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Research Article

Boundary Value Problems for Delay Differential Systems

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Conditions are derived of the existence of solutions of linear Fredholm's boundary-value problems for systems of ordinary differential equations with constant coefficients and a single delay, assuming that these solutions satisfy the initial and boundary conditions. Utilizing a delayed matrix exponential and a method of pseudoinverse by Moore-Penrose matrices led to an explicit and analytical form of a criterion for the existence of solutions in a relevant space and, moreover, to the construction of a family of linearly independent solutions of such problems in a general case with the number of boundary conditions (defined by a linear vector functional) not coinciding with the number of unknowns of a differential system with a single delay. As an example of application of the results derived, the problem of bifurcation of solutions of boundary-value problems for systems of ordinary differential equations with a small parameter and with a finite number of measurable delays of argument is considered.

1. Introduction

First we mention auxiliary results regarding the theory of differential equations with delay. Consider a system of linear differential equations with concentrated delay

$$\dot{z}(t) - A(t)z(h(t)) = g(t), \quad \text{if } t \in [a, b], \tag{1.1}$$

assuming that

$$z(s) := \psi(s), \quad \text{if } s \notin [a,b], \tag{1.2}$$

where A is an $n \times n$ real matrix, and g is an n-dimensional real column vector, with components in the space $L_p[a,b]$ (where $p \in [1,\infty)$) of functions integrable on $[a,b]$ with the degree p; the delay $h(t) \leq t$ is a function $h : [a,b] \to \mathbb{R}$ measurable on $[a,b]$; $\psi : \mathbb{R} \setminus [a,b] \to \mathbb{R}^n$ is a given vector function with components in $L_p[a,b]$. Using the denotations

$$(S_h z)(t) := \begin{cases} z(h(t)), & \text{if } h(t) \in [a,b], \\ \theta, & \text{if } h(t) \notin [a,b], \end{cases} \tag{1.3}$$

$$\psi^h(t) := \begin{cases} \theta, & \text{if } h(t) \in [a,b], \\ \psi(h(t)), & \text{if } h(t) \notin [a,b], \end{cases} \tag{1.4}$$

where θ is an n-dimensional zero column vector, and assuming $t \in [a,b]$, it is possible to rewrite (1.1), (1.2) as

$$(Lz)(t) := \dot{z}(t) - A(t)(S_h z)(t) = \varphi(t), \quad t \in [a,b], \tag{1.5}$$

where φ is an n-dimensional column vector defined by the formula

$$\varphi(t) := g(t) + A(t)\psi^h(t) \in L_p[a,b]. \tag{1.6}$$

We will investigate (1.5) assuming that the operator L maps a Banach space $D_p[a,b]$ of absolutely continuous functions $z : [a,b] \to \mathbb{R}^n$ into a Banach space $L_p[a,b]$ $(1 \leq p < \infty)$ of function $\varphi : [a,b] \to \mathbb{R}^n$ integrable on $[a,b]$ with the degree p ; the operator S_h maps the space $D_p[a,b]$ into the space $L_p[a,b]$. Transformations of (1.3), (1.4) make it possible to add the initial vector function $\psi(s)$, $s < a$ to nonhomogeneity, thus generating an additive and homogeneous operation not depending on ψ, and without the classical assumption regarding the continuous connection of solution $z(t)$ with the initial function $\psi(t)$ at $t = a$.

A solution of differential system (1.5) is defined as an n-dimensional column vector function $z \in D_p[a,b]$, absolutely continuous on $[a,b]$ with a derivative $\dot{z}$ in a Banach space $L_p[a,b]$ $(1 \leq p < \infty)$ of functions integrable on $[a,b]$ with the degree p, satisfying (1.5) almost everywhere on $[a,b]$. Throughout this paper we understand the notion of a solution of a differential system and the corresponding boundary value problem in the sense of the above definition.

Such treatment makes it possible to apply the well-developed methods of linear functional analysis to (1.5) with a linear and bounded operator L. It is well known (see, e.g., [1–4]) that a nonhomogeneous operator equation (1.5) with delayed argument is solvable in the space $D_p[a,b]$ for an arbitrary right-hand side $\varphi \in L_p[a,b]$ and has an n-dimensional family of solutions ($\dim \ker L = n$) in the form

$$z(t) = X(t)c + \int_a^b K(t,s)\varphi(s)ds, \quad \forall c \in \mathbb{R}^n, \tag{1.7}$$

where the kernel $K(t,s)$ is an $n \times n$ Cauchy matrix defined in the square $[a,b] \times [a,b]$ which is, for every $s \le t$, a solution of the matrix Cauchy problem:

$$(LK(\cdot,s))(t) := \frac{\partial K(t,s)}{\partial t} - A(t)(S_h K(\cdot,s))(t) = \Theta, \qquad K(s,s) = I, \tag{1.8}$$

where $K(t,s) \equiv \Theta$ if $a \le t < s \le b$, and Θ is the $n \times n$ null matrix. A fundamental $n \times n$ matrix $X(t)$ for the homogeneous ($\varphi \equiv \theta$) (1.5) has the form $X(t) = K(t,a)$, $X(a) = I$.

A serious disadvantage of this approach, when investigating the above-formulated problem, is the necessity to find the Cauchy matrix $K(t,s)$ [5, 6]. It exists but, as a rule, can only be found numerically. Therefore, it is important to find systems of differential equations with delay such that this problem can be solved directly. Below, we consider the case of a system with what is called a single delay [7]. In this case, the problem of how to construct the Cauchy matrix is solved *analytically* thanks to a delayed matrix exponential, as defined below.

2. A Delayed Matrix Exponential

Consider a Cauchy problem for a linear nonhomogeneous differential system with constant coefficients and with a single delay τ

$$\dot{z}(t) = Az(t-\tau) + g(t), \tag{2.1}$$

$$z(s) = \psi(s), \quad \text{if } s \in [-\tau, 0] \tag{2.2}$$

with $n \times n$ constant matrix A, $g : [0,\infty) \to \mathbb{R}^n$, $\psi : [-\tau,0] \to \mathbb{R}^n$, $\tau > 0$ and an unknown vector solution $z : [-\tau,\infty) \to \mathbb{R}^n$. Together with a nonhomogeneous problem (2.1), (2.2), we consider a related homogeneous problem

$$\dot{z}(t) = Az(t-\tau), \tag{2.3}$$

$$z(s) = \psi(s), \quad \text{if } s \in [-\tau, 0]. \tag{2.4}$$

Denote by e_τ^{At} a matrix function called a delayed matrix exponential (see [7]) and defined as

$$e_\tau^{At} := \begin{cases} \Theta, & \text{if } -\infty < t < -\tau, \\ I, & \text{if } -\tau \le t < 0, \\ I + A\dfrac{t}{1!}, & \text{if } 0 \le t < \tau, \\ I + A\dfrac{t}{1!} + A^2\dfrac{(t-\tau)^2}{2!}, & \text{if } \tau \le t < 2\tau, \\ \cdots \\ I + A\dfrac{t}{1!} + \cdots + A^k\dfrac{(t-(k-1)\tau)^k}{k!}, & \text{if } (k-1)\tau \le t < k\tau, \\ \cdots . \end{cases} \tag{2.5}$$

This definition can be reduced to the following expression:

$$e_\tau^{At} = \sum_{n=0}^{[t/\tau]+1} A^n \frac{(t-(n-1)\tau)^n}{n!}, \tag{2.6}$$

where $[t/\tau]$ is the greatest integer function. The delayed matrix exponential equals a unit matrix I on $[-\tau, 0]$ and represents a fundamental matrix of a homogeneous system with a single delay.

We mention some of the properties of e_τ^{At} given in [7]. Regarding the system without delay ($\tau = 0$), the delayed matrix exponential does not have the form of a matrix series, but it is a matrix polynomial, depending on the time interval in which it is considered. It is easy to prove directly that the delayed matrix exponential $X(t) := e_\tau^{A(t-\tau)}$ satisfies the relations

$$\dot{X}(t) = AX(t-\tau), \quad \text{for } t \ge 0, \qquad X(s) = 0, \quad \text{for } s \in [\tau, 0), \qquad X(0) = I. \tag{2.7}$$

By integrating the delayed matrix exponential, we get

$$\int_0^t e_\tau^{As} ds = I\frac{t}{1!} + A\frac{(t-\tau)^2}{2!} + \cdots + A^k\frac{(t-(k-1)\tau)^{k+1}}{(k+1)!}, \tag{2.8}$$

where $k = [t/\tau] + 1$. If, moreover, the matrix A is regular, then

$$\int_0^t e_\tau^{As} ds = A^{-1} \cdot \left(e_\tau^{A(t-\tau)} - e_\tau^{A\tau} \right). \tag{2.9}$$

Delayed matrix exponential e_τ^{At}, $t > 0$ is an infinitely many times continuously differentiable function except for the nodes $k\tau$, $k = 0, 1, \ldots$ where there is a discontinuity of the derivative of order $(k+1)$:

$$\lim_{t \to k\tau - 0} \left(e_\tau^{At}\right)^{(k+1)} = 0, \qquad \lim_{t \to k\tau + 0} \left(e_\tau^{At}\right)^{(k+1)} = A^{k+1}. \tag{2.10}$$

The following results (proved in [7] and being a consequence of (1.7) with $K(t,s) = e_\tau^{A(t-\tau-s)}$ as well) hold.

Theorem 2.1. (A) *The solution of a homogeneous system* (2.3) *with a single delay satisfying the initial condition* (2.4) *where* $\psi(s)$ *is an arbitrary continuously differentiable vector function can be represented in the form*

$$z(t) = e_\tau^{At}\psi(-\tau) + \int_{-\tau}^{0} e_\tau^{A(t-\tau-s)}\psi'(s)ds. \tag{2.11}$$

(B) *A particular solution of a nonhomogeneous system* (2.1) *with a single delay satisfying the zero initial condition* $z(s) = 0$ *if* $s \in [-\tau, 0]$ *can be represented in the form*

$$z(t) = \int_0^t e_\tau^{A(t-\tau-s)} g(s)ds. \tag{2.12}$$

(C) *A solution of a Cauchy problem of a nonhomogeneous system with a single delay* (2.1) *satisfying a constant initial condition*

$$z(s) = \psi(s) = c \in \mathbb{R}^n, \quad \text{if } s \in [-\tau, 0] \tag{2.13}$$

has the form

$$z(t) = e_\tau^{A(t-\tau)}c + \int_0^t e_\tau^{A(t-\tau-s)} g(s)ds. \tag{2.14}$$

3. Main Results

Without loss of generality, let $a = 0$. The problem (2.1), (2.2) can be transformed ($h(t) := t - \tau$) to an equation of type (1.1) (see (1.5)):

$$\dot{z}(t) - A(S_h z)(t) = \varphi(t), \quad t \in [0, b], \tag{3.1}$$

where, in accordance with (1.3), (1.4),

$$(S_h z)(t) = \begin{cases} z(t-\tau), & \text{if } t-\tau \in [0,b], \\ \theta, & \text{if } t-\tau \notin [0,b], \end{cases}$$
$$\varphi(t) = \; g(t) + A\ \varphi^h(t) \in L_p[0,b], \tag{3.2}$$
$$\varphi^h(t) = \begin{cases} \theta, & \text{if } t-\tau \in [0,b], \\ \varphi(t-\tau), & \text{if } t-\tau \notin [0,b]. \end{cases}$$

A general solution of a Cauchy problem for a nonhomogeneous system (3.1) with a single delay satisfying a constant initial condition

$$z(s) = \varphi(s) = c \in \mathbb{R}^n, \quad \text{if } s \in [-\tau, 0] \tag{3.3}$$

has the form (1.7):

$$z(t) = X(t)c + \int_0^b K(t,s)\varphi(s)ds, \quad \forall c \in \mathbb{R}^n, \tag{3.4}$$

where, as can easily be verified (in view of the above-defined delayed matrix exponential) by substituting into (3.1),

$$X(t) = \mathrm{e}_\tau^{A(t-\tau)}, \qquad X(0) = e_\tau^{-A\tau} = I \tag{3.5}$$

is a normal fundamental matrix of the homogeneous system related to (3.1) (or (2.1)) with the initial data $X(0) = I$, and the Cauchy matrix $K(t,s)$ has the form

$$\begin{gathered} K(t,s) = e_\tau^{A(t-\tau-s)}, \quad \text{if } 0 \le s < t \le b, \\ K(t,s) \equiv \Theta, \quad \text{if } 0 \le t < s \le b. \end{gathered} \tag{3.6}$$

Obviously,

$$K(t,0) = e_\tau^{A(t-\tau)} = X(t), \qquad K(0,0) = e_\tau^{A(-\tau)} = X(0) = I, \tag{3.7}$$

and, therefore, the initial problem (3.1) for systems of ordinary differential equations with constant coefficients and a single delay, satisfying a constant initial condition, has an n-parametric family of linearly independent solutions

$$z(t) = e_\tau^{A(t-\tau)}c + \int_0^t e_\tau^{A(t-\tau-s)}\varphi(s)ds, \quad \forall c \in \mathbb{R}^n. \tag{3.8}$$

Now we will consider a general Fredholm boundary value problem for system (3.1).

3.1. Fredholm Boundary Value Problem

Using the results in [8, 9], it is easy to derive statements for a general boundary value problem if the number m of boundary conditions does not coincide with the number n of unknowns in a differential system with a single delay.

We consider a boundary value problem

$$\begin{gathered}\dot{z}(t) - Az(t-\tau) = g(t), \quad \text{if } t \in [0,b], \\ z(s) := \psi(s), \quad \text{if } s \notin [0,b],\end{gathered} \tag{3.9}$$

assuming that

$$\ell z = \ \alpha \in \mathbb{R}^m \tag{3.10}$$

or, using (3.2), in an equivalent form

$$\dot{z}(t) - A(S_h z)(t) = \ \varphi(t), \quad t \in [0,b], \tag{3.11}$$

$$\ell z = \ \alpha \in \mathbb{R}^m, \tag{3.12}$$

where α is an m-dimensional constant vector column, and $\ell : D_p[0,b] \ \rightarrow \mathbb{R}^m$ is a linear vector functional. It is well known that, for functional differential equations, such problems are of Fredholm's type (see, e.g., [1, 9]). We will derive the necessary and sufficient conditions and a representation (in an *explicit analytical* form) of the solutions $z \in D_p[0,b]$, $\dot{z} \in L_p[0,b]$ of the boundary value problem (3.11), (3.12).

We recall that, because of properties (3.6)–(3.7), a general solution of system (3.11) has the form

$$z(t) = e_\tau^{A(t-\tau)}c + \int_0^b K(t,s)\varphi(s)ds, \quad \forall c \in \mathbb{R}^n. \tag{3.13}$$

In the algebraic system

$$Qc = \alpha - \ell \int_0^b K(\cdot,s)\varphi(\mathrm{s})ds, \tag{3.14}$$

derived by substituting (3.13) into boundary condition (3.12); the constant matrix

$$Q := \ell X(\cdot) = \ell e_\tau^{A(\cdot-\tau)} \tag{3.15}$$

has a size of $m \times n$. Denote

$$\operatorname{rank} Q = n_1, \tag{3.16}$$

where, obviously, $n_1 \le \min(m, n)$. Adopting the well-known notation (e.g., [9]), we define an $n \times n$-dimensional matrix

$$P_Q := I - Q^+ Q \tag{3.17}$$

which is an orthogonal projection projecting space $\mathbb{R}^n$ to $\ker Q$ of the matrix Q where I is an $n \times n$ identity matrix and an $m \times m$-dimensional matrix

$$P_{Q^*} := I_m - QQ^+ \tag{3.18}$$

which is an orthogonal projection projecting space $\mathbb{R}^m$ to $\ker Q^*$ of the transposed matrix $Q^* = Q^T$ where I_m is an $m \times m$ identity matrix and Q^+ is an $n \times m$-dimensional matrix pseudoinverse to the $m \times n$-dimensional matrix Q. Using the property

$$\operatorname{rank} P_{Q^*} = m - \operatorname{rank} Q^* = d := m - n_1, \tag{3.19}$$

where $\operatorname{rank} Q^* = \operatorname{rank} Q = n_1$, we will denote by $P_{Q_d^*}$ a $d \times m$-dimensional matrix constructed from d linearly independent rows of the matrix P_{Q^*}. Moreover, taking into account the property

$$\operatorname{rank} P_Q = n - \operatorname{rank} Q = r = n - n_1, \tag{3.20}$$

we will denote by P_{Q_r} an $n \times r$-dimensional matrix constructed from r linearly independent columns of the matrix P_Q.

Then (see [9, page 79, formulas (3.43), (3.44)]) the condition

$$P_{Q_d^*}\left(\alpha - \ell \int_0^b K(\cdot, s)\varphi(s)\,ds\right) = \theta_d \tag{3.21}$$

is necessary and sufficient for algebraic system (3.14) to be solvable where θ_d is (throughout the paper) a d-dimensional column zero vector. If such condition is true, system (3.14) has a solution

$$c = P_{Q_r} c_r + Q^+\left(\alpha - \ell \int_0^b K(\cdot, s)\varphi(s)\,ds\right), \quad \forall c_r \in \mathbb{R}^r. \tag{3.22}$$

Substituting the constant $c \in \mathbb{R}^n$ defined by (3.22) into (3.13), we get a formula for a general solution of problem (3.11), (3.12):

$$z(t) = z(t, c_r) := X(t) P_{Q_r} c_r + (G\varphi)(t) + X(t) Q^+ \alpha, \quad \forall c_r \in \mathbb{R}^r, \tag{3.23}$$

where $(G\varphi)(t)$ is a generalized Green operator. If the vector functional ℓ satisfies the relation [9, page 176]

$$\ell \int_0^b K(\cdot, s)\varphi(s)ds = \int_0^b \ell K(\cdot, s)\varphi(s)ds, \tag{3.24}$$

which is assumed throughout the rest of the paper, then the generalized Green operator takes the form

$$(G\varphi)(t) := \int_0^b G(t,s)\varphi(s)ds, \tag{3.25}$$

where

$$G(t,s) := K(t,s) - e_\tau^{A(t-\tau)}Q^+\ell K(\cdot, s) \tag{3.26}$$

is a generalized Green matrix, corresponding to the boundary value problem (3.11), (3.12), and the Cauchy matrix $K(t,s)$ has the form of (3.6). Therefore, the following theorem holds (see [10]).

Theorem 3.1. *Let Q be defined by* (3.15) *and* $\operatorname{rank} Q = n_1$. *Then the homogeneous problem*

$$\begin{aligned} \dot{z}(t) - A(S_h z)(t) &= \theta, \quad t \in [0,b], \\ \ell z &= \theta_m \in \mathbb{R}^m \end{aligned} \tag{3.27}$$

corresponding to the problem (3.11), (3.12) *has exactly $r = n - n_1$ linearly independent solutions*

$$z(t, c_r) = X(t)P_{Q_r}c_r = e_\tau^{A(t-\tau)}P_{Q_r}c_r, \quad \forall c_r \in \mathbb{R}^r. \tag{3.28}$$

Nonhomogeneous problem (3.11), (3.12) *is solvable if and only if $\varphi \in L_p[0,b]$ and $\alpha \in \mathbb{R}^m$ satisfy d linearly independent conditions* (3.21). *In that case, this problem has an r-dimensional family of linearly independent solutions represented in an explicit analytical form* (3.23).

The case of $\operatorname{rank} Q = n$ implies the inequality $m \geq n$. If $m > n$, the boundary value problem is overdetermined, the number of boundary conditions is more than the number of unknowns, and Theorem 3.1 has the following corollary.

Corollary 3.2. *If* $\operatorname{rank} Q = n$, *then the homogeneous problem* (3.27) *has only the trivial solution. Nonhomogeneous problem* (3.11), (3.12) *is solvable if and only if $\varphi \in L_p[0,b]$ and $\alpha \in \mathbb{R}^m$ satisfy d linearly independent conditions* (3.21) *where $d = m - n$. Then the unique solution can be represented as*

$$z(t) = (G\varphi)(t) + X(t)Q^+\alpha. \tag{3.29}$$

The case of rank $Q = m$ is interesting as well. Then the inequality $m \leq n$, holds. If $m < n$ the boundary value problem is not fully defined. In this case, Theorem 3.1 has the following corollary.

Corollary 3.3. *If* rank $Q = m$, *then the homogeneous problem* (3.27) *has an r-dimensional* ($r = n - m$) *family of linearly independent solutions*

$$z(t, c_r) = X(t) P_{Q_r} c_r = e_\tau^{A(t-\tau)} P_{Q_r} c_r, \quad \forall c_r \in \mathbb{R}^r. \tag{3.30}$$

Nonhomogeneous problem (3.11), (3.12) *is solvable for arbitrary $\varphi \in L_p[0, b]$ and $\alpha \in R^m$ and has an r-parametric family of solutions*

$$z(t, c_r) = X(t) P_{Q_r} c_r + \left(G\varphi\right)(t) + X(t) Q^+ \alpha, \quad \forall c_r \in \mathbb{R}^r. \tag{3.31}$$

Finally, combining both particular cases mentioned in Corollaries 3.2 and 3.3, we get a noncritical case.

Corollary 3.4. *If* rank $Q = m = n$ *(i.e., $Q^+ = Q^{-1}$), then the homogeneous problem* (3.27) *has only the trivial solution. The nonhomogeneous problem* (3.11), (3.12) *is solvable for arbitrary $\varphi \in L_p[0, b]$ and $\alpha \in R^n$ and has a unique solution*

$$z(t) = \left(G\varphi\right)(t) + X(t) Q^{-1} \alpha, \tag{3.32}$$

where

$$\left(G\varphi\right)(t) := \int_0^b G(t, s) \varphi(s) ds \tag{3.33}$$

is a Green operator, and

$$G(t, s) := K(t, s) - e_\tau^{A(t-\tau)} Q^{-1} \ell K(\cdot, s) \tag{3.34}$$

is a related Green matrix, corresponding to the problem (3.11), (3.12).

4. Perturbed Boundary Value Problems

As an example of application of Theorem 3.1, we consider the problem of bifurcation from point $\varepsilon = 0$ of solutions $z : [0, b] \to \mathbb{R}^n$, $b > 0$ satisfying, for a.e. $t \in [0, b]$, systems of ordinary differential equations

$$\dot{z}(t) = Az(h_0(t)) + \varepsilon \sum_{i=1}^{k} B_i(t) z(h_i(t)) + g(t), \tag{4.1}$$

where A is $n \times n$ constant matrix, $B(t) = (B_1(t), \ldots, B_k(t))$ is an $n \times N$ matrix, $N = nk$, consisting of $n \times n$ matrices $B_i : [0, b] \to \mathbb{R}^{n \times n}$, $i = 1, 2, \ldots, k$, having entries in $L_p[0, b]$,

ε is a small parameter, delays $h_i : [0,b] \to \mathbb{R}$ are measurable on $[0,b]$, $h_i(t) \le t$, $t \in [0,b]$, $i = 0,1,\dots,k$, $g : [0,b] \to \mathbb{R}$, $g \in L_p[0,b]$, and satisfying the initial and boundary conditions

$$z(s) = \psi(s), \quad \text{if } s < 0, \;\; \ell z = \alpha, \tag{4.2}$$

where $\alpha \in \mathbb{R}^m$, $\psi : \mathbb{R} \setminus [0,b] \to \mathbb{R}^n$ is a given vector function with components in $L_p[a,b]$, and $\ell : D_p[0,b] \to \mathbb{R}^m$ is a linear vector functional. Using denotations (1.3), (1.4), and (1.6), it is easy to show that the perturbed nonhomogeneous linear boundary value problem (4.1), (4.2) can be rewritten as

$$\dot{z}(t) = A(S_{h_0}z)(t) + \varepsilon B(t)(S_h z)(t) + \varphi(t,\varepsilon), \quad \ell z = \alpha. \tag{4.3}$$

In (4.3) we specify $h_0 : [0,b] \to \mathbb{R}$ as a single delay defined by formula $h_0(t) := t - \tau$ $(\tau > 0)$;

$$(S_h z)(t) = \operatorname{col}[(S_{h_1}z)(t), \dots, (S_{h_k}z)(t)] \tag{4.4}$$

is an N-dimensional column vector, and $\varphi(t,\varepsilon)$ is an n-dimensional column vector given by

$$\varphi(t,\varepsilon) = g(t) + A\ \psi^{h_0}(t) + \varepsilon \sum_{i=1}^{k} B_i(t)\psi^{h_i}(t). \tag{4.5}$$

It is easy to see that $\varphi \in L_p[0,b]$. The operator S_h maps the space D_p into the space

$$L_p^N = \underbrace{L_p \times \cdots \times L_p}_{k \text{ times}}, \tag{4.6}$$

that is, $S_h : D_p \to L_p^N$. Using denotation (1.3) for the operator $S_{h_i} : D_p \to L_p$, we have the following representation:

$$(S_{h_i}z)(t) = \int_0^b \chi_{h_i}(t,s)\dot{z}(s)ds + \chi_{h_i}(t,0)z(0), \tag{4.7}$$

where

$$\chi_{h_i}(t,s) = \begin{cases} 1, & \text{if } (t,s) \in \Omega_i, \\ 0, & \text{if } (t,s) \notin \Omega_i \end{cases} \tag{4.8}$$

is the characteristic function of the set

$$\Omega_i := \{(t,s) \in [0,b] \times [0,b] : 0 \le s \le h_i(t) \le b\}, \quad i = 1,2,\dots,k. \tag{4.9}$$

Assume that nonhomogeneities $\varphi(t,0) \in L_p[0,b]$ and $\alpha \in \mathbb{R}^m$ are such that the shortened boundary value problem

$$\dot{z}(t) = A(S_{h_0}z)(t) + \varphi(t,0), \quad lz = \alpha, \tag{4.10}$$

being a particular case of (4.3) for $\varepsilon = 0$, does not have a solution. In such a case, according to Theorem 3.1, the solvability criterion (3.21) does not hold for problem (4.10). Thus, we arrive at the following question.

Is it possible to make the problem (4.10) *solvable by means of linear perturbations and, if this is possible, then of what kind should the perturbations B_i and the delays h_i, $i = 1,2,\ldots,k$ be for the boundary value problem* (4.3) *to be solvable?*

We can answer this question with the help of the $d \times r$-matrix

$$B_0 := \int_0^b H(s)B(s)\left(S_h X P_{Q_r}\right)(s)ds = \int_0^b H(s)\sum_{i=1}^k B_i(s)\left(S_{h_i} X P_{Q_r}\right)(s)ds, \tag{4.11}$$

where

$$H(s) := P_{Q_d^*}\ell K(\cdot,s) = P_{Q_d^*}\ell e_\tau^{A(\cdot-\tau-s)}, \qquad X(t) := e_\tau^{A(t-\tau)}, \qquad Q := \ell X = \ell e_\tau^{A(\cdot-\tau)}, \tag{4.12}$$

constructed by using the coefficients of the problem (4.3).

Using the Vishik and Lyusternik method [11] and the theory of generalized inverse operators [9], we can find bifurcation conditions. Below we formulate a statement (proved using [8] and [9, page 177]) which partially answers the above problem. Unlike an earlier result [9], this one is derived in an *explicit analytical* form. We remind that the notion of a solution of a boundary value problem was specified in part 1.

Theorem 4.1. *Consider system*

$$\dot{z}(t) = Az(t-\tau) + \varepsilon\sum_{i=1}^k B_i(t)z(h_i(t)) + g(t), \tag{4.13}$$

where A is $n \times n$ constant matrix, $B(t) = (B_1(t),\ldots,B_k(t))$ is an $n \times N$ matrix, $N = nk$, consisting of $n \times n$ matrices $B_i : [0,b] \to \mathbb{R}^{n\times n}$, $i = 1,2,\ldots,k$, having entries in $L_p[0,b]$, ε is a small parameter, delays $h_i : [0,b] \to \mathbb{R}$ are measurable on $[0,b]$, $h_i(t) \le t$, $t \in [0,b]$, $g : [0,b] \to \mathbb{R}$, $g \in L_p[0,b]$, with the initial and boundary conditions

$$z(s) = \psi(s), \quad \text{if } s < 0, \ \ell z = \alpha, \tag{4.14}$$

where $\alpha \in \mathbb{R}^m$, $\psi : \mathbb{R} \setminus [0,b] \to \mathbb{R}^n$ is a given vector function with components in $L_p[a,b]$, and $\ell : D_p[0,b] \to \mathbb{R}^m$ is a linear vector functional, and assume that

$$\varphi(t,0) = g(t) + A\psi^{h_0}(t), \quad h_0(t) := t - \tau \tag{4.15}$$

(satisfying $\varphi \in L_p[0,b]$*) and* α *are such that the shortened problem*

$$\dot{z}(t) = A(S_{h_0}z)(t) + \varphi(t,0), \quad \ell z = \alpha \tag{4.16}$$

does not have a solution. If

$$\operatorname{rank} B_0 = d \quad \text{or} \quad P_{B_0^*} := I_d - B_0 B_0^+ = 0, \tag{4.17}$$

then the boundary value problem (4.13), (4.14) *has a set of* $\rho := n - m$ *linearly independent solutions in the form of the series*

$$\begin{gathered} z(t,\varepsilon) = \sum_{i=-1}^{\infty} \varepsilon^i z_i(t, c_\rho), \\ z(\cdot,\varepsilon) \in D_p[0,b], \qquad \dot{z}(\cdot,\varepsilon) \in L_p[0,b], \qquad z(t,\cdot) \in C(0,\varepsilon_*], \end{gathered} \tag{4.18}$$

converging for fixed $\varepsilon \in (0,\varepsilon_*]$*, where* ε_* *is an appropriate constant characterizing the domain of the convergence of the series* (4.18)*, and* $z_i(t,c_\rho)$ *are suitable coefficients.*

Remark 4.2. Coefficients $z_i(t,c_\rho)$, $i = -1,\dots,\infty$, in (4.18) can be determined. The procedure describing the method of their deriving is a crucial part of the proof of Theorem 4.1 where we give their form as well.

Proof. Substitute (4.18) into (4.3) and equate the terms that are multiplied by the same powers of ε. For ε^{-1}, we obtain the homogeneous boundary value problem

$$\dot{z}_{-1}(t) = A(S_{h_0}z_{-1})(t), \quad \ell z_{-1} = 0, \tag{4.19}$$

which determines $z_{-1}(t)$.

By Theorem 3.1, the homogeneous boundary value problem (4.19) has an r-parametric ($r = n - n_1$) family of solutions $z_{-1}(t) : z_{-1}(t,c_{-1}) = X(t)P_{Q_r}(t)c_{-1}$ where the r-dimensional column vector $c_{-1} \in \mathbb{R}^r$ can be determined from the solvability condition of the problem for $z_0(t)$.

For ε^0, we get the boundary value problem

$$\dot{z}_0(t) = A(S_{h_0}z_0)(t) + B(t)(S_h z_{-1})(t) + \varphi(t,0), \quad \ell z_0 = \alpha, \tag{4.20}$$

which determines $z_0(t) := z_0(t,c_0)$.

It follows from Theorem 3.1 that the solvability criterion (3.21) for problem (4.20) has the form

$$P_{Q_d^*}\alpha - \int_0^b H(s)\left(\varphi(s,0) + B(s)(S_h X P_{Q_r})(s)c_{-1}\right)ds = 0, \tag{4.21}$$

from which we receive, with respect to $c_{-1} \in \mathbb{R}^r$, an algebraic system

$$B_0 c_{-1} = P_{Q_d^*}\alpha - \int_0^b H(s)\varphi(s,0)ds. \tag{4.22}$$

The right-hand side of (4.22) is nonzero only in the case that the shortened problem does not have a solution. The system (4.22) is solvable for arbitrary $\varphi(t,0) \in L_p[0,b]$ and $\alpha \in \mathbb{R}^m$ if the condition (4.17) is satisfied [9, page 79]. In this case, system (4.22) becomes resolvable with respect to $c_{-1} \in \mathbb{R}^r$ up to an arbitrary constant vector $P_{B_0}c \in \mathbb{R}^r$ from the null-space of matrix B_0 and

$$c_{-1} = -B_0^+\left(P_{Q_d^*}\alpha - \int_0^b H(s)\varphi(s,0)ds\right) + P_{B_0}c \quad \left(P_{B_0} = I_r - B_0^+B_0\right). \tag{4.23}$$

This solution can be rewritten in the form

$$c_{-1} = \overline{c}_{-1} + P_{B_\rho}c_\rho, \quad \forall c_\rho \in \mathbb{R}^\rho, \tag{4.24}$$

where

$$\overline{c}_{-1} = -B_0^+\left(P_{Q_d^*}\alpha - \int_0^b H(s)\varphi(s,0)ds\right), \tag{4.25}$$

and P_{B_ρ} is an $r \times \rho$-dimensional matrix whose columns are a complete set of ρ linearly independent columns of the $r \times r$-dimensional matrix P_{B_0} with

$$\rho := \operatorname{rank} P_{B_0} = r - \operatorname{rank} B_0 = r - d = n - m. \tag{4.26}$$

So, for the solutions of the problem (3.14), we have the following formulas:

$$\begin{gathered} z_{-1}(t, c_\rho) = \overline{z}_{-1}(t, \overline{c}_{-1}) + X(t)P_{Q_r}P_{B_\rho}c_\rho, \quad \forall c_\rho \in \mathbb{R}^\rho, \\ \overline{z}_{-1}(t, \overline{c}_{-1}) = X(t)P_{Q_r}\overline{c}_{-1}. \end{gathered} \tag{4.27}$$

Assuming that (3.24) and (4.17) hold, the boundary value problem (4.20) has the r-parametric family of solutions

$$\begin{aligned} z_0(t, c_0) &= X(t)P_{Q_r}c_0 + X(t)Q^+\alpha \\ &\quad + \int_0^b G(t,s)\left[\varphi(s,0) + B(s)S_h\left(\overline{z}_{-1}(\cdot, \overline{c}_{-1}) + X(\cdot)P_{Q_r}P_{B_\rho}c_\rho\right)(s)\right]ds. \end{aligned} \tag{4.28}$$

Here, c_0 is an r-dimensional constant vector, which is determined at the next step from the solvability condition of the boundary value problem for $z_1(t)$.

For ε^1, we get the boundary value problem

$$\dot{z}_1(t) = A(S_{h_0} z_1)(t) + B(t)(S_h z_0)(t) + \sum_{i=1}^{k} B_i(t)\varphi^{h_i}(t), \quad \ell z_1 = 0, \tag{4.29}$$

which determines $z_1(t) := z_1(t, c_1)$. The solvability criterion for the problem (4.29) has the form (in computations below we need a composition of operators and the order of operations is following the inner operator S_h which acts to matrices and vector function having an argument denoted by $'' \cdot ''$ and the outer operator S_h which acts to matrices having an argument denoted by $'' \star ''$)

$$\begin{aligned}
&\int_0^b H(s)\sum_{i=1}^{k} B_i(s)\varphi^{h_i}(s)ds \\
&\quad + \int_0^b H(s)B(s)S_h \\
&\quad \times \Bigg(X(\star)P_{Q_r}c_0 + X(\star)Q^{+}\alpha \\
&\qquad + \int_0^b G(\star, s_1)\Big[\varphi(s_1, 0) + B(s)S_h\Big(\overline{z}_{-1}(\cdot, \overline{c}_{-1}) + X(\cdot)P_{Q_r}P_{B_\rho}c_\rho\Big)(s_1)\Big]ds_1\Bigg)(s)ds = 0
\end{aligned} \tag{4.30}$$

or, equivalently, the form

$$\begin{aligned}
B_0 c_0 = &-\int_0^b H(s)\sum_{i=1}^{k} B_i(s)\varphi^{h_i}(s)ds \\
&- \int_0^b H(s)B(s)S_h \\
&\quad \times \Bigg(X(\star)Q^{+}\alpha \\
&\qquad + \int_0^b G(\star, s_1)\Big[\varphi(s_1, 0) + B(s_1)S_h\Big(\overline{z}_{-1}(\cdot, \overline{c}_{-1}) + X(\cdot)P_{Q_r}P_{B_\rho}c_\rho\Big)(s_1)\Big]ds_1\Bigg)(s)ds.
\end{aligned} \tag{4.31}$$

Assuming that (4.17) holds, the algebraic system (4.31) has the following family of solutions:

$$c_0 = \overline{c}_0 + \left[I_r - B_0^{+}\int_0^b H(s)B(s)S_h\left(\int_0^b G(\star, s_1)B(s_1)\left(S_h X(\cdot)P_{Q_r}\right)(s_1)ds_1\right)(s)ds\right]P_{B_\rho}c_\rho, \tag{4.32}$$

where

$$\overline{c}_0 = -B_0^+ \int_0^b H(s) \sum_{i=1}^{k} B_i(s) \varphi^{h_i}(s) ds$$
$$- B_0^+ \int_0^b H(s) B(s) S_h \tag{4.33}$$
$$\times \left(X(\star) Q^+ \alpha + \int_0^b G(\star, s_1) \left[\varphi(s_1, 0) + B(s_1) (S_h \overline{z}_{-1}(\cdot, \overline{c}_{-1}))(s_1) \right] ds_1 \right)(s) ds.$$

So, for the ρ-parametric family of solutions of the problem (4.20), we have the following formula:

$$z_0(t, c_\rho) = \overline{z}_0(t, \overline{c}_0) + \overline{X}_0(t) P_{B_\rho} c_\rho, \quad \forall c_\rho \in \mathbb{R}^\rho, \tag{4.34}$$

where

$$\overline{z}_0(t, \overline{c}_0) = X(t) P_{Q_r} \overline{c}_0 + X(t) Q^+ \alpha + \int_0^b G(t, s) \left[\varphi(s, 0) + B(s) (S_h \overline{z}_{-1}(\cdot, \overline{c}_{-1}))(s) \right] ds,$$
$$\overline{X}_0(t) = X(t) P_{Q_r} \left[I_r - B_0^+ \int_0^b H(s) B(s) S_h \left(\int_0^b G(\star, s_1) B(s_1) (S_h X(\cdot) P_{Q_r})(s_1) ds_1 \right)(s) ds \right]$$
$$+ \int_0^b G(t, s) B(s) (S_h X(\cdot) P_{Q_r})(s) ds. \tag{4.35}$$

Again, assuming that (4.17) holds, the boundary value problem (4.29) has the r-parametric family of solutions

$$z_1(t, c_1) = X(t) P_{Q_r} c_1 + \int_0^b G(t, s) B(s) S_h \left(\overline{z}_0(\cdot, \overline{c}_0) + \overline{X}_0(\cdot) P_{B_\rho} c_\rho \right)(s) ds. \tag{4.36}$$

Here, c_1 is an r-dimensional constant vector, which is determined at the next step from the solvability condition of the boundary value problem for $z_2(t)$:

$$\dot{z}_2(t) = A(S_{h_0} z_2)(t) + B(t)(S_h z_1)(t), \quad \ell z_2 = 0. \tag{4.37}$$

The solvability criterion for the problem (4.37) has the form

$$\int_0^b H(s) B(s) S_h \left(X(\star) P_{Q_r} c_1 + \int_0^b G(\star, s_1) B(s_1) S_h \left(\overline{z}_0(\cdot, \overline{c}_0) + \overline{X}_0(\cdot) P_{B_\rho} c_\rho \right)(s_1) ds_1 \right)(s) ds = 0 \tag{4.38}$$

or, equivalently, the form

$$B_0c_1 = -\int_0^b H(s)B(s)\left(S_h\left(\int_0^b G(\star,s_1)B(s_1)S_h\left(\overline{z}_0(\cdot,\overline{c}_0)+\overline{X}_0(\cdot)P_{B_\rho}c_\rho\right)(s_1)ds_1\right)\right)(s)ds. \tag{4.39}$$

Under condition (4.17), the last equation has the ρ-parametric family of solutions

$$c_1 = \overline{c}_1 + \left[I_r - B_0^+\int_0^b H(s)B(s)\left(S_h\left(\int_0^b G(\star,s_1)B(s_1)\left(S_h\overline{X}_0(\cdot)\right)(s_1)ds_1\right)\right)(s)ds\right]P_{B_\rho}c_\rho, \tag{4.40}$$

where

$$\overline{c}_1 = -B_0^+\int_0^b H(s)B(s)\left(S_h\left(\int_0^b G(\star,s_1)B(s_1)(S_h\overline{z}_0(\cdot,\overline{c}_0))(s_1)ds_1\right)\right)(s)ds. \tag{4.41}$$

So, for the coefficient $z_1(t,c_1) = z_1(t,c_\rho)$, we have the following formula:

$$z_1(t,c_\rho)\overline{z}_1(t,\overline{c}_1) + \overline{X}_1(t)P_{B_\rho}c_\rho, \quad \forall c_\rho \in \mathbb{R}^\rho, \tag{4.42}$$

where

$$\begin{aligned}
\overline{z}_1(t,\overline{c}_1) &= X(t)P_{Q_r}\overline{c}_1 + \int_0^b G(t,s)B(s)(S_h\overline{z}_0(\cdot,\overline{c}_0))(s)ds,\\
\overline{X}_1(t) &= X(t)P_{Q_r}\left[I_r - B_0^+\int_0^b H(s)B(s)S_h\left(\int_0^b G(\star,s_1)B(s_1)\left(S_h\overline{X}_0(\cdot)\right)(s_1)ds_1\right)(s)ds\right]\\
&\quad + \int_0^b G(t,s)B(s)\left(S_h\overline{X}_0(\cdot)\right)(s)ds.
\end{aligned} \tag{4.43}$$

Continuing this process, by assuming that (4.17) holds, it follows by induction that the coefficients $z_i(t,c_i) = z_i(t,c_\rho)$ of the series (4.18) can be determined, from the relevant boundary value problems as follows:

$$z_i(t,c_\rho) = \overline{z}_i(t,\overline{c}_i) + \overline{X}_i(t)P_{B_\rho}c_\rho, \quad \forall c_\rho \in \mathbb{R}^\rho, \tag{4.44}$$

where

$$\begin{aligned}
\overline{z}_i(t,\overline{c}_i) &= X(t)P_{Q_r}\overline{c}_1 + \int_0^b G(t,s)B(s)S_h\overline{z}_{i-1}(\cdot,\overline{c}_{i-1})(s)ds,\\
\overline{c}_i &= -B_0^+\int_0^b H(s)B(s)\left(S_h\left(\int_0^b G(\star,s_1)B(s_1)S_h\overline{z}_{i-1}(\cdot,\overline{c}_{i-1})(s_1)ds_1\right)\right)(s)ds, \quad i=2,\dots,\\
\overline{X}_i(t) &= X(t)P_{Q_r}\left[I_r - B_0^+\int_0^b H(s)B(s)S_h\left(\int_0^b G(\star,s_1)B(s_1)\left(S_h\overline{X}_{i-1}(\cdot)\right)(s_1)ds_1\right)(s)ds\right]\\
&\quad + \int_0^b G(t,s)B(s)\left(S_h\overline{X}_{i-1}(\cdot)\right)(s)ds, \quad i=0,1,2,\dots,
\end{aligned} \tag{4.45}$$

and $\overline{X}_{-1}(t) = X(t)P_{Q_r}$.

The convergence of the series (4.18) can be proved by traditional methods of majorization [9, 11].

In the case $m = n$, the condition (4.17) is equivalent with $\det B_0 \neq 0$, and problem (4.13), (4.14) has a unique solution. □

Example 4.3. Consider the linear boundary value problem for the delay differential equation

$$\begin{gathered}
\dot{z}(t) = z(t-\tau) + \varepsilon\sum_{i=1}^{k} B_i(t)z(h_i(t)) + g(t), \quad h_i(t) \le t \in [0,T],\\
z(s) = \psi(s), \quad \text{if } s<0, \text{ and } z(0) = z(T),
\end{gathered} \tag{4.46}$$

where, as in the above, $B_i, g, \psi \in L_p[0,T]$ and $h_i(t)$ are measurable functions. Using the symbols S_{h_i} and ψ^{h_i} (see (1.3), (1.4), (1.6), and (4.7)), we arrive at the following operator system:

$$\begin{gathered}
\dot{z}(t) = z(t-\tau) + \varepsilon B(t)(S_h z)(t) + \varphi(t,\varepsilon),\\
\ell z := z(0) - z(T) = 0,
\end{gathered} \tag{4.47}$$

where $B(t) = (B_1(t),\dots,B_k(t))$ is an $n \times N$ matrix ($N = nk$), and

$$\varphi(t,\varepsilon) = g(t) + \psi^{h_0}(t) + \varepsilon\sum_{i=1}^{k} B_i(t)\psi^{h_i}(t) \in L_p[0,T]. \tag{4.48}$$

Under the condition that the generating boundary value problem has no solution, we consider the simplest case of $T \le \tau$. Using the delayed matrix exponential (2.5), it is easy to see that, in this case, $X(t) = e_\tau^{I(t-\tau)} = I$ is a normal fundamental matrix for the homogeneous unperturbed system $\dot{z}(t) = z(t-\tau)$, and

$$\begin{gathered}
Q := \ell X(\cdot) = e_\tau^{-I\tau} - e_\tau^{I(T-\tau)} = 0, \\
P_Q = P_{Q^*} = I \quad (r = n,\ d = m = n), \\
K(t,s) \begin{cases} e_\tau^{I(t-\tau-s)} = I, & \text{if } 0 \le s \le t \le T, \\ \Theta, & \text{if } s > t, \end{cases} \\
\ell K(\cdot, s) = K(0,s) - K(T,s) = -I, \\
H(\tau) = P_{Q^*}\ell K(\cdot, s) = -I, \\
(S_{h_i} I)(t) = \chi_{h_i}(t,0) I = I \cdot \begin{cases} 1, & \text{if } 0 \le h_i(t) \le T, \\ 0, & \text{if } h_i(t) < 0. \end{cases}
\end{gathered} \tag{4.49}$$

Then the $n \times n$ matrix B_0 has the form

$$\begin{aligned}
B_0 &= \int_0^T H(s)B(s)(S_h I)(s)ds = -\int_0^T \sum_{i=1}^k B_i(s)(S_{h_i} I)(s)ds \\
&= -\sum_{i=1}^k \int_0^T B_i(s)\chi_{h_i}(s,0)ds.
\end{aligned} \tag{4.50}$$

If $\det B_0 \ne 0$, problem (4.46) has a unique solution $z(t,\varepsilon)$ with the properties

$$z(\cdot,\varepsilon) \in D_p[0,T], \qquad \dot{z}(\cdot,\varepsilon) \in L_p[0,T], \qquad z(t,\cdot) \in C(0,\varepsilon_*]. \tag{4.51}$$

Let, say, $h_i(t) := t - \Delta_i$ where $0 < \Delta_i = \text{const} < T$, $i = 1,\ldots,k$, then

$$\chi_{h_i}(t,0) = \begin{cases} 1, & \text{if } 0 \le h_i(t) = t - \Delta_i \le T, \\ 0, & \text{if } h_i(t) = t - \Delta_i \ < 0, \end{cases} \tag{4.52}$$

or, equivalently,

$$\chi_{h_i}(t,0) = \begin{cases} 1, & \text{if } \Delta_i \le t \le T + \Delta_{i,} \\ 0, & \text{if } t < \Delta_i. \end{cases} \tag{4.53}$$

Now the matrix B_0 turns into

$$B_0 = -\sum_{i=1}^k \int_0^T B_i(s)\chi_{h_i}(s,0)ds = -\sum_{i=1}^k \int_{\Delta_i}^T B_i(s)ds, \tag{4.54}$$

and the boundary value problem (4.46) is uniquely solvable if

$$\det\left[-\sum_{i=1}^{k}\int_{\Delta_i}^{T} B_i(s)ds\right] \neq 0. \tag{4.55}$$

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