D*, vol. 157, no. 1-2, pp. 75–89, 2001.
[7] A. Constantin and J. Escher, "Particle trajectories in solitary water waves," *Bulletin of American Mathematical Society*, vol. 44, no. 3, pp. 423–431, 2007.
[8] A. Constantin, "The trajectories of particles in Stokes waves," *Inventiones Mathematicae*, vol. 166, no. 3, pp. 523–535, 2006.
[9] A. Constantin and J. Escher, "Wave breaking for nonlinear nonlocal shallow water equations," *Acta Mathematica*, vol. 181, no. 2, pp. 229–243, 1998.
[10] A. Constantin and W. A. Strauss, "Stability of a class of solitary waves in compressible elastic rods," *Physics Letters A*, vol. 270, no. 3-4, pp. 140–148, 2000.
[11] H.-H. Dai, "Model equations for nonlinear dispersive waves in a compressible Mooney-Rivlin rod," *Acta Mechanica*, vol. 127, no. 1–4, pp. 193–207, 1998.
[12] A. Bressan and A. Constantin, "Global conservative solutions of the Camassa-Holm equation," *Archive for Rational Mechanics and Analysis*, vol. 183, no. 2, pp. 215–239, 2007.
[13] A. Bressan and A. Constantin, "Global dissipative solutions of the Camassa-Holm equation," *Analysis and Applications*, vol. 5, no. 1, pp. 1–27, 2007.
[14] A. Parker, "Wave dynamics for peaked solitons of the Camassa-Holm equation," *Chaos, Solitons and Fractals*, vol. 35, no. 2, pp. 220–237, 2008.
[15] R. Beals, D. H. Sattinger, and J. Szmigielski, "Multipeakons and the classical moment problem," *Advances in Mathematics*, vol. 154, no. 2, pp. 229–257, 2000.
[16] R. Beals, D. H. Sattinger, and J. Szmigielski, "Multi-peakons and a theorem of Stieltjes," *Inverse Problems*, vol. 15, no. 1, pp. L1–L4, 1999.
[17] A. Constantin and R. S. Johnson, "Propagation of very long water waves, with vorticity, over variable depth, with applications to tsunamis," *Fluid Dynamics Research*, vol. 40, no. 3, pp. 175–211, 2008.
[18] H. Holden and X. Raynaud, "Global conservative solutions of the Camassa-Holm equation—a Lagrangian point of view," *Communications in Partial Differential Equations*, vol. 32, no. 10–12, pp. 1511–1549, 2007.
[19] A. Constantin, "On the Cauchy problem for the periodic Camassa-Holm equation," *Journal of Differential Equations*, vol. 141, no. 2, pp. 218–235, 1997.
[20] A. Constantin, "On the inverse spectral problem for the Camassa-Holm equation," *Journal of Functional Analysis*, vol. 155, no. 2, pp. 352–363, 1998.
[21] A. Constantin, V. S. Gerdjikov, and R. I. Ivanov, "Inverse scattering transform for the Camassa-Holm equation," *Inverse Problems*, vol. 22, no. 6, pp. 2197–2207, 2006.
[22] W. Rui, B. He, Y. Long, and C. Chen, "The integral bifurcation method and its application for solving a family of third-order dispersive PDEs," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 69, no. 4, pp. 1256–1267, 2008.

[23] W. Rui, S. Xie, Y. Long, and B. He, "Integral bifurcation method and its application for solving the modified equal width wave equation and its variants," *Rostocker Mathematisches Kolloquium*, no. 62, pp. 87–106, 2007.

[24] J. Lenells, "Infinite propagation speed of the Camassa-Holm equation," *Journal of Mathematical Analysis and Applications*, vol. 325, no. 2, pp. 1468–1478, 2007.

[25] D. Cohen, B. Owren, and X. Raynaud, "Multi-symplectic integration of the Camassa-Holm equation," *Journal of Computational Physics*, vol. 227, no. 11, pp. 5492–5512, 2008.

[26] Y. Meng and L. Tian, "Boundary control of the viscous generalized Camassa-Holm equation," *International Journal of Nonlinear Science*, vol. 8, no. 2, pp. 173–181, 2009.

[27] Y. Long, Z. Li, and W. Rui, "New travelling wave solutions for a nonlinearly dispersive wave equation of Camassa-Holm equation type," *Applied Mathematics and Computation*, vol. 217, no. 4, pp. 1315–1320, 2010.

[28] W.-G. Rui and Y. Long, "New periodic loop solitons of the generalized KdV equation," *International Journal of Nonlinear Sciences and Numerical Simulation*, vol. 9, no. 4, pp. 441–444, 2008.

[29] D. P. Ding and L. X. Tian, "The attractor on dissipative Camassa-Holm equation," *Acta Mathematicae Applicatae Sinica*, vol. 27, no. 3, pp. 536–545, 2004.

[30] D. Ding and L. Tian, "The study of solution of dissipative Camassa-Holm equation on total space," *International Journal of Nonlinear Science*, vol. 1, no. 1, pp. 37–42, 2006.

[31] E. A. Olson, "Well posedness for a higher order modified Camassa-Holm equation," *Journal of Differential Equations*, vol. 246, no. 10, pp. 4154–4172, 2009.

[32] J. Zhou and L. Tian, "Blow-up of solution of an initial boundary value problem for a generalized Camassa-Holm equation," *Physics Letters A*, vol. 372, no. 20, pp. 3659–3666, 2008.

[33] L. Tian, C. Shen, and D. Ding, "Optimal control of the viscous Camassa-Holm equation," *Nonlinear Analysis: Real World Applications*, vol. 10, no. 1, pp. 519–530, 2009.

[34] L. Tian, G. Gui, and Y. Liu, "On the well-posedness problem and the scattering problem for the Dullin-Gottwald-Holm equation," *Communications in Mathematical Physics*, vol. 257, no. 3, pp. 667–701, 2005.

[35] M. Chen, S.-Q. Liu, and Y. Zhang, "A two-component generalization of the Camassa-Holm equation and its solutions," *Letters in Mathematical Physics*, vol. 75, no. 1, pp. 1–15, 2006.

[36] A. Constantin and R. I. Ivanov, "On an integrable two-component Camassa-Holm shallow water system," *Physics Letters A*, vol. 372, no. 48, pp. 7129–7132, 2008.

[37] Z. Popowicz, "A two-component generalization of the Degasperis-Procesi equation," *Journal of Physics A*, vol. 39, no. 44, pp. 13717–13726, 2006.

[38] D. Henry, "Infinite propagation speed for a two component Camassa-Holm equation," *Discrete and Continuous Dynamical Systems. Series B*, vol. 12, no. 3, pp. 597–606, 2009.

[39] J. Escher, O. Lechtenfeld, and Z. Y. Yin, "Well-posedness and blow-up phenomena for the twocomponent Camassa-Holm equation," *Discrete and Continuous Dynamical Systems*, vol. 19, no. 3, pp. 493–513, 2007.

[40] C. Guan and Z. Yin, "Global existence and blow-up phenomena for an integrable two-component Camassa-Holm shallow water system," *Journal of Differential Equations*, vol. 248, no. 8, pp. 2003–2014, 2010.

[41] J. B. Li and Y. S. Li, "Bifurcations of travelling wave solutions for a two-component Camassa-Holm equation," *Acta Mathematica Sinica*, vol. 24, no. 8, pp. 1319–1330, 2008.

[42] Y. Wang, J. Huang, and L. Chen, "Global conservative solutions of the two-component Camassa-Holm shallow water system," *International Journal of Nonlinear Science*, vol. 9, pp. 379–384, 2010.

[43] H. Aratyn, J. F. Gomes, and A. H. Zimerman, "Deformations of $N = 2$ superconformal algebra and supersymmetric two-component Camassa-Holm equation," *Journal of Physics A*, vol. 40, no. 17, pp. 4511–4527, 2007.

[44] A. Constantin and B. Kolev, "Geodesic flow on the diffeomorphism group of the circle," *Commentarii Mathematici Helvetici*, vol. 78, no. 4, pp. 787–804, 2003.

[45] A. Constantin, "Existence of permanent and breaking waves for a shallow water equation: a geometric approach," *Annales de l'Institut Fourier*, vol. 50, no. 2, pp. 321–362, 2000.

[46] Y. Fu and C. Qu, "Well posedness and blow-up solution for a new coupled Camassa-Holm equations with peakons," *Journal of Mathematical Physics*, vol. 50, no. 1, Article ID 012906, 25 pages, 2009.

[47] Y. Li and X. Yang, "Existence and regularity of the global attractor for a weakly damped forced shallow water equation in $H^1(R)$," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 71, no. 11, pp. 5587–5598, 2009.

[48] L. Tian, R. Tian, and J. Fan, "The global attractor for the viscous weakly damped forced Korteweg-de Vries equations in $H^1(R)$," *International Journal of Nonlinear Science*, vol. 5, no. 1, pp. 3–10, 2008.

[49] Y. Zhang and C. Zhong, "Existence of global attractors for a nonlinear wave equation," *Applied Mathematics Letters*, vol. 18, no. 1, pp. 77–84, 2005.

[50] L. Tian and R. Tian, "The attractor for the two-dimensional weakly damped KdV equation in belt field," *Nonlinear Analysis: Real World Applications*, vol. 9, no. 3, pp. 912–919, 2008.

[51] Y. Xu and L. Tian, "Attractor for a coupled nonhomogeneous Camassa-Holm equation," *International Journal of Nonlinear Science*, vol. 9, pp. 112–117, 2010.

[52] L. Tian and J. Fan, "The attractor on viscosity Degasperis-Procesi equation," *Nonlinear Analysis: Real World Applications*, vol. 9, no. 4, pp. 1461–1473, 2007.

[53] L. Tian and Y. Gao, "The global attractor of the viscous Fornberg-Whitham equation," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 71, no. 11, pp. 5176–5186, 2009.

[54] J. Zhou and L. Tian, "A type of bounded traveling wave solutions for the Fornberg-Whitham equation," *Journal of Mathematical Analysis and Applications*, vol. 346, no. 1, pp. 255–261, 2008.

[55] B. L. Guo, *Infinite Dimensional Dynamical System*, Beijing, China, 1st edition, 2000.