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Research Article

On the Symmetric Properties of Higher-Order Twisted q-Euler Numbers and Polynomials

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In 2009, Kim et al. gave some identities of symmetry for the twisted Euler polynomials of higher-order, recently. In this paper, we extend our result to the higher-order twisted q-Euler numbers and polynomials. The purpose of this paper is to establish various identities concerning higher-order twisted q-Euler numbers and polynomials by the properties of p-adic invariant integral on $\mathbb{Z}_p$. Especially, if $q = 1$, we derive the result of Kim et al. (2009).

1. Introduction

Let p be a fixed odd prime number. Throughout this paper, the symbols $\mathbb{Z}$, $\mathbb{Z}_p, \mathbb{Q}_p, \mathbb{C}$, and $\mathbb{C}_p$ will denote the ring of rational integers, the ring of p-adic integers, the field of p-adic rational numbers, the complex number field, and the completion of the algebraic closure of $\mathbb{Q}_p$, respectively. Let $\mathbb{N}$ be the set of natural numbers and $\mathbb{Z}_+ = \mathbb{N}\bigcup\{0\}$. Let v_p be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-v_p(p)} = 1/p$.

When one talks of q-extension, q is variously considered as an indeterminate, a complex $q \in \mathbb{C}$, or a p-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$, one normally assumes that $|q| < 1$. If $q \in \mathbb{C}_p$, then we assume that $|q-1|_p < p^{-1/(p-1)}$ so that $q^x = \exp(x \log q)$ for each $x \in \mathbb{Z}_p$. We use the following notation:

$$[x]_q = \frac{1-q^x}{1-q}, \quad [x]_{-q} = \frac{1-(-q)^x}{1+q} \quad \forall x \in \mathbb{Z}_p. \tag{1.1}$$

For a fixed positive integer d with $(p, d) = 1$, set

$$X = X_d = \frac{\varprojlim_{n} \mathbb{Z}}{dp^n\mathbb{Z}}, \qquad X_1 = \mathbb{Z}_p,$$
$$X^* = \bigcup_{\substack{0<a<dp,\\ (a,p)=1}} \left(a + dp\mathbb{Z}_p\right), \tag{1.2}$$
$$a + dp^n\mathbb{Z}_p = \left\{x \in X \mid x \equiv a\left(\operatorname{mod} dp^n\right)\right\},$$

where $a \in \mathbb{Z}$ satisfies the condition $0 \leq a < dp^n$. For any $n \in \mathbb{N}$,

$$\mu_q\left(a + dp^n\mathbb{Z}_p\right) = \frac{q^a}{\left[dp^n\right]_q} \tag{1.3}$$

(see [1–13]) is known to be a distribution on X.

We say that f is a uniformly differentiable function at $a \in \mathbb{Z}_p$ and denote this property by $f \in UD(\mathbb{Z}_p)$ if the difference quotients

$$F_f(x, y) = \frac{f(x) - f(y)}{x - y} \tag{1.4}$$

have a limit $f'(a)$ as $(x, y) \to (a, a)$.

For $f \in UD(\mathbb{Z}_p)$, the fermionic p-adic invariant q-integral on $\mathbb{Z}_p$ is defined as

$$I_{-q}(f) = \int_{\mathbb{Z}_p} f(x)d\mu_{-q}(x) = \lim_{n\to\infty} \frac{1}{\left[p^n\right]_{-q}} \sum_{x=0}^{p^n-1} f(x)(-q)^x \tag{1.5}$$

(see [14]). Let us define the fermionic p-adic invariant integral on $\mathbb{Z}_p$ as follows:

$$I_{-1}(f) = \lim_{q\to 1} I_{-q}(f) = \int_{\mathbb{Z}_p} f(x)d\mu_{-1}(x) = \lim_{n\to\infty} \sum_{x=0}^{p^n-1} f(x)(-1)^x \tag{1.6}$$

(see [1–12, 14–20]). From the definition of q-integral, we have

$$I_{-1}(f_1) + I_{-1}(f) = 2f(0), \quad \text{where } f_1(x) = f(x+1). \tag{1.7}$$

For $n \in \mathbb{N}$, let T_p be the p-adic locally constant space defined by

$$T_p = \bigcup_{n \geq 1} \mathbb{C}_{p^n} = \lim_{n \to \infty} \mathbb{C}_{p^n} = \mathbb{C}_{p^\infty}, \tag{1.8}$$

where $\mathbb{C}_{p^n} = \{\zeta \in \mathbb{C}_p \mid \zeta^{p^n} = 1 \text{ for some } n \geq 0\}$ is the cyclic group of order p^n.

It is well known that the twisted q-Euler polynomials of order k are defined as

$$e^{xt}\left(\frac{2}{e^t\zeta q + 1}\right)^k = \sum_{n=0}^{\infty} E_{n,\zeta,q}^{(k)}(x)\frac{t^n}{n!}, \quad \zeta \in T_p, \tag{1.9}$$

and $E_{n,\zeta,q}^{(k)} = E_{n,\zeta,q}^{(k)}(0)$ are called the twisted q-Euler numbers of order k. When $k = 1$, the polynomials and numbers are called the twisted q-Euler polynomials and numbers, respectively. When $k = 1$ and $q = 1$, the polynomials and numbers are called the twisted Euler polynomials and numbers, respectively. When $k = 1$, $q = 1$, and $\zeta = 1$, the polynomials and numbers are called the ordinary Euler polynomials and numbers, respectively.

In [15], Kim et al. gave some identities of symmetry for the twisted Euler polynomials of higher order, recently. In this paper, we extend our result to the higher-order twisted q-Euler numbers and polynomials.

The purpose of this paper is to establish various identities concerning higher-order twisted q-Euler numbers and polynomials by the properties of p-adic invariant integral on $\mathbb{Z}_p$. Especially, if $q = 1$, we derive the result of [15].

2. Some Identities of the Higher-Order Twisted q-Euler Numbers and Polynomials

Let $w_1, w_2 \in \mathbb{N}$ with $w_1 \equiv 1 (\mathrm{mod}\ 2)$ and $w_2 \equiv 1 (\mathrm{mod}\ 2)$.
For $\zeta \in T_p$ and $m \in \mathbb{N}$, we set

$$\begin{aligned} R_q^{(m)}(w_1, w_2 : \zeta) = {} & \frac{\int_{\mathbb{Z}_p^m} e^{(\sum_{i=1}^m x_i + w_2 x) w_1 t} \zeta^{(\sum_{i=1}^m x_i) w_1} q^{(\sum_{i=1}^m x_i) w_1} d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_m)}{\int_{\mathbb{Z}_p} e^{w_1 w_2 x t} \zeta^{w_1 w_2 x} q^{w_1 w_2 x} d\mu_{-1}(x)} \\ & \times \int_{\mathbb{Z}_p^m} e^{(\sum_{i=1}^m x_i + w_1 y) w_2 t} \zeta^{(\sum_{i=1}^m x_i) w_2} q^{(\sum_{i=1}^m x_i) w_2} d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_m), \end{aligned} \tag{2.1}$$

where

$$\int_{\mathbb{Z}_p^m} f(x_1, \ldots, x_m) d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_m) = \underbrace{\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p}}_{m\text{-times}} f(x_1, \ldots, x_m) d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_m). \tag{2.2}$$

In (2.1), we note that $R_q^{(m)}(w_1, w_2 : \zeta)$ is symmetric in w_1 and w_2.

From (2.1), we derive that

$$R_q^{(m)}(w_1,w_2:\zeta)=e^{w_1w_2xt}\int_{\mathbb{Z}_p^m}e^{(\sum_{i=1}^m x_i)w_1t}\zeta^{(\sum_{i=1}^m x_i)w_1}q^{(\sum_{i=1}^m x_i)w_1}d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_m)$$
$$\times\frac{\int_{\mathbb{Z}_p}e^{w_2x_mt}\zeta^{w_2x_m}q^{w_2x_m}d\mu_{-1}(x_m)}{\int_{\mathbb{Z}_p}e^{w_1w_2xt}\zeta^{w_1w_2x}q^{w_1w_2x}d\mu_{-1}(x)}$$
$$\times e^{w_1w_2yt}\int_{\mathbb{Z}_{p^{m-1}}}e^{(\sum_{i=1}^{m-1} x_i)w_2t}\zeta^{(\sum_{i=1}^{m-1} x_i)w_2}q^{(\sum_{i=1}^{m-1} x_i)w_2}d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_{m-1}). \tag{2.3}$$

From the definition of q-integral, we also see that

$$\int_{\mathbb{Z}_{p^m}}e^{(\sum_{i=1}^m x_i)w_1t}\zeta^{(\sum_{i=1}^m x_i)w_1}q^{(\sum_{i=1}^m x_i)w_1}d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_m)e^{w_1w_2xt}$$
$$=\left(\frac{2}{e^{w_1t}\zeta^{w_1}q^{w_1}+1}\right)^m e^{w_1w_2xt}=\sum_{n=0}^{\infty}E_{n,\zeta^{w_1},q^{w_1}}^{(m)}(w_2x)\frac{{w_1}^nt^n}{n!}. \tag{2.4}$$

It is easy to see that

$$\frac{\int_{\mathbb{Z}_p}e^{xt}\zeta^xq^xd\mu(x)}{\int_{\mathbb{Z}_p}e^{w_1xt}\zeta^{w_1x}q^{w_1x}d\mu(x)}=\sum_{l=0}^{w_1-1}(-1)^l\zeta^lq^le^{lt}=\sum_{k=0}^{\infty}T_{k,q}(w_1-1:\zeta)\frac{t^k}{k!}, \tag{2.5}$$

where $T_{k,q}(w_1-1:\zeta)=\sum_{l=0}^{w_1-1}(-1)^l\zeta^lq^ll^k$.

From (2.3), (2.4), and (2.5), we can derive

$$R_q^{(m)}(w_1,w_2:\zeta)$$
$$=\left(\sum_{l=0}^{\infty}E_{l,\zeta^{w_1},q^{w_1}}^{(m)}(w_2x)\frac{w_1^lt^l}{l!}\right)\left(\sum_{k=0}^{\infty}T_{k,q^{w_2}}(w_1-1:\zeta^{w_2})\ \frac{w_2^kt^k}{k!}\right)\left(\sum_{i=0}^{\infty}E_{i,\zeta^{w_2},q^{w_2}}^{(m-1)}(w_1y)\frac{{w_2}^it^i}{i!}\right)$$
$$=\sum_{n=0}^{\infty}\left\{\sum_{j=0}^{n}\binom{n}{j}w_2^jw_1^{n-j}E_{n-j,\zeta^{w_1},q^{w_1}}^{(m)}(w_2x)\sum_{k=0}^{j}T_{k,q^{w_2}}(w_1-1:\zeta^{w_2})\binom{j}{k}E_{j-k,\zeta^{w_2},q^{w_2}}^{(m-1)}(w_1y)\right\}\frac{t^n}{n!}. \tag{2.6}$$

From the symmetry of $R_q^{(m)}(w_1,w_2:\zeta)$ in w_1 and w_2, we also see that

$$R_q^{(m)}(w_1,w_2:\zeta)$$
$$=\sum_{n=0}^{\infty}\left\{\sum_{j=0}^{n}\binom{n}{j}w_1^jw_2^{n-j}E_{n-j,\zeta^{w_2},q^{w_2}}^{(m)}(w_1x)\sum_{k=0}^{j}T_{k,q^{w_1}}(w_2-1:\zeta^{w_1})\binom{j}{k}E_{j-k,\zeta^{w_1},q^{w_1}}^{(m-1)}(w_2y)\right\}\frac{t^n}{n!}. \tag{2.7}$$

Comparing the coefficients on the both sides of (2.6) and (2.7), we obtain an identity for the twisted q-Euler polynomials of higher order as follows.

Theorem 2.1. *Let $w_1, w_2 \in \mathbb{N}$ with $w_1 \equiv 1(\operatorname{mod} 2)$ and $w_2 \equiv 1(\operatorname{mod} 2)$.*
For $n \in \mathbb{Z}_+$ and $m \in \mathbb{N}$, we have

$$\begin{aligned}&\sum_{j=0}^{n}\binom{n}{j}w_2^j w_1^{n-j}E^{(m)}_{n-j,\zeta^{w_1},q^{w_1}}(w_2x)\sum_{k=0}^{j}T_{k,q^{w_2}}(w_1-1:\zeta^{w_2})\binom{j}{k}E^{(m-1)}_{j-k,\zeta^{w_2},q^{w_2}}(w_1y)\\&\quad=\sum_{j=0}^{n}\binom{n}{j}w_1^j w_2^{n-j}E^{(m)}_{n-j,\zeta^{w_2},q^{w_2}}(w_1x)\sum_{k=0}^{j}T_{k,q^{w_1}}(w_2-1:\zeta^{w_1})\binom{j}{k}E^{(m-1)}_{j-k,\zeta^{w_1},q^{w_1}}(w_2y).\end{aligned}\tag{2.8}$$

Remark 2.2. Taking $m=1$ and $y=0$ in Theorem 2.1, we can derive the following identity:

$$\begin{aligned}&\sum_{j=0}^{n}\binom{n}{j}w_2^j w_1^{n-j}E_{n-j,\zeta^{w_1},q^{w_1}}(w_2x)\sum_{k=0}^{j}T_{k,q^{w_2}}(w_1-1:\zeta^{w_2})\binom{j}{k}\\&\quad=\sum_{j=0}^{n}\binom{n}{j}w_1^j w_2^{n-j}E_{n-j,\zeta^{w_2},q^{w_2}}(w_1x)\sum_{k=0}^{j}T_{k,q^{w_1}}(w_2-1:\zeta^{w_1})\binom{j}{k}.\end{aligned}\tag{2.9}$$

Moreover, if we take $x=0$ and $y=0$ in Theorem 2.1, then we have the following identity for the twisted q-Euler numbers of higher order.

Corollary 2.3. *Let $w_1, w_2 \in \mathbb{N}$ with $w_1 \equiv 1(\operatorname{mod} 2)$ and $w_2 \equiv 1(\operatorname{mod} 2)$. For $n \in \mathbb{Z}_+$ and $m \in \mathbb{N}$, we have*

$$\begin{aligned}&\sum_{j=0}^{n}\binom{n}{j}w_2^j w_1^{n-j}E^{(m)}_{n-j,\zeta^{w_1},q^{w_1}}\sum_{k=0}^{j}T_{k,q^{w_2}}(w_1-1:\zeta^{w_2})\binom{j}{k}E^{(m-1)}_{j-k,\zeta^{w_2},q^{w_2}}\\&\quad=\sum_{j=0}^{n}\binom{n}{j}w_1^j w_2^{n-j}E^{(m)}_{n-j,\zeta^{w_2},q^{w_2}}\sum_{k=0}^{j}T_{k,q^{w_1}}(w_2-1:\zeta^{w_1})\binom{j}{k}E^{(m-1)}_{j-k,\zeta^{w_1},q^{w_1}}.\end{aligned}\tag{2.10}$$

We also note that taking $m=1$ in Corollary 1 shows the following identity:

$$\begin{aligned}&\sum_{j=0}^{n}\binom{n}{j}w_2^j w_1^{n-j}E_{n-j,\zeta^{w_1},q^{w_1}}\sum_{k=0}^{j}T_{k,q^{w_2}}(w_1-1:\zeta^{w_2})\binom{j}{k}\\&\quad=\sum_{j=0}^{n}\binom{n}{j}w_1^j w_2^{n-j}E_{n-j,\zeta^{w_2},q^{w_2}}\sum_{k=0}^{j}T_{k,q^{w_1}}(w_2-1:\zeta^{w_1})\binom{j}{k}.\end{aligned}\tag{2.11}$$

Now we will derive another interesting identities for the twisted q-Euler numbers and polynomials of higher order. From (2.3), we can derive that

$$\begin{aligned}
&R_q^{(m)}(w_1, w_2 : \zeta)\\
&= \left\{\sum_{i=0}^{w_1-1}(-1)^i q^{w_2 i}\zeta^{w_2 i}\right\}\left\{\sum_{k=0}^{\infty}\left(E_{k,\zeta^{w_1},q^{w_1}}^{(m)}\left(\frac{w_2}{w_1}i + w_2 x\right)w_1^k\frac{t^k}{k!}\right\}\left\{\sum_{l=0}^{\infty}\left(E_{l,\zeta^{w_2},q^{w_2}}^{(m-1)}(w_1 y)w_2^l\right)\frac{t^l}{l!}\right\}\\
&= \sum_{n=0}^{\infty}\left\{\sum_{k=0}^{n}\binom{n}{k}w_1^k w_2^{n-k}E_{n-k,\zeta^{w_2},q^{w_2}}^{(m-1)}(w_1 y)\sum_{i=0}^{w_1-1}(-1)^i\zeta^{w_2 i}q^{w_2 i}E_{k,\zeta^{w_1},q^{w_1}}^{(m)}\left(w_2 x + \frac{w_2}{w_1}i\right)\right\}\frac{t^n}{n!}.
\end{aligned} \tag{2.12}$$

From the symmetry of $R_q^{(m)}(w_1, w_2 : \zeta)$ in w_1 and w_2, we see that

$$\begin{aligned}
&R_q^{(m)}(w_1, w_2 : \zeta)\\
&= \sum_{n=0}^{\infty}\left\{\sum_{k=0}^{n}\binom{n}{k}w_2^k w_1^{n-k}E_{n-k,\zeta^{w_1},q^{w_1}}^{(m-1)}(w_2 y)\sum_{i=0}^{w_2-1}(-1)^i\zeta^{w_1 i}q^{w_1 i}E_{k,\zeta^{w_2},q^{w_2}}^{(m)}\left(w_1 x + \frac{w_1}{w_2}i\right)\right\}\frac{t^n}{n!}.
\end{aligned} \tag{2.13}$$

Comparing the coefficients on the both sides of (2.12) and (2.13), we obtain the following theorem which shows the relationship between the power sums and the twisted q-Euler polynomials.

Theorem 2.4. *Let $w_1, w_2 \in \mathbb{N}$ with $w_1 \equiv 1 (\mathrm{mod}\, 2)$ and $w_2 \equiv 1 (\mathrm{mod}\, 2)$. For $n \in \mathbb{Z}_+$ and $m \in \mathbb{N}$, we have*

$$\begin{aligned}
&\sum_{k=0}^{n}\binom{n}{k}w_1^k w_2^{n-k}E_{n-k,\zeta^{w_2},q^{w_2}}^{(m-1)}(w_1 y)\sum_{i=0}^{w_1-1}(-1)^i\zeta^{w_2 i}q^{w_2 i}E_{k,\zeta^{w_1},q^{w_1}}^{(m)}\left(w_2 x + \frac{w_2}{w_1}i\right)\\
&\quad= \sum_{k=0}^{n}\binom{n}{k}w_2^k w_1^{n-k}E_{n-k,\zeta^{w_1},q^{w_1}}^{(m-1)}(w_2 y)\sum_{i=0}^{w_2-1}(-1)^i\zeta^{w_1 i}q^{w_1 i}E_{k,\zeta^{w_2},q^{w_2}}^{(m)}\left(w_1 x + \frac{w_1}{w_2}i\right).
\end{aligned} \tag{2.14}$$

Remark 2.5. Let $m = 1$ and $y = 0$ in Theorem 2. Then it follows that

$$\begin{aligned}
&\sum_{k=0}^{n}\binom{n}{k}w_1^k w_2^{n-k}\sum_{i=0}^{w_1-1}(-1)^i\zeta^{w_2 i}q^{w_2 i}E_{k,\zeta^{w_1},q^{w_1}}\left(w_2 x + \frac{w_2}{w_1}i\right)\\
&\quad= \sum_{k=0}^{n}\binom{n}{k}w_2^k w_1^{n-k}\sum_{i=0}^{w_2-1}(-1)^i\zeta^{w_1 i}q^{w_1 i}E_{k,\zeta^{w_2},q^{w_2}}\left(w_1 x + \frac{w_1}{w_2}i\right).
\end{aligned} \tag{2.15}$$

Moreover, if we take $x = 0$ and $y = 0$ in Theorem 2.4, then we have the following identity for the twisted q-Euler numbers of higher order.

Corollary 2.6. *Let $w_1, w_2 \in \mathbb{N}$ with $w_1 \equiv 1 (\mathrm{mod}\, 2), w_2 \equiv 1 (\mathrm{mod}\, 2)$. For $n \in \mathbb{Z}_+$ and $m \in \mathbb{N}$, we have*

$$\begin{aligned}\sum_{k=0}^{n}\binom{n}{k} w_1^k w_2^{n-k} E_{n-k,\zeta^{w_2},q^{w_2}}^{(m-1)} \sum_{i=0}^{w_1-1}(-1)^i \zeta^{w_2 i} q^{w_2 i} E_{k,\zeta^{w_1},q^{w_1}}^{(m)}\left(\frac{w_2}{w_1} i\right) \\ = \sum_{k=0}^{n}\binom{n}{k} w_2^k w_1^{n-k} E_{n-k,\zeta^{w_1},q^{w_1}}^{(m-1)} \sum_{i=0}^{w_2-1}(-1)^i \zeta^{w_1 i} q^{w_1 i} E_{k,\zeta^{w_2},q^{w_2}}^{(m)}\left(\frac{w_1}{w_2} i\right).\end{aligned} \tag{2.16}$$

If we take $m = 1$ in Corollary 2.3, we derive the following identity for the twisted q-Euler polynomials: for $w_1, w_2 \in \mathbb{N}$ with $w_1 \equiv 1 (\mathrm{mod}\, 2), w_2 \equiv 1 (\mathrm{mod}\, 2)$, and $n \in \mathbb{Z}_+$,

$$\begin{aligned}\sum_{k=0}^{n}\binom{n}{k} w_1^k w_2^{n-k} \sum_{i=0}^{w_1-1}(-1)^i \zeta^{w_2 i} q^{w_2 i} E_{k,\zeta^{w_1},q^{w_1}}\left(\frac{w_2}{w_1} i\right) \\ = \sum_{k=0}^{n}\binom{n}{k} w_2^k w_1^{n-k} \sum_{i=0}^{w_2-1}(-1)^i \zeta^{w_1 i} q^{w_1 i} E_{k,\zeta^{w_2},q^{w_2}}\left(\frac{w_1}{w_2} i\right).\end{aligned} \tag{2.17}$$

Remark 2.7. If $q = 1$, we can observe the result of [15].

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