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Research Article

Dynamical Properties in a Fourth-Order Nonlinear Difference Equation

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The rule of trajectory structure for fourth-order nonlinear difference equation $x_{n+1} = (x^a_{n-2} + x_{n-3})/(x^a_{n-2}x_{n-3} + 1)$, $n = 0, 1, 2, \ldots$, where $a \in [0, 1)$ and the initial values $x_{-3}, x_{-2}, x_{-1}, x_0 \in [0, \infty)$, is described clearly out in this paper. Mainly, the lengths of positive and negative semicycles of its nontrivial solutions are found to occur periodically with prime period 15. The rule is $4^+, 3^-, 1^+, 2^-, 2^+, 1^-, 1^+, 1^-$ in a period. By utilizing this rule its positive equilibrium point is verified to be globally asymptotically stable.

1. Introduction

In this paper we consider the following fourth-order nonlinear difference equation:

$$x_{n+1} = \frac{x^a_{n-2} + x_{n-3}}{x^a_{n-2}x_{n-3} + 1}, \quad n = 0, 1, 2, \ldots, \tag{1.1}$$

where $a \in [0, 1)$ and the initial values $x_{-3}, x_{-2}, x_{-1}, x_0 \in (0, \infty)$.

When $a = 0$, (1.1) becomes the trivial case $x_{n+1} = 1$, $n = 0, 1, \ldots$. Hence, we will assume in the sequel that $0 < a < 1$.

When $a \in (0,\ 1)$, (1.1) is not a rational difference equation but a nonlinear one. So far, there have not been any effective general methods to deal with the global behavior of nonlinear difference equations of order greater than one. Therefore, to study the qualitative properties of nonlinear difference equations with higher order is theoretically meaningful.

In this paper, it is of key for us to find that the lengths of positive and negative semi-cycles of nontrivial solutions of (1.1) occur periodically with prime period 15 with the rule $4^+, 3^-, 1^+, 2^-, 2^+, 1^-, 1^+, 1^-$ and in a period. With the help of this rule and utilizing the monotonicity of solution the positive equilibrium point of the equation is verified to be globally asymptotically stable.

Essentially, we derive the following results for solutions of (1.1).

Theorem CL. *The rule of the trajectory structure of* (1.1) *is that all of its solutions asymptotically approach its equilibrium; furthermore, any one of its solutions is either*

(1) *eventually trivial*

(2) *nonoscillatory and eventually negative (i.e., $x_n \geq 1$) or*

(3) *strictly oscillatory with the lengths of positive and negative semi-cycles periodically successively occurring with prime period 15 and the rule to be $4^+, 3^-, 1^+, 2^-, 2^+, 1^-, 1^+, 1^-$ in a period.*

It follows from the results stated below that Theorem CL is true.

It is easy to see that the positive equilibrium $\overline{x}$ of (1.1) *satisfies*

$$\overline{x} = \frac{\overline{x}^a + \overline{x}}{\overline{x}^a\overline{x} + 1}, \tag{1.2}$$

from which one can see that (1.1) *has a unique equilibrium $\overline{x} = 1$.*

In the following, we state some main definitions used in this paper.

Definition 1.1. A positive semi-cycle $\{x_n\}_{n=-3}^{\infty}$ of a solution of (1.1) consists of a "string" of terms $x_l, x_{l+1}, \ldots, x_m$, all greater than or equal to the equilibrium $\overline{x}$, with $l \geq -3$ and $m \leq \infty$ such that

$$\begin{gathered} \text{either } l = -3 \text{ or } l > -3, x_{l-1} < \overline{x}\,, \\ \text{either } m = \infty \text{ or } m < \infty, x_{m+1} < \overline{x}. \end{gathered} \tag{1.3}$$

A negative semi-cycle of a solution $\{x_n\}_{n=-3}^{\infty}$ of (1.1) consists of a "string" of term $x_l, x_{l+1}, \ldots, x_m$, all less than $\overline{x}$, with $l \geq -3$ and $m \leq \infty$ such that

$$\begin{gathered} \text{either } l = -3 \text{ or } l > -3, x_{l-1} \geq \overline{x}\,, \\ \text{either } m = \infty \text{ or } m < \infty, x_{m+1} \geq \overline{x}. \end{gathered} \tag{1.4}$$

The length of a semi-cycle is the number of the total terms contained in it.

Definition 1.2. A solution $\{x_n\}_{n=-3}^{\infty}$ of (1.1) is said to be eventually trivial if x_n is eventually equal to $\overline{x} = 1$; Otherwise, the solution is said to be nontrivial. A solution $\{x_n\}_{n=-3}^{\infty}$ of (1.1) is said to be eventually positive (negative) if x_n is eventually greater (less) than $\overline{x} = 1$.

For the other concepts in this paper and related work, see [1–3] and [4–11], respectively.

2. Three Lemmas

Before drawing a qualitatively clear picture for the solutions of (1.1), we first establish three basic lemmas which will play key roles in the proof of our main results.

Lemma 2.1. *A solution $\{x_n\}_{n=-3}^{\infty}$ of* (1.1) *is eventually trivial if and only if*

$$(x_{-3}-1)(x_{-2}-1)(x_{-1}-1)(x_0-1)=0. \tag{2.1}$$

Proof. Sufficiency. Assume that (2.1) holds. Then it follows from (1.1) that the following conclusions hold:

(i) if $x_{-3}-1=0$, then $x_n=1$ for $n\geq 7$;

(ii) if $x_{-2}-1=0$, then $x_n=1$ for $n\geq 4$;

(iii) if $x_{-1}-1=0$, then $x_n=1$ for $n\geq 5$;

(iv) if $x_0-1=0$, then $x_n=1$ for $n\geq 6$.

Necessity. Conversely, assume that

$$(x_{-3}-1)(x_{-2}-1)(x_{-1}-1)(x_0-1)\neq 0. \tag{2.2}$$

Then one can show that

$$x_n\neq 1 \quad \text{for any } n\geq 1. \tag{2.3}$$

Assume the contrary that for some $N\geq 1$,

$$x_N=1 \quad \text{and that} \quad x_n\neq 1 \quad \text{for } -3\leq n\leq N-1. \tag{2.4}$$

Clearly,

$$1=x_N=\frac{x_{N-3}^a+x_{N-4}}{x_{N-3}^a x_{N-4}+1}, \tag{2.5}$$

which implies that $(x_{N-3}^a-1)(x_{N-4}-1)=0$, which contradicts (2.4). □

Remark 2.2. Lemma 2.1 actually demonstrates that a solution $\{x_n\}_{n=-3}^{\infty}$ of (1.1) is eventually nontrivial if and only if

$$(x_{-3}-1)(x_{-2}-1)(x_{-1}-1)(x_0-1)\neq 0. \tag{2.6}$$

Therefore, if a solution $\{x_n\}_{n=-3}^{\infty}$ is nontrivial, then $x_n\neq 1$ for $n\geq -3$.

Lemma 2.3. *Let $\{x_n\}_{n=-3}^{\infty}$ be a nontrivial positive solution of* (1.1). *Then the following conclusions are true:*

(a) $(x_{n+1}-1)(x_{n-2}-1)(x_{n-3}-1)<0$ *for* $n\geq 0$;

(b) $(x_{n+1}-x_{n-3})(x_{n-3}-1)<0$ *for* $n\geq 0$.

Proof. In view of (1.1), we can see that

$$x_{n+1}-1=-\frac{(x_{n-2}^{a}-1)(x_{n-3}-1)}{x_{n-2}^{a}x_{n-3}+1},\quad n=0,1,\ldots,$$
$$x_{n+1}-x_{n-3}=\frac{x_{n-2}^{a}(1-x_{n-3})(1+x_{n-3})}{x_{n-2}^{a}x_{n-3}+1},\quad n=0,1,\ldots \tag{2.7}$$

from which inequalities (a) and (b) follow. So the proof is complete. □

Lemma 2.4. *There exist nonoscillatory solutions of* (1.1)*, which must be eventually negative. There do not exist eventually positive non-oscillatory solutions of* (1.1).

Proof. Consider a solution of (1.1) with $x_{-3}<1$, $x_{-2}<1$, $x_{-1}<1$, and $x_0<1$. We then know from Lemma 2.3(a) that $x_n<1$ for $n\geq -3$. So, this solution is just a non-oscillatory solution, and furthermore, eventually negative. Suppose that there exist eventually positive nonoscillatory solutions of (1.1). Then, there exists a positive integer N such that $x_n>1$ for $n\geq N$. Thereout, for $n\geq N+3$, $(x_{n+1}-1)(x_{n-2}-1)(x_{n-3}-1)>0$. This contradicts Lemma 2.3(a). So, there do not exist eventually positive non-oscillatory solutions of (1.1), as desired. □

3. Main Results and Their Proofs

First we analyze the structure of the semi-cycles of nontrivial solutions of (1.1). Here we confine us to consider the situation of the strictly oscillatory solution of (1.1).

Theorem 3.1. *Let* $\{x_n\}_{n=-3}^{\infty}$ *be any strictly oscillatory solution of* (1.1)*. Then, the lengths of positive and negative semi-cycles of the solution periodically successively occur with prime period 15. And in a period, the rule is* $4^+, 3^-, 1^+, 2^-, 2^+, 1^-, 1^+, 1^-$.

Proof. By Lemma 2.3(a), one can see that the length of a positive semi-cycle is not larger than 4, whereas, the length of a negative semi-cycle is at most 3. Based on the strictly oscillatory character of the solution, we see, for some integer $p\geq 0$, that one of the following four cases must occur.

Case 1. $x_{p-3}>1$, $x_{p-2}<1$, $x_{p-1}>1$, $x_p>1$.

Case 2. $x_{p-3}>1$, $x_{p-2}<1$, $x_{p-1}>1$, $x_p<1$.

Case 3. $x_{p-3}>1$, $x_{p-2}<1$, $x_{p-1}<1$, $x_p>1$.

Case 4. $x_{p-3}>1$, $x_{p-2}<1$, $x_{p-1}<1$, $x_p<1$.

If Case 1 occurs, it follows from Lemma 2.3(a) that $x_{p-3}>1$, $x_{p-2}<1$, $x_{p-1}>1$, $x_p>1$, $x_{p+1}>1$, $x_{p+2}>1$, $x_{p+3}<1$, $x_{p+4}<1$, $x_{p+5}<1$, $x_{p+6}>1$, $x_{p+7}<1$, $x_{p+8}<1$, $x_{p+9}>1$, $x_{p+10}>1$, $x_{p+11}<1$, $x_{p+12}>1$, $x_{p+13}<1$, $x_{p+14}>1$, $x_{p+15}>1$, $x_{p+16}>1$, $x_{p+17}>1$, $x_{p+18}<1$, $x_{p+19}<1$, $x_{p+20}<1$, $x_{p+21}>1$, $x_{p+22}<1$, $x_{p+23}<1$, $x_{p+24}>1$, $x_{p+25}>1$, $x_{p+26}<1$, $x_{p+27}>1$,

$x_{p+28} < 1$, $x_{p+29} > 1$, $x_{p+30} > 1$, $x_{p+31} > 1$, $x_{p+32} > 1$, $x_{p+33} < 1$, $x_{p+34} < 1$, $x_{p+35} < 1$, $x_{p+36} > 1$, $x_{p+37} < 1$, $x_{p+38} < 1$, $x_{p+39} > 1$, $x_{p+40} > 1$, $x_{p+41} < 1, \ldots$, which means that the rule for the lengths of positive and negative semi-cycles of the solution of (1.1) to successively occur is $\ldots, 4^+, 3^-, 1^+, 2^-, 2^+, 1^-, 1^+, 1^-, 4^+, 3^-, 1^+, 2^-, 2^+, 1^-, 1^+, 1^-, \ldots$.

If Case 2 happens, then Lemma 2.3(a) tells us that $x_{p-3} > 1$, $x_{p-2} < 1$, $x_{p-1} > 1$, $x_p < 1$, $x_{p+1} > 1$, $x_{p+2} > 1$, $x_{p+3} > 1$, $x_{p+4} > 1$, $x_{p+5} < 1$, $x_{p+6} < 1$, $x_{p+7} < 1$, $x_{p+8} > 1$, $x_{p+9} < 1$, $x_{p+10} < 1$, $x_{p+11} > 1$, $x_{p+12} > 1$, $x_{p+13} < 1$, $x_{p+14} > 1$, $x_{p+15} < 1$, $x_{p+16} > 1$, $x_{p+17} > 1$, $x_{p+18} > 1$, $x_{p+19} > 1$, $x_{p+20} < 1$, $x_{p+21} < 1$, $x_{p+22} < 1$, $x_{p+23} > 1$, $x_{p+24} < 1$, $x_{p+25} < 1$, $x_{p+26} > 1$, $x_{p+27} > 1$, $x_{p+28} < 1$, $x_{p+29} > 1$, $x_{p+30} < 1$, $x_{p+31} > 1$, $x_{p+32} > 1$, $x_{p+33} > 1$, $x_{p+34} > 1$, $x_{p+35} < 1$, $x_{p+36} < 1$, $x_{p+37} < 1$, $x_{p+38} > 1$, $x_{p+39} < 1$, $x_{p+40} < 1$, $x_{p+41} > 1$, $x_{p+42} > 1, \ldots$. This shows that the rule for the numbers of terms of positive and negative semi-cycles of the solution of (1.1) to successively occur still is $\ldots, 4^+, 3^-, 1^+, 2^-, 2^+, 1^-, 1^+, 1^-, 4^+, 3^-, 1^+, 2^-, 2^+, 1^-, 1^+, 1^-, \ldots$.

If Case 3 happens, then Lemma 2.3(a) implies that $x_{p-3} > 1$, $x_{p-2} < 1$, $x_{p-1} < 1$, $x_p > 1$, $x_{p+1} > 1$, $x_{p+2} < 1$, $x_{p+3} > 1$, $x_{p+4} < 1$, $x_{p+5} > 1$, $x_{p+6} > 1$, $x_{p+7} > 1$, $x_{p+8} > 1$, $x_{p+9} < 1$, $x_{p+10} < 1$, $x_{p+11} < 1$, $x_{p+12} > 1$, $x_{p+13} < 1$, $x_{p+14} < 1$, $x_{p+15} > 1$, $x_{p+16} > 1$, $x_{p+17} < 1$, $x_{p+18} > 1$, $x_{p+19} < 1$, $x_{p+20} > 1$, $x_{p+21} > 1$, $x_{p+22} > 1$, $x_{p+23} > 1$, $x_{p+24} > 1$, $x_{p+25} < 1$, $x_{p+26} < 1$, $x_{p+27} > 1$, $x_{p+28} < 1$, $x_{p+29} < 1$, $x_{p+30} > 1$, $x_{p+31} > 1$, $x_{p+32} < 1$, $x_{p+33} > 1$, $x_{p+34} < 1$, $x_{p+35} > 1$, $x_{p+36} > 1$, $x_{p+37} > 1$, $x_{p+38} > 1$, $x_{p+39} < 1$, $x_{p+40} < 1$, $x_{p+41} < 1$, $x_{p+42} > 1, \ldots$. This shows that the rule for the numbers of terms of positive and negative semi-cycles of the solution of (1.1) to successively occur still is $\ldots, 4^+, 3^-, 1^+, 2^-, 2^+, 1^-, 1^+, 1^-, 4^+, 3^-, 1^+, 2^-, 2^+, 1^-, 1^+, 1^-, \ldots$.

If Case 4 happens, then it is to be seen from Lemma 2.3(a) that $x_{p-3} > 1$, $x_{p-2} < 1$, $x_{p-1} < 1$, $x_p < 1$, $x_{p+1} > 1$, $x_{p+2} < 1$, $x_{p+3} < 1$, $x_{p+4} > 1$, $x_{p+5} > 1$, $x_{p+6} < 1$, $x_{p+7} > 1$, $x_{p+8} < 1$, $x_{p+9} > 1$, $x_{p+10} > 1$, $x_{p+11} > 1$, $x_{p+12} > 1$, $x_{p+13} < 1$, $x_{p+14} < 1$, $x_{p+15} < 1$, $x_{p+16} > 1$, $x_{p+17} < 1$, $x_{p+18} < 1$, $x_{p+19} > 1$, $x_{p+20} > 1$, $x_{p+21} < 1$, $x_{p+22} > 1$, $x_{p+23} < 1$, $x_{p+24} > 1$, $x_{p+25} > 1$, $x_{p+26} > 1$, $x_{p+27} > 1$, $x_{p+28} < 1$, $x_{p+29} < 1$, $x_{p+30} < 1$, $x_{p+31} > 1$, $x_{p+32} < 1$, $x_{p+33} < 1$, $x_{p+34} > 1$, $x_{p+35} > 1$, $x_{p+36} < 1$, $x_{p+37} > 1$, $x_{p+38} < 1$, $x_{p+39} > 1$, $x_{p+40} > 1$, $x_{p+41} > 1$, $x_{p+42} > 1, \ldots$. This shows that the rule for the numbers of terms of positive and negative semi-cycles of the solution of (1.1) to successively occur still is $\ldots, 4^+, 3^-, 1^+, 2^-, 2^+, 1^-, 1^+, 1^-, 4^+, 3^-, 1^+, 2^-, 2^+, 1^-, 1^+, 1^- \ldots$.

Hence, the proof is complete. □

Now, we present the global asymptotical stable results for (1.1).

Theorem 3.2. *Assume that $a \in [0, 1)$. Then the unique positive equilibrium of* (1.1) *is globally asymptotically stable.*

Proof. When $a = 0$, (1.1) is trivial. So, we only consider the case $a > 0$ and prove that the positive equilibrium point $\overline{x}$ of (1.1) is both locally asymptotically stable and globally attractive. The linearized equation of (1.1) about the positive equilibrium $\overline{x} = 1$ is

$$y_{n+1} = 0 \cdot y_n + 0 \cdot y_{n-1} + 0 \cdot y_{n-2} + 0 \cdot y_{n-3}, \quad n = 0, 1, \ldots. \tag{3.1}$$

By virtue of [3, Remark 1.3.7, page 13], $\overline{x}$ is locally asymptotically stable. It remains to be verified that every positive solution $\{x_n\}_{n=-3}^{\infty}$ of (1.1) converges to $\overline{x}$ as $n \to \infty$. Namely, we want to prove that

$$\lim_{n \to \infty} x_n = \overline{x} = 1. \tag{3.2}$$

If the initial values of the solutions satisfy (2.1), that is to say, the solution is a trivial solution, then Lemma 2.1 says that the solution is eventually equal to 1 and of course (3.2) holds.

If the solution is a nontrivial solution, then we can further divide the solution into two cases.

(a) non-oscillatory solution;

(b) oscillatory solution.

If Case (a) happens, then it follows from Lemma 2.3 that the solution is actually an eventually negative one. According to Lemma 2.3(b), we see that x_{4n}, x_{4n-1}, x_{4n-2} and x_{4n-3} are eventually increasing and bounded from the upper by the constant 1. So the limits

$$\lim_{n\to\infty} x_{4n} = G, \qquad \lim_{n\to\infty} x_{4n+1} = L, \qquad \lim_{n\to\infty} x_{4n+2} = M, \qquad \lim_{n\to\infty} x_{4n+3} = N \tag{3.3}$$

exist and are finite. Noting that

$$x_{4n+1} = \frac{x_{4n-2}^a + x_{4n-3}}{x_{4n-2}^a x_{4n-3} + 1}, \qquad x_{4n} = \frac{x_{4n-3}^a + x_{4n-4}}{x_{4n-3}^a x_{4n-4} + 1}, \qquad x_{4n+2} = \frac{x_{4n-1}^a + x_{4n-2}}{x_{4n-1}^a x_{4n-2} + 1},$$
$$x_{4n+3} = \frac{x_{4n}^a + x_{4n-1}}{x_{4n}^a x_{4n-1} + 1} \tag{3.4}$$

and taking the limits on both sides of the above equalities, respectively, one may obtain

$$L = \frac{M^a + L}{M^a L + 1}, \qquad G = \frac{L^a + G}{L^a G + 1}, \qquad M = \frac{N^a + M}{N^a M + 1}, \qquad N = \frac{G^a + N}{G^a N + 1}. \tag{3.5}$$

Solving these equations, we get $G = L = M = N = 1$, which shows that (3.2) is true.

If case (b) happens, the solution is strictly oscillatory.

Consider now x_n to be strictly oscillatory about the positive equilibrium point $\overline{x}$ of (1.1). By virtue of Theorem 3.1, one understands that the lengths of positive and negative semi-cycles of the solution periodically successively occur, and in a period, the rule is $4^+, 3^-, 1^+, 2^-, 2^+, 1^-, 1^+, 1^-$.

For simplicity, for some integer $p \geq 0$, we denote by $\{x_p, x_{p+1}, x_{p+2,xp+3}\}^+$ the terms of a positive semi-cycle of length four, followed by $\{x_{p+4}, x_{p+5}, x_{p+6}\}^-$ negative semi-cycle with length three, then a positive semi-cycle $\{x_{p+7}\}^+$, a negative semi-cycle $\{x_{p+8}, x_{p+9}\}^-$, a positive semi-cycle $\{x_{p+10}, x_{p+11}\}^+$, a negative semi-cycle $\{x_{p+12}\}^-$, a positive semi-cycle $\{x_{p+13}\}^+$, and a negative semi-cycle $\{x_{p+14}\}^-$. Namely, the rule for the lengths of negative and positive semi-cycles to occur successively can be periodically expressed as follows: $\{x_{p+15n}, x_{p+15n+1}, x_{p+15n+2}, x_{p+15n+3}\}^+$, $\{x_{p+15n+4}, x_{p+15n+5}, x_{p+15n+6}\}^-$, $\{x_{p+15n+7}\}^+$, $\{x_{p+15n+8}, x_{p+15n+9}\}^-$, $\{x_{p+15n+10}, x_{p+15n+11}\}^+$, $\{x_{p+15n+12}\}^-$, $\{x_{p+15n+13}\}^+$, $\{x_{p+15n+14}\}^-$, and $n = 0, 1, \ldots$.

From Lemma 2.3(b), we may immediately obtain the following results:

(i) $x_{p+15n+15} < x_{p+15n+11} < x_{p+15n+7} < x_{p+15n+3}$; $x_{p+15n+16} < x_{p+15n+13}$;

(ii) $1 > x_{p+15n+12} > x_{p+15\mathrm{n}+8} > x_{p+15n+4}$; $1 > x_{p+15n+9} > x_{p+15n+5}$.

Also, the following inequalities hold:

(iii) $x_{p+15n+18}x_{p+15n+14} < 1$; $x_{p+15n+14}x_{p+15n+10} > 1$;

(iv) $x_{p+15n+6}x_{p+15n+10} < 1$; $x_{p+15n+5}x_{p+15n+1} > 1$;

(v) $x_{p+15n+6}x_{p+15n+2} > 1$;

(vi) $x_{p+15n+13}x_{p+14n+9} < 1$; $x_{p+15n+14}x_{p+15n+11} > 1$; $x_{p+15n+4}x_{p+15n} > 1$;

(vii) $x_{p+15n+6}x_{p+15n+3} > 1$; $x_{p+15n+11}x_{p+15n+8} < 1$.

In fact, from the observation that

$$\begin{aligned} x_{p+15n+18} &= \frac{x^a_{p+15n+15} + x_{p+15n+14}}{x^a_{p+15n+15}x_{p+15n+14} + 1} \\ &< \frac{x^a_{p+15n+15} + x_{p+15n+14}}{x^a_{p+15n+15}x_{p+15n+14} + x^2_{p+15n+14}} \\ &= \frac{1}{x_{p+15n+14}}, \end{aligned} \tag{3.6}$$

we know that the first inequality in (iii) is true. The other inequalities in (iii)–(vi) can be similarly proved. Noticing that $0 \le a < 1$ and from that the observation

$$\begin{aligned} x_{p+15n+6} &= \frac{x^a_{p+15n+3} + x_{p+15n+2}}{x^a_{p+15n+3}x_{p+15n+2} + 1} \\ &> \frac{x^a_{p+15n+3} + x_{p+15n+2}}{x^a_{p+15n+3}x_{p+15n+2} + x^{2a}_{p+15n+3}} \\ &= \frac{1}{x^a_{p+15n+3}} \\ &> \frac{1}{x_{p+15n+3}}, \end{aligned} \tag{3.7}$$

we know that the first inequality in (vii) holds. The other inequality in (vii) can be analogously proved.

Combining the above inequalities, one can derive that

$$1 < x_{p+15n+18} < \frac{1}{x_{p+15n+14}} < x_{p+15n+10} < \frac{1}{x_{p+15n+6}} < x_{p+15n+3}, \tag{3.8}$$

$$1 < x_{p+15n+15} < x_{p+15n+11} < \frac{1}{x_{p+15n+8}} < \frac{1}{x_{p+15n+4}} < x_{p+15n}, \tag{3.9}$$

$$1 < x_{p+15n+16} < x_{p+15n+13} < \frac{1}{x_{p+15n+9}} < \frac{1}{x_{p+15n+5}} < x_{p+15n+1}. \tag{3.10}$$

It follows from (3.8) that $\{x_{p+15n+3}\}_{n=0}^{\infty}$ is decreasing with lower bound 1. So, the limit

$$\lim_{n\to\infty} x_{p+15n+3} = L \tag{3.11}$$

exists and is finite. Accordingly, by view of (3.8), we obtain

$$\lim_{n\to\infty} x_{p+15n+10} = L, \qquad \lim_{n\to\infty} x_{p+15n+14} = \lim_{n\to\infty} x_{p+15n+6} = \frac{1}{L}. \tag{3.12}$$

It is easy to see from (3.9) that $\{x_{p+15n}\}_{n=0}^{\infty}$ is decreasing with lower bound 1. So, the limit

$$\lim_{n\to\infty} x_{p+15n} = M \tag{3.13}$$

exists and is finite. Thereout, in light of (3.9), one has

$$\lim_{n\to\infty} x_{p+15n+11} = M, \qquad \lim_{n\to\infty} x_{p+15n+8} = \lim_{n\to\infty} x_{p+15n+4} = \frac{1}{M}. \tag{3.14}$$

It follows from (3.10) that $\{x_{p+15n+1}\}_{n=0}^{\infty}$ is decreasing with lower bound 1. So, the limit

$$\lim_{n\to\infty} x_{p+15n+1} = N \tag{3.15}$$

exists and is finite. Accordingly, by view of (3.10), we obtain

$$\lim_{n\to\infty} x_{p+15n+13} = N, \qquad \lim_{n\to\infty} x_{p+15n+9} = \lim_{n\to\infty} x_{p+15n+5} = \frac{1}{N}. \tag{3.16}$$

Taking the limits on both sides of $x_{p+15n+18} = (x_{p+15n+15}^{a} + x_{p+15n+14})/(x_{p+15n+15}^{a} x_{p+15n+14} + 1)$, one has $L = (M^a + 1/L)/(M^a/L + 1)$, which gives rise to $L = 1$. We further obtain from (i) and (3.12) that

$$\lim_{n\to\infty} x_{p+15n+k} = 1, \quad k = 3, 6, 7, 10, 11, 14, 15. \tag{3.17}$$

Hence, $M = 1$. Therefore,

$$\lim_{n\to\infty} x_{p+15n+k} = 1, \quad k = 3, 4, 6, 7, 8, 10, 11, 14, 15. \tag{3.18}$$

It is easy to derive from (v) that $1 > x_{p+15n+2} > 1/(x_{p+15n+6})$. Noticing that $\lim_{n\to\infty} x_{p+15n+6} = 1$, one can see that $\lim_{n\to\infty} x_{p+15n+2} = 1$.

Similarly, taking the limits on both sides of $x_{p+15n+13} = (x_{p+15n+10}^{a} + x_{p+15n+9})/(x_{p+15n+10}^{a} x_{p+15n+9} + 1)$, one has $\lim_{n\to\infty} x_{p+15n+13} = N = 1$. Finally, by taking the limits on both sides of $x_{p+15n+12} = (x^a_{p+15n+9} + x_{p+15n+8})/(x^a_{p+15n+9}x_{p+15n+8} + 1)$, one has $\lim_{n\to\infty} x_{p+15n+12} = 1$.

Up to now, we have shown that

$$\lim_{n\to\infty} x_{p+15n+k} = 1, \quad k = 1, 2, \ldots, 15. \tag{3.19}$$

So, the proof for Theorem 3.2 is complete. □

Remark 3.3. One can see from the process of proofs stated previously that these results in this paper also hold for $a = 1$.

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