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Research Article

A Note on Symmetric Properties of the Twisted q-Bernoulli Polynomials and the Twisted Generalized q-Bernoulli Polynomials

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We define the twisted q-Bernoulli polynomials and the twisted generalized q-Bernoulli polynomials attached to χ of higher order and investigate some symmetric properties of them. Furthermore, using these symmetric properties of them, we can obtain some relationships between twisted q-Bernoulli numbers and polynomials and between twisted generalized q-Bernoulli numbers and polynomials.

1. Introduction

Let p be a fixed prime number. Throughout this paper $\mathbb{Z}_p$, $\mathbb{Q}_p$, and $\mathbb{C}_p$ will, respectively, denote the ring of p-adic rational integers, the field of p-adic rational numbers, and the completion of algebraic closure of $\mathbb{Q}_p$. Let v_p be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-v_p(p)} = p^{-1}$. When one talks of q-extension, q is variously considered as an indeterminate, a complex number $q \in \mathbb{C}$, or a p-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$, one normally assumes $|q| < 1$. If $q \in \mathbb{C}_p$, then we assume $|q-1|_p < p^{-1/(p-1)}$, so that $q^x = \exp(x \log q)$ for $|x|_p \leq 1$ (cf. [1–32]).

For $N, d \in \mathbb{N}$, we set

$$X = X_d = \frac{\lim_{\overleftarrow{N}} \mathbb{Z}}{dp^N \mathbb{Z}}, \qquad X_1 = \mathbb{Z}_p \tag{1.1}$$

(see [1–13]). The Bernoulli numbers B_n and polynomials $B_n(x)$ are defined by the generating function as

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}, \tag{1.2}$$

$$\frac{t}{e^t - 1} e^{xt} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!} \tag{1.3}$$

(cf. [17, 18, 21, 24, 26]). Let UD(X) be the set of uniformly differentiable functions on X. For $f \in \mathrm{UD}(X)$, the p-adic invariant integral on $\mathbb{Z}_p$ is defined as

$$I(f) = \int_X f(x)dx = \lim_{N \to \infty} \frac{1}{dp^N} \sum_{x=0}^{dp^N - 1} f(x). \tag{1.4}$$

Note that $\int_X f(x)dx = \int_{\mathbb{Z}_p} f(x)dx$ (see [27]). Let $f_n(x)$ be a translation with $f_n(x) = f(x+n)$. We note that

$$I(f_n) = I(f) + \sum_{i=0}^{n-1} f'(i) \tag{1.5}$$

(cf. [1–32]). Kim [18] studied the symmetric properties of the q-Bernoulli numbers and polynomials as follows:

$$\frac{t + \log q}{qe^t - 1} e^{xt} = \sum_{n=0}^{\infty} B_n^q(x) \frac{t^n}{n!}. \tag{1.6}$$

In this paper, we define the twisted q-Bernoulli polynomials and the twisted generalized q-Bernoulli polynomials attached to χ of higher order and investigate some symmetric properties of them. Furthermore, using these symmetric properties of them, we can obtain some relationships between the twisted q-Bernoulli numbers and polynomials and between the twisted generalized q-Bernoulli numbers and polynomials attached to χ of higher order.

2. The Twisted q-Bernoulli Polynomials

Let $C_{p^\infty} = \bigcup_{n\geq 1} C_{p^n} = \lim_{n\to\infty} C_{p^n}$ be the locally constant space, where $C_{p^n} = \{\xi \mid \xi^{p^n} = 1\}$ is the cyclic group of order p^n. For $w \in C_{p^\infty}$, we denote the locally constant function by

$$\phi_w : \mathbb{Z}_p \longrightarrow \mathbb{C}_p, \qquad x \longmapsto w^x \tag{2.1}$$

(cf. [2, 3, 21, 24]). If we take $f(x) = \phi_w(x) q^x e^{tx}$, then

$$\int_{\mathbb{Z}_p} e^{xt} w^x q^x dx = \frac{\log q + t}{wqe^t - 1}. \tag{2.2}$$

Now we define the q-extension of twisted Bernoulli numbers and polynomials as follows:

$$\frac{\log q + t}{wqe^t - 1} = \sum_{n=0}^{\infty} B_{n,w}^{q} \frac{t^n}{n!}, \tag{2.3}$$

$$\frac{\log q + t}{wqe^t - 1} e^{tx} = \sum_{n=0}^{\infty} B_{n,w}^{q}(x) \frac{t^n}{n!} \tag{2.4}$$

(see [31]). From (1.5), (2.2), (2.3), and (2.4), we can derive

$$\int_{\mathbb{Z}_p} w^y q^y (x+y)^n dy = B_{n,w}^{q}(x), \qquad \int_{\mathbb{Z}_p} w^y q^y y^n dy = B_{n,w}^{q}. \tag{2.5}$$

By (1.5), we can see that

$$\begin{aligned}
&\frac{1}{\log q + t}\left(\int_{\mathbb{Z}_p} w^{n+x} q^{n+x} e^{(n+x)t} dx - \int_{\mathbb{Z}_p} w^x q^x e^{xt} dx\right) \\
&\quad = \frac{w^n q^n e^{nt} - 1}{t + \log q} \int_{\mathbb{Z}_p} w^x q^x e^{xt} dx \\
&\quad = \frac{w^n q^n e^{nt} - 1}{wqe^t - 1} \\
&\quad = \sum_{i=0}^{n-1} w^i q^i e^{it} \\
&\quad = \sum_{k=0}^{\infty} \left(\sum_{i=0}^{n-1} i^k w^i q^i\right) \frac{t^k}{k!}.
\end{aligned} \tag{2.6}$$

In (1.4), it is easy to show that

$$\frac{1}{\log q+t}\left(\int_{\mathbb{Z}_p} w^{n+x}q^{n+x}e^{(n+x)t}dx-\int_{\mathbb{Z}_p} w^{x}q^{x}e^{xt}dx\right)=\frac{n\int_{\mathbb{Z}_p} w^{x}q^{x}e^{xt}dx}{\int_{\mathbb{Z}_p} w^{nx}q^{nx}e^{nxt}dx}. \tag{2.7}$$

For each integer $k \geq 0$, let

$$S_{k,w}^{q}(n)=0^k+1^k wq+2^k w^2q^2+\cdots+n^k w^n q^n. \tag{2.8}$$

From (2.6), (2.7), and (2.8), we derive

$$\frac{1}{\log q+t}\left(\int_{\mathbb{Z}_p} w^{n+x}q^{n+x}e^{(n+x)t}dx-\int_{\mathbb{Z}_p} w^{x}q^{x}e^{xt}dx\right)=\frac{n\int_{\mathbb{Z}_p} w^{x}q^{x}e^{xt}dx}{\int_{\mathbb{Z}_p} w^{nx}q^{nx}e^{nxt}dx}=\sum_{k=0}^{\infty}S_{k,w}^{q}(n-1)\frac{t^k}{k!}. \tag{2.9}$$

From (2.9), we note that

$$B_{k,w}^{q}(n)-B_{k,w}^{q}=kS_{k-1,w}^{q}(n-1)+\log qS_{k,w}^{q}(n-1), \tag{2.10}$$

for all $k, n \in \mathbb{N}$. Let $u_1, u_2 \in \mathbb{N}$ and $w \in C_{p^\infty}$; then we have

$$\frac{\int_{\mathbb{Z}_p} w^{u_1x_1+u_2x_2}q^{u_1x_1+u_2x_2}e^{u_1x_1+u_2x_2}dx_1dx_2}{\int_{\mathbb{Z}_p} w^{u_1u_2x}q^{u_1u_2x}e^{u_1u_2xt}dx}=(t+\log q)\frac{w^{u_1u_2}q^{u_1u_2}e^{u_1t}-1}{w^{u_2}q^{u_2}e^{u_2t}-1}. \tag{2.11}$$

By (2.9), we see that

$$\frac{u_1\int_{\mathbb{Z}_p} w^{x}q^{x}e^{xt}dx}{\int_{\mathbb{Z}_p} w^{u_1x}q^{u_1x}e^{u_1xt}dx}=\sum_{l=0}^{\infty}\left(\sum_{k=0}^{u_1-1}k^l w^k q^k\right)\frac{t^l}{l!}=\sum_{l=0}^{\infty}S_{l,w}^{q}(u_1-1)\frac{t^l}{l!}. \tag{2.12}$$

Let $T_w(u_1, u_2; x, t)$ be as follows:

$$T_w(u_1,u_2;x,t)=\frac{\int_{\mathbb{Z}_p} w^{u_1x_1+u_2x_2}q^{u_1x_1+u_2x_2}e^{(u_1x_1+u_2x_2+u_1u_2x)t}dx_1dx_2}{\int_{\mathbb{Z}_p} w^{u_1u_2x}q^{u_1u_2x}e^{u_1u_2xt}dx}. \tag{2.13}$$

Then we have

$$T_w(u_1,u_2;x,t)=\frac{(t+\log q)e^{u_1u_2t}\left(w^{u_1u_2}q^{u_1u_2}e^{u_1u_2t}-1\right)}{\left(w^{u_1}q^{u_1}e^{u_1t}-1\right)\left(w^{u_2}q^{u_2}e^{u_2t}-1\right)}. \tag{2.14}$$

From (2.13), we derive

$$T_w(u_1,u_2;x,t)=\left(\frac{1}{u_1}\int_{\mathbb{Z}_p}w^{u_1x_1}q^{u_1x_1}e^{u_1(x_1+u_2x)t}dx_1\right)\left(\frac{u_1\int_{\mathbb{Z}_p}w^{u_2x_2}q^{u_2x_2}e^{u_2x_2t}}{\int_{\mathbb{Z}_p}w^{u_1u_2x}q^{u_1u_2x}e^{u_1u_2xt}dx}\right). \tag{2.15}$$

By (2.4), (2.12), and (2.15), we can see that

$$\begin{aligned}T_w(u_1,u_2;x,t)&=\frac{1}{u_1}\left(\sum_{i=0}^{\infty}B_{i,w^{u_1}}^{q^{u_1}}(u_2x)\frac{u_1^it^i}{i!}\right)\left(\sum_{l=0}^{\infty}S_{l,w^{u_2}}^{q^{u_2}}(u_1-1)\frac{u_2^lt^l}{l!}\right)\\&=\sum_{n=0}^{\infty}\left(\sum_{i=0}^{n}\binom{n}{i}B_{i,w^{u_1}}^{q^{u_1}}(u_2x)S_{n-i,w^{u_2}}^{q^{u_2}}(u_1-1)u_1^{i-1}u_2^{n-i}\right)\frac{t^n}{n!}.\end{aligned} \tag{2.16}$$

By the symmetry of p-adic invariant integral on $\mathbb{Z}_p$, we also see that

$$\begin{aligned}T_w(u_1,u_2;x,t)&=\left(\frac{1}{u_2}\int_{\mathbb{Z}_p}w^{u_2x_2}q^{u_2x_2}e^{u_2(x_2+u_1x)t}dx_2\right)\left(\frac{u_2\int_{\mathbb{Z}_p}w^{u_1x_1}q^{u_1x_1}e^{u_1x_1t}}{\int_{\mathbb{Z}_p}w^{u_1u_2x}q^{u_1u_2x}e^{u_1u_2xt}dx}\right)\\&=\sum_{n=0}^{\infty}\left(\sum_{i=0}^{n}\binom{n}{i}B_{i,w^{u_2}}^{q^{u_2}}(u_1x)S_{n-i,w^{u_1}}^{q^{u_1}}(u_2-1)u_2^{i-1}u_1^{n-i}\right)\frac{t^n}{n!}.\end{aligned} \tag{2.17}$$

By comparing the coefficients of $t^n/n!$ on both sides of (2.16) and (2.17), we obtain the following theorem.

Theorem 2.1. *Let $u_1,u_2,n\in\mathbb{N}$. Then for all $x\in\mathbb{Z}_p$,*

$$\sum_{i=0}^{n}\binom{n}{i}B_{i,w^{u_1}}^{q^{u_1}}(u_2x)S_{n-i,w^{u_2}}^{q^{u_2}}(u_1-1)u_1^{i-1}u_2^{n-i}=\sum_{i=0}^{n}\binom{n}{i}B_{i,w^{u_2}}^{q^{u_2}}(u_1x)S_{n-i,w^{u_1}}^{q^{u_1}}(u_2-1)u_2^{i-1}u_1^{n-i}, \tag{2.18}$$

where $\binom{n}{i}$ is the binomial coefficient.

From Theorem 2.1, if we take $u_2=1$, then we have the following corollary.

Corollary 2.2. *For $m\geq 0$, one we has*

$$B_{i,w}^{q}(u_1x)=\sum_{i=0}^{n}\binom{n}{i}B_{i,w^{u_1}}^{q^{u_1}}(x)S_{n-i,w}^{q}(u_1-1)u_1^{i-1}, \tag{2.19}$$

where $\binom{n}{i}$ is the binomial coefficient.

By (2.17), (2.18), and (2.19), we can see that

$$
\begin{aligned}
T_w(u_1,u_2;x,t) &= \left(\frac{e^{u_1u_2xt}}{u_1}\int_{\mathbb{Z}_p} w^{u_1x}q^{u_1x_1}e^{u_1x_1t}dx_1\right)\left(\frac{u_1\int_{\mathbb{Z}_p} w^{u_2x_2}q^{u_2x_2}e^{u_2x_2t}dx_2}{\int_{\mathbb{Z}_p} w^{u_1u_2x}q^{u_1u_2x}e^{u_1u_2xt}dx}\right)\\
&= \left(\frac{e^{u_1u_2xt}}{u_1}\int_{\mathbb{Z}_p} w^{u_1x}q^{u_1x_1}e^{u_1x_1t}dx_1\right)\left(\sum_{i=0}^{u_1-1} w^{u_2i}q^{u_2i}e^{u_2it}\right)\\
&= \frac{1}{u_1}\sum_{i=0}^{u_1-1} w^{u_2i}q^{u_2i}\int_{\mathbb{Z}_p} w^{u_1x}q^{u_1x}e^{(x_1+u_2x+(u_2/u_1)i)tu_1}dx_1\\
&= \sum_{n=0}^{\infty}\sum_{i=0}^{u_1-1} B_{n,w^{u_1}}^{q^{u_1}}\left(u_2x+\frac{u_2}{u_1}i\right)u_1^{n-1}w^{u_2i}q^{u_2i}\frac{t^n}{n!}.
\end{aligned}
\tag{2.20}
$$

From the symmetry of $T_w(u_1,u_2;x,t)$, we can also derive

$$
T_w(u_1,u_2;x,t) = \sum_{n=0}^{\infty}\sum_{i=0}^{u_2-1} B_{n,w^{u_2}}^{q^{u_2}}\left(u_1x+\frac{u_1}{u_2}i\right)u_2^{n-1}w^{u_1i}q^{u_1i}\frac{t^n}{n!}. \tag{2.21}
$$

By comparing the coefficients of $t^n/n!$ on both sides of (2.20) and (2.21), we obtain the following theorem.

Theorem 2.3. *For $m \in \mathbb{Z}_+$, $u_1, u_2 \in \mathbb{N}$, we have*

$$
\sum_{i=0}^{u_1-1} B_{n,w^{u_1}}^{q^{u_1}}\left(u_2x+\frac{u_2}{u_1}i\right)u_1^{n-1}w^{u_2i}q^{u_2i} = \sum_{i=0}^{u_2-1} B_{n,w^{u_2}}^{q^{u_2}}\left(u_1x+\frac{u_1}{u_2}i\right)u_2^{n-1}w^{u_1i}q^{u_1i}. \tag{2.22}
$$

We note that by setting $u_2 = 1$ in Theorem 2.3, we get the following multiplication theorem for the twisted q-Bernoulli polynomials.

Theorem 2.4. *For $m \in \mathbb{Z}_+$, $u_1 \in \mathbb{N}$, one has*

$$
B_{n,w}^{q}(u_1x) = u_1^{n-1}\sum_{i=0}^{u_1-1} B_{n,w^{u_1}}^{q^{u_1}}\left(x+\frac{i}{u_1}\right)w^iq^i. \tag{2.23}
$$

Remark 2.5. [18], Kim suggested open questions related to finding symmetric properties for Carlitz q-Bernoulli numbers. In this paper, we give the symmetric property for q-Bernoulli numbers in the viewpoint to give the answer of Kim's open questions.

3. The Twisted Generalized Bernoulli Polynomials Attached to χ of Higher Order

In this section, we consider the generalized Bernoulli numbers and polynomials and then define the twisted generalized Bernoulli polynomials attached to χ of higher order by using multivariate p-adic invariant integrals on $\mathbb{Z}_p$. Let χ be Dirichlet's character with conductor $d \in \mathbb{N}$. Then the generalized Bernoulli numbers $B_{n,\chi}$ and polynomials $B_{n,\chi}(x)$ attached to χ are defined as

$$\frac{t\sum_{a=0}^{d-1}\chi(a)e^{at}}{e^{dt}-1} = \sum_{n=0}^{\infty} B_{n,\chi}\frac{t^n}{n!}, \tag{3.1}$$

$$\frac{t\sum_{a=0}^{d-1}\chi(a)e^{at}}{e^{dt}-1}e^{xt} = \sum_{n=0}^{\infty} B_{n,\chi}(x)\frac{t^n}{n!} \tag{3.2}$$

(cf. [2, 18, 23, 27]).

Let $C_{p^\infty} = \bigcup_{n\geq 1} C_{p^n} = \lim_{n\to\infty} C_{p^n}$ be the locally constant space, where $C_{p^n} = \{w \mid w^{p^n} = 1\}$ is the cyclic group of order p^n. For $w \in C_{p^\infty}$, we denote the locally constant function by

$$\phi_w : \mathbb{Z}_p \longrightarrow \mathbb{C}_p, \qquad x \longrightarrow w^x \tag{3.3}$$

(cf. [2, 3, 21, 23, 24]). If we take $f(x) = \chi(x)e^{tx}\phi_w(x)q^x$, for $q \in \mathbb{C}_p$ with $|q-1|_p < 1$, then it is obvious from (3.1) that

$$\int_X \chi(x)e^{tx}w^x q^x dx = \frac{(t+\log q)\sum_{a=0}^{d-1}\chi(a)w^a q^a e^{at}}{w^d q^d e^{dt}-1}. \tag{3.4}$$

Now we define the twisted generalized Bernoulli numbers $B_{n,\chi,w}^q$ and polynomials $B_{n,\chi,w}^q(x)$ attached to χ as follows:

$$\frac{(t+\log q)\sum_{a=0}^{d-1}\chi(a)w^a q^a e^{at}}{w^d q^d e^{dt}-1} = \sum_{n=0}^{\infty} B_{n,\chi,w}^q\frac{t^n}{n!}, \tag{3.5}$$

$$\frac{(t+\log q)\sum_{a=0}^{d-1}\chi(a)w^a q^a e^{at}e^{xt}}{w^d q^d e^{dt}-1} = \sum_{n=0}^{\infty} B_{n,\chi,w}^q(x)\frac{t^n}{n!} \tag{3.6}$$

for each $w \in C_{p^\infty}$ (see [31, 32]). By (3.5) and (3.6),

$$\begin{gathered}\int_X \chi(x)x^n w^x q^x dx = B_{n,\chi,w}^q,\\ \int_X \chi(y)(x+y)^n w^y q^y dy = B_{n,\chi,w}^q(x).\end{gathered} \tag{3.7}$$

Thus we have

$$\begin{aligned}
&\frac{1}{\log q+t}\left(\int_X \chi(x)e^{(nd+x)t}w^{n+x}q^{n+x}\,dx-\int_X \chi(x)e^{xt}w^{x}q^{x}\,dx\right)\\
&\quad=\frac{nd\int_X \chi(x)e^{xt}w^{x}q^{x}\,dx}{\int_X e^{ndxt}w^{ndx}q^{ndx}\,dx}\\
&\quad=\frac{w^{nd}q^{nd}e^{ndt}-1}{w^{d}q^{d}e^{dt}-1}\sum_{i=0}^{d-1}\chi(i)e^{it}w^{i}q^{i}.
\end{aligned}\tag{3.8}$$

Then

$$\begin{aligned}
&\frac{1}{\log q+t}\left(\int_X \chi(x)e^{(nd+x)t}w^{n+x}q^{n+x}\,dx-\int_X \chi(x)e^{xt}w^{x}q^{x}\,dx\right)\\
&\quad=\sum_{l=0}^{nd-1}\chi(l)e^{lt}w^{l}q^{l}=\sum_{k=0}^{\infty}\sum_{l=0}^{nd-1}\chi(l)l^{k}w^{l}q^{l}\frac{t^k}{k!}.
\end{aligned}\tag{3.9}$$

Let us define the p-adic twisted q-function $T_{k,w}^{q}(\chi,n)$ as follows:

$$T_{k,w}^{q}(\chi,n)=\sum_{l=0}^{n}\chi(l)l^{k}w^{l}q^{l}.\tag{3.10}$$

By (3.9) and (3.10), we see that

$$\frac{1}{\log q+t}\left(\int_X \chi(x)e^{(nd+x)t}w^{nd+x}q^{nd+x}\,dx-\int_X \chi(x)e^{xt}w^{x}q^{x}\,dx\right)=\sum_{k=0}^{\infty}T_{k,w}^{q}(\chi,nd-1)\frac{t^k}{k!}.\tag{3.11}$$

Thus,

$$\left(\int_X \chi(x)(nd+x)^{k}w^{n+x}q^{n+x}\,dx-\int_X \chi(x)x^{k}w^{x}q^{x}\,dx\right)=(t+\log q)T_{k,w}^{q}(\chi,nd-1),\tag{3.12}$$

for all $k,n,d\in\mathbb{N}$. This means that

$$B_{k,\chi,w}^{q}(nd)-B_{n,\chi,w}^{q}=(t+\log q)T_{k,w}^{q}(\chi,nd-1),\tag{3.13}$$

for all $k, n, d \in \mathbb{N}$. For all $u_1, u_2, d \in \mathbb{N}$, we have

$$\begin{aligned} &\frac{d\int_X\int_X \chi(x_1)\chi(x_2)e^{(w_1x_1+w_2x_2)t}w^{u_1x_1+u_2x_2}q^{u_1x_1+u_2x_2}dx_1dx_2}{\int_X e^{du_1u_2xt}w^{du_1u_2x}q^{du_1u_2x}dx} \\ &\quad= \frac{(t+\log q)\left(e^{du_1u_2t}w^{du_1u_2}q^{du_1u_2}-1\right)}{\left(e^{du_1t}w^{du_1}q^{du_1}-1\right)\left(e^{du_2t}w^{du_2}q^{du_2}-1\right)} \\ &\quad\quad\times\left(\sum_{a=0}^{d-1}\chi(a)e^{u_1at}w^{u_1a}q^{u_1a}\right)\left(\sum_{b=0}^{d-1}\chi(b)e^{u_2bt}w^{u_2b}q^{u_2b}\right). \end{aligned} \tag{3.14}$$

The twisted generalized Bernoulli numbers $B_{n,\chi,w}^{(k,q)}$ and polynomials $B_{n,\chi,w}^{(k,q)}(x)$ attached to χ of order k are defined as

$$\left(\frac{(t+\log q)\sum_{a=0}^{d-1}\chi(a)w^aq^ae^{at}}{w^dq^de^{dt}-1}\right)^k = \sum_{n=0}^{\infty}B_{n,\chi,w}^{(k,q)}\frac{t^n}{n!}, \tag{3.15}$$

$$\left(\frac{(t+\log q)\sum_{a=0}^{d-1}\chi(a)w^aq^ae^{at}}{w^dq^de^{dt}-1}\right)^k e^{xt} = \sum_{n=0}^{\infty}B_{n,\chi,w}^{(k,q)}(x)\frac{t^n}{n!} \tag{3.16}$$

for each $w \in C_{p^\infty}$. For $u_1, u_2 \in \mathbb{N}$, we set

$$\begin{aligned} &K_w^q(m,\chi;u_1,u_2) \\ &\quad= \frac{d\int_{X^m}\prod_{i=1}^m\chi(x_i)e^{(\sum_{i=1}^m x_i+u_2x)u_1t}w^{(\sum_{i=1}^m x_i+u_2x)u_1}q^{(\sum_{i=1}^m x_i+u_2x)u_1}dx_1\cdots dx_m}{\int_X e^{du_1u_2xt}w^{du_1u_2x}q^{du_1u_2x}dx} \\ &\quad\quad\times\int_{X^m}\prod_{i=1}^m\chi(x_i)e^{(\sum_{i=1}^m x_i+u_1y)u_2t}w^{(\sum_{i=1}^m x_i+u_1y)u_2}q^{(\sum_{i=1}^m x_i+u_1y)u_1}dx_1\cdots dx_m, \end{aligned} \tag{3.17}$$

where $\int_{X^m} f(x_1\cdots x_m)dx_1\cdots dx_m = \int_X\cdots\int_X f(x_1,\ldots,x_m)dx_1\cdots dx_m$. In (3.17), we note that $\mathrm{K}_w^q(m,\chi;u_1,u_2)$ is symmetric in u_1, u_2. From (3.17), we have

$$\begin{aligned} K_w^q(m,\chi;u_1,u_2) &= \int_{X^m}\prod_{i=1}^m\chi(x_i)e^{(\sum_{i=1}^m x_i)u_2t}w^{(\sum_{i=1}^m x_i)u_2}q^{(\sum_{i=1}^m x_i)u_2}dx_1\cdots dx_m \\ &\quad\times e^{u_1u_2xt}w^{u_1u_2x}q^{u_1u_2x}\left(\frac{d\int_X\chi(x_m)e^{u_2x_mt}w^{u_2x_m}q^{u_2x_m}dx_m}{\int_X e^{du_1u_2x}q^{du_1u_2x}dx}\right) \\ &\quad\times\int_{X^{m-1}}\prod_{i=1}^{m-1}\chi(x_i)e^{(\sum_{i=1}^{m-1} x_i)u_2t}w^{(\sum_{i=1}^{m-1} x_i)u_2}q^{(\sum_{i=1}^{m-1} x_i)u_2}dx_1\cdots dx_{m-1} \\ &\quad\times e^{u_1u_2yt}w^{u_1u_2y}q^{u_1u_2y}. \end{aligned} \tag{3.18}$$

Thus we can obtain

$$\frac{u_1 d\int_X \chi(x)e^{xt}w^x q^x dx}{\int_X e^{du_2xt}w^{du_2x}q^{du_2x}dx} = \sum_{k=0}^{\infty}\left(\sum_{i=0}^{u_1d-1}\chi(i)i^k w^i q^i\right)\frac{t^k}{k!} = \sum_{k=0}^{\infty}T_{k,w}^{q}(\chi, u_1 d-1)\frac{t^k}{k!},$$

$$\begin{aligned} &e^{u_1u_2xt}w^{u_1u_2x}q^{u_1u_2x}\int_{X^m}\prod_{i=1}^{m}\chi(x_i)e^{(\sum_{i=1}^m x_i)u_1t}w^{(\sum_{i=1}^m x_i)u_1}q^{(\sum_{i=1}^m x_i)u_1}dx_1\cdots dx_m \\ &\quad = e^{u_1u_2xt}w^{u_1u_2x}q^{u_1u_2x}\left(\frac{u_1}{e^{du_1t}w^{du_1}q^{du_1}-1}\sum_{a=0}^{d-1}\chi(a)e^{u_1at}w^{u_1a}q^{u_1a}\right) \\ &\quad = \sum_{n=0}^{\infty}B_{n,\chi,w}^{(m,q)}(u_2x)u_1^n\frac{t^n}{n!}. \end{aligned} \tag{3.19}$$

From (3.19), we derive

$$\begin{aligned} &K_w^q(m,\chi;u_1,u_2) \\ &\quad = \sum_{l=0}^{\infty}B_{l,\chi,w}^{(m,q)}(u_1x)u_1^l\frac{t^l}{l!}\sum_{k=0}^{\infty}T_{k,w}^{q}(\chi,u_1d-1)\frac{t^k}{k!}\left(\sum_{i=0}^{\infty}B_{i,\chi,w}^{(m-1,q)}(u_1y)\frac{u_2^i t^i}{i!}\right)\frac{1}{u_1} \\ &\quad = \sum_{n=0}^{\infty}\sum_{j=0}^{n}\binom{n}{j}u_2^j u_1^{n-j-1}B_{n-j,\chi,w}^{(m,q)}(u_2x)\times\sum_{k=0}^{j}T_{k,w}^{q}(\chi,u_1d-1)\binom{j}{k}B_{j-k,\chi,w}^{(m-1,q)}(u_1y)\frac{t^n}{n!}. \end{aligned} \tag{3.20}$$

By the symmetry of $K_w^q(m,\chi;u_1,u_2)$ in u_1 and u_2, we can see that

$$\begin{aligned} &K_w^q(m,\chi;u_1,u_2) \\ &\quad = \sum_{n=0}^{\infty}\sum_{j=0}^{n}\binom{n}{j}u_1^j u_2^{n-j-1}B_{n-j,\chi,w}^{(m,q)}(u_1x)\times\sum_{k=0}^{j}T_{k,w}^{q}(\chi,u_2d-1)\binom{j}{k}B_{j-k,\chi,w}^{(m-1,q)}(u_2y)\frac{t^n}{n!}. \end{aligned} \tag{3.21}$$

By comparing the coefficients on both sides of (3.20) and (3.21), we see the following theorem.

Theorem 3.1. *For $d, u_1, u_2, m \in \mathbb{N}$, $n \in \mathbb{Z}$, one has*

$$\begin{aligned} &\sum_{j=0}^{n}\binom{n}{j}u_2^j u_1^{n-j-1}B_{n-j,\chi,w}^{(m,q)}(u_2x)\sum_{k=0}^{j}T_{k,w}^{q}(\chi,u_1d-1)\binom{j}{k}B_{j-k,\chi,w}^{(m-1,q)}(u_1y) \\ &\quad = \sum_{j=0}^{n}\binom{n}{j}u_1^j u_2^{n-j-1}B_{n-j,\chi,w}^{(m,q)}(u_1x)\sum_{k=0}^{j}T_{k,w}^{q}(\chi,u_2d-1)\binom{j}{k}B_{j-k,\chi,w}^{(m-1,q)}(u_2y). \end{aligned} \tag{3.22}$$

Remark 3.2. If we take $y = 0$ and $m = 1$ in (3.22), then we have

$$\begin{aligned}&\sum_{j=0}^{n}\binom{n}{j}u_2^{j}u_1^{n-j-1}B_{n-j,\chi,w}^{q}(u_2x)\sum_{k=0}^{j}T_{k,w}^{q}(\chi,u_1d-1)\binom{j}{k}\\&\quad=\sum_{j=0}^{n}\binom{n}{j}u_1^{j}u_2^{n-j-1}B_{n-j,\chi,w}^{q}(u_1x)\sum_{k=0}^{j}T_{k,w}^{q}(\chi,u_2d-1)\binom{j}{k}.\end{aligned}\tag{3.23}$$

Now we can also calculate

$$K_w^q(m,\chi;u_1,u_2)=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}u_1^{k-1}u_2^{n-k}B_{n-k,\chi,w}^{(m-1,q)}(u_1y)\sum_{i=0}^{du_1-1}B_{i,\chi,w}^{(m,q)}\left(u_2x+\frac{u_2}{u_1}i\right)\right)\frac{t^n}{n!}.\tag{3.24}$$

From the symmetric property of $K_w^q(m,\chi;u_1,u_2)$ in u_1 and u_2, we derive

$$K_w^q(m,\chi;u_1,u_2)=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}u_2^{k-1}u_1^{n-k}B_{n-k,\chi,w}^{(m-1,q)}(u_2y)\sum_{i=0}^{du_2-1}B_{i,\chi,w}^{(m,q)}\left(u_1x+\frac{u_1}{u_2}i\right)\right)\frac{t^n}{n!}.\tag{3.25}$$

By comparing the coefficients on both sides of (3.24) and (3.26), we obtain the following theorem.

Theorem 3.3. *For $d,u_1,u_2,m\in\mathbb{N}$, $n\in\mathbb{Z}$, we have*

$$\begin{aligned}&\sum_{k=0}^{n}\binom{n}{k}u_1^{k-1}u_2^{n-k}B_{n-k,\chi,w}^{(m-1,q)}(u_1y)\sum_{i=0}^{du_1-1}B_{k,\chi,w}^{(m,q)}\left(u_2x+\frac{u_2}{u_1}i\right)\\&\quad=\sum_{k=0}^{n}\binom{n}{k}u_2^{k-1}u_1^{n-k}B_{n-k,\chi,w}^{(m-1,q)}(u_2y)\sum_{i=0}^{du_2-1}B_{k,\chi,w}^{(m,q)}\left(u_1x+\frac{u_1}{u_2}i\right).\end{aligned}\tag{3.26}$$

Remark 3.4. If we take $y = 0$ and $m = 1$ in (3.26), then one has

$$u_1^{n-1}\sum_{i=0}^{du_1-1}B_{n,\chi,w}^{q}\left(u_2x+\frac{u_2}{u_1}i\right)=u_2^{n-1}\sum_{i=0}^{du_2-1}B_{n,\chi,w}^{q}\left(u_1x+\frac{u_1}{u_2}i\right).\tag{3.27}$$

Remark 3.5. In our results for $q = 1$, we can also derive similar results, which were treated in [27]. In this paper, we used the p-adic integrals to derive the symmetric properties of the q-Bernoulli polynomials. By using the symmetric properties of p-adic integral on X, we can easily derive many interesting symmetric properties related to Bernoulli numbers and polynomials.

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