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Research Article

Nonoscillation of First-Order Dynamic Equations with Several Delays

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For dynamic equations on time scales with positive variable coefficients and several delays, we prove that nonoscillation is equivalent to the existence of a positive solution for the generalized characteristic inequality and to the positivity of the fundamental function. Based on this result, comparison tests are developed. The nonoscillation criterion is illustrated by examples which are neither delay-differential nor classical difference equations.

1. Introduction

Oscillation of first-order delay-difference and differential equations has been extensively studied in the last two decades. As is well known, most results for delay differential equations have their analogues for delay difference equations. In [1], Hilger revealed this interesting connection, and initiated studies on a new *time-scale theory*. With this new theory, it is now possible to unify most of the results in the discrete and the continuous calculus; for instance, some results obtained separately for delay difference equations and delay-differential equations can be incorporated in the general type of equations called *dynamic equations*.

The objective of this paper is to unify some results obtained in [2, 3] for the delay difference equation

$$\Delta x(t) + \sum_{i=1}^{n} A_i(t) x(\alpha_i(t)) = 0 \quad \text{for } t \in \{t_0, t_0 + 1, \dots\}, \tag{1.1}$$

where Δ is the forward difference operator defined by $\Delta x(t) := x(t+1) - x(t)$, and the delay differential equation

$$x'(t) + \sum_{i=1}^{n} A_i(t)x(\alpha_i(t)) = 0 \quad \text{for } t \in [t_0, \infty). \tag{1.2}$$

Although we further assume familiarity of readers with the notion of time scales, we would like to mention that any nonempty, closed subset $\mathbb{T}$ of $\mathbb{R}$ is called a *time scale*, and that the *forward jump operator* $\sigma : \mathbb{T} \to \mathbb{T}$ is defined by $\sigma(t) := (t, \infty)_{\mathbb{T}}$ for $t \in \mathbb{T}$, where the interval with a subscript $\mathbb{T}$ is used to denote the intersection of the real interval with the set $\mathbb{T}$. Similarly, the *backward jump operator* $\rho : \mathbb{T} \to \mathbb{T}$ is defined to be $\rho(t) := \sup(-\infty, t)_{\mathbb{T}}$ for $t \in \mathbb{T}$, and the *graininess* $\mu : \mathbb{T} \to \mathbb{R}_0^+$ is given by $\mu(t) := \sigma(t) - t$ for $t \in \mathbb{T}$. The readers are referred to [4] for an introduction to the time-scale calculus.

Let us now present some oscillation and nonoscillation results on delay dynamic equations, and from now on, we will without further more mentioning suppose that the time scale $\mathbb{T}$ is unbounded from above because of the definition of oscillation. The object of the present paper is to study nonoscillation of the following delay dynamic equation:

$$x^{\Delta}(t) + \sum_{i \in [1,n]_{\mathbb{N}}} A_i(t)x(\alpha_i(t)) = 0 \quad \text{for } t \in [t_0, \infty)_{\mathbb{T}}, \tag{1.3}$$

where $n \in \mathbb{N}$, $t_0 \in \mathbb{T}$, for all $i \in [1,n]_{\mathbb{N}}$, $A_i \in \mathrm{C}_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}}, \mathbb{R})$, α_i is a delay function satisfying $\alpha_i \in \mathrm{C}_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}}, \mathbb{T})$, $\lim_{t\to\infty}\alpha_i(t) = \infty$, and $\alpha_i(t) \le t$ for all $t \in [t_0,\infty)_{\mathbb{T}}$. Let us denote

$$\alpha_{\min}(t) := \min_{i\in[1,n]_{\mathbb{N}}} \{\alpha_i(t)\} \quad \text{for } t \in [t_0,\infty)_{\mathbb{T}}, \quad t_{-1} := \inf_{t\in[t_0,\infty)_{\mathbb{T}}} \{\alpha_{\min}(t)\}, \tag{1.4}$$

then t_{-1} is finite, since $\alpha_{\min}$ asymptotically tends to infinity. By a *solution* of (1.3), we mean a function $x : [t_{-1}, \infty)_{\mathbb{T}} \to \mathbb{R}$ such that $x \in \mathrm{C}^1_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}}, \mathbb{R})$ and (1.3) is satisfied on $[t_0,\infty)_{\mathbb{T}}$ identically. For a given function $\varphi \in \mathrm{C}_{\mathrm{rd}}([t_{-1}, t_0]_{\mathbb{T}}, \mathbb{R})$, (1.3) admits a unique solution satisfying $x = \varphi$ on $[t_{-1}, t_0]_{\mathbb{T}}$ (see [5, Theorem 3.1]). As usual, a solution of (1.3) is called *eventually positive* if there exists $s \in [t_0,\infty)_{\mathbb{T}}$ such that $x > 0$ on $[s,\infty)_{\mathbb{T}}$, and if $-x$ is eventually positive, then x is called *eventually negative*. A solution, which is neither eventually positive nor eventually negative, is called *oscillatory*, and (1.3) is said to be *oscillatory* provided that every solution of (1.3) is oscillatory.

In the papers [6, 7], the authors studied oscillation of (1.3) and proved the following oscillation criterion.

Theorem A (see [6, Theorem 1] and [7, Theorem 1])**.** *Suppose that $A \in C_{rd}([t_0,\infty)_{\mathbb{T}}, \mathbb{R}_0^+)$. If*

$$\liminf_{\substack{t\in T\\ t\to\infty}} \inf_{\substack{-\lambda A \in \mathcal{R}^+([\alpha(t),t)_{\mathbb{T}},\mathbb{R})\\ \lambda\in\mathbb{R}^+}} \left\{ \frac{e_{\ominus-\lambda A}(t, \alpha(t))}{\lambda} \right\} > 1, \tag{1.5}$$

then every solution of the equation

$$x^{\Delta}(t) + A(t)x(\alpha(t)) = 0 \quad \textit{for } t \in [t_0,\infty)_{\mathbb{T}} \tag{1.6}$$

is oscillatory.

Theorem A is the generalization of the well-known oscillation results stated for $\mathbb{T} = \mathbb{Z}$ and $\mathbb{T} = \mathbb{R}$ in the literature (see [8, Theorems 2.3.1 and 7.5.1]). In [9], Bohner et al. used an iterative method to advance the sufficiency condition in Theorem A, and in [10, Theorem 3.2] Agwo extended Theorem A to (1.3). Further, in [11], Şahiner and Stavroulakis gave the generalization of a well-known oscillation criterion, which is stated below.

Theorem B (see [11, Theorem 2.4]). *Suppose that $A \in C_{rd}([t_0, \infty)_{\mathbb{T}}, \mathbb{R}_0^+)$ and*

$$\limsup_{\substack{t\in T\\ t\to\infty}} \int_{\alpha(t)}^{\sigma(t)} A(\eta)\Delta\eta > 1. \tag{1.7}$$

Then every solution of (1.6) *is oscillatory.*

The present paper is mainly concerned with the existence of nonoscillatory solutions. So far, only few sufficient nonoscillation conditions have been known for dynamic equations on time scales. In particular, the following theorem, which is a sufficient condition for the existence of a nonoscillatory solution of (1.3), was proven in [7].

Theorem C (see [7, Theorem 2]). *Suppose that $A \in C_{rd}([t_0, \infty)_{\mathbb{T}}, \mathbb{R}_0^+)$ and there exist a constant $\lambda \in \mathbb{R}^+$ and a point $t_1 \in [t_0, \infty)_{\mathbb{T}}$ such that*

$$-\lambda A \in \mathcal{R}^+([t_1, \infty)_{\mathbb{T}}, \mathbb{R}), \quad \lambda \geq e_{\ominus-\lambda A}(t, \alpha(t)) \quad \forall t \in [t_2, \infty)_{\mathbb{T}}, \tag{1.8}$$

where $t_2 \in [t_1, \infty)_{\mathbb{T}}$ satisfies $\alpha(t) \geq t_1$ for all $t \in [t_2, \infty)_{\mathbb{T}}$. Then, (1.6) *has a nonoscillatory solution.*

In [10, Theorem 3.1, and Corollary 3.3], Agwo extended Theorem C to (1.3).

Theorem D (see [10, Corollary 3.3]). *Suppose that $A_i \in C_{\mathrm{rd}}([t_0, \infty)_{\mathbb{T}}, \mathbb{R}_0^+)$ for all $i \in [1, n]_{\mathbb{N}}$ and there exist a constant $\lambda \in \mathbb{R}^+$ and $t_1 \in [t_0, \infty)_{\mathbb{T}}$ such that $-\lambda A \in \mathcal{R}^+([t_1, \infty)_{\mathbb{T}}, \mathbb{R})$ and for all $t \in [t_1, \infty)_{\mathbb{T}}$*

$$\lambda \geq \frac{1}{A(t)} \sum_{i\in[1,n]_{\mathbb{N}}} A_i(t) e_{\ominus-\lambda A}(t, \alpha_i(t)), \tag{1.9}$$

where $A := \sum_{i\in[1,n]_{\mathbb{N}}} A_i$ on $[t_0, \infty)_{\mathbb{T}}$. Then, (1.3) *has a nonoscillatory solution.*

As was mentioned above, there are presently only few results on nonoscillation of (1.3); the aim of the present paper is to partially fill up this gap. To this end, we present a nonoscillation criterion; based on it, comparison theorems on oscillation and nonoscillation of solutions to (1.3) are obtained. Thus, solutions of two different equations and/or two different solutions of the same equation are compared, which allows to deduce oscillation and nonoscillation results.

The paper is organized as follows. In Section 2, some important auxiliary results, definitions and lemmas which will be needed in the sequel are introduced. Section 3 contains a nonoscillation criterion which is the main result of the present paper. Section 4 presents comparison theorems. All results are illustrated by examples on “nonstandard” time scales (which lead to neither differential nor classical difference equations).

2. Definitions and Preliminaries

Consider now the following delay dynamic initial value problem:

$$\begin{aligned} &x^{\Delta}(t) + \sum_{i\in[1,n]_{\mathbb{N}}} A_i(t)x(\alpha_i(t)) = f(t) \quad \text{for } t \in [t_0,\infty)_{\mathbb{T}} \\ &x(t_0) = x_0, \quad x(t) = \varphi(t) \quad \text{for } t \in [t_{-1},t_0)_{\mathbb{T}}, \end{aligned} \tag{2.1}$$

where $n \in \mathbb{N}$, $t_0 \in \mathbb{T}$ is the initial point, $x_0 \in \mathbb{R}$ is the initial value, $\varphi \in \mathrm{C}_{\mathrm{rd}}([t_{-1},t_0)_{\mathbb{T}},\mathbb{R})$ is the initial function such that φ has a finite left-sided limit at the initial point provided that it is left-dense, $f \in \mathrm{C}_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}},\mathbb{R})$ is the forcing term, and $A_i \in \mathrm{C}_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}},\mathbb{R})$ is the coefficient corresponding to the delay function α_i for all $i \in [1,n]_{\mathbb{N}}$. We assume that for all $i \in [1,n]_{\mathbb{N}}$, $A_i \in \mathrm{C}_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}},\mathbb{R})$, α_i is a delay function satisfying $\alpha_i \in \mathrm{C}_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}},\mathbb{T})$, $\lim_{t\to\infty}\alpha_i(t) = \infty$ and $\alpha_i(t) \le t$ for all $t \in [t_0,\infty)_{\mathbb{T}}$. We recall that $t_{-1} := \min_{i\in[1,n]_{\mathbb{N}}}\{\inf_{t\in[t_0,\infty)_{\mathbb{T}}}\alpha_i(t)\}$ is finite, since $\lim_{t\to\infty}\alpha_i(t) = \infty$ for all $i \in [1,n]_{\mathbb{N}}$.

For convenience in the notation and simplicity in the proofs, we suppose that functions vanish out of their specified domains, that is, let $f : D \to \mathbb{R}$ be defined for some $D \subset \mathbb{R}$, then it is always understood that $f(t) = \chi_D(t)f(t)$ for $t \in \mathbb{R}$, where χ_D is the characteristic function of D defined by $\chi_D(t) \equiv 1$ for $t \in D$ and $\chi_D(t) \equiv 0$ for $t \notin D$.

Definition 2.1. Let $s \in \mathbb{T}$, and $s_{-1} := \inf_{t\in[s,\infty)_{\mathbb{T}}}\{\alpha_{\min}(t)\}$. The solution $\mathcal{X} = \mathcal{X}(\cdot,s) : [s_{-1},\infty)_{\mathbb{T}} \to \mathbb{R}$ of the initial value problem

$$\begin{aligned} &x^{\Delta}(t) + \sum_{i\in[1,n]_{\mathbb{N}}} A_i(t)x(\alpha_i(t)) = 0 \quad \text{for } t \in [s,\infty)_{\mathbb{T}} \\ &x(t) = \chi_{\{s\}}(t) \quad \text{for } t \in [s_{-1},s]_{\mathbb{T}}, \end{aligned} \tag{2.2}$$

which satisfies $\mathcal{X}(\cdot,s) \in \mathrm{C}^1_{\mathrm{rd}}([s,\infty)_{\mathbb{T}},\mathbb{R})$, is called the *fundamental solution* of (2.1).

The following lemma (see [5, Lemma 3.1]) is extensively used in the sequel; it gives a solution representation formula for (2.1) in terms of the fundamental solution.

Lemma 2.2. *Let x be a solution of* (2.1), *then x can be written in the following form:*

$$\begin{aligned} x(t) = {} & x_0\mathcal{X}(t,t_0) + \int_{t_0}^{t} \mathcal{X}(t,\sigma(\eta))f(\eta)\Delta\eta \\ & - \sum_{i\in[1,n]_{\mathbb{N}}} \int_{t_0}^{t} \mathcal{X}(t,\sigma(\eta))A_i(\eta)\varphi(\alpha_i(\eta))\Delta\eta \quad \textit{for } t \in [t_0,\infty)_{\mathbb{T}}. \end{aligned} \tag{2.3}$$

As functions are assumed to vanish out of their domains, $\varphi(\alpha_i(t)) = 0$ if $\alpha_i(t) \ge t_0$ for $t \in [t_0,\infty)_{\mathbb{T}}$.

Proof. As the uniqueness for the solution of (2.1) was proven in [5], it suffices to show that

$$y(t) := \begin{cases} x_0\mathcal{X}(t,t_0) + \displaystyle\int_{t_0}^{t} \mathcal{X}(t,\sigma(\eta))f(\eta)\Delta\eta \\ \qquad -\displaystyle\int_{t_0}^{t} \mathcal{X}(t,\sigma(\eta)) \sum_{i\in[1,n]_{\mathbb{N}}} A_i(\eta)\varphi(\alpha_i(\eta))\Delta\eta, & t \in [t_0,\infty)_{\mathbb{T}}, \\ x_0, & t = t_0, \\ \varphi(t), & t \in [t_{-1},t_0)_{\mathbb{T}} \end{cases} \tag{2.4}$$

defined by the right hand side in (2.3) solves (2.1). For $t \in [t_0,\infty)_{\mathbb{T}}$, set $I(t) = \{j \in [1,n]_{\mathbb{N}} : \chi_{[t_0,\infty)_{\mathbb{T}}}(\alpha_j(t)) = 1\}$ and $J(t) := \{j \in [1,n]_{\mathbb{N}} : \chi_{[t_{-1},t_0)_{\mathbb{T}}}(\alpha_j(t)) = 1\}$. Considering the definition of the fundamental solution $\mathcal{X}$, we have

$$\begin{aligned} y^{\Delta}(t) = {}& x_0\mathcal{X}^{\Delta}(t,t_0) + \int_{t_0}^{t} \mathcal{X}^{\Delta}(t,\sigma(\eta))f(\eta)\Delta\eta + \mathcal{X}(\sigma(t),\sigma(t))f(t) \\ & - \int_{t_0}^{t} \mathcal{X}^{\Delta}(t,\sigma(\eta)) \sum_{i\in[1,n]_{\mathbb{N}}} A_i(\eta)\varphi(\alpha_i(\eta))\Delta\eta - \mathcal{X}(\sigma(t),\sigma(t)) \sum_{i\in[1,n]_{\mathbb{N}}} A_i(t)\varphi(\alpha_i(t)) \\ = {}& -\sum_{j\in I(t)} A_j(t)\Bigg[x_0\mathcal{X}(\alpha_j(t),t_0) + \int_{t_0}^{t} \mathcal{X}(\alpha_j(t),\sigma(\eta))f(\eta)\Delta\eta \\ & - \int_{t_0}^{t} \mathcal{X}(\alpha_j(t),\sigma(\eta)) \sum_{i\in[1,n]_{\mathbb{N}}} A_i(\eta)\varphi(\alpha_i(\eta))\Delta\eta\Bigg] - \sum_{j\in J(t)} A_j(t)\varphi(\alpha_j(t)) + f(t) \end{aligned} \tag{2.5}$$

for all $t \in [t_0,\infty)_{\mathbb{T}}$. After making some arrangements, we get

$$\begin{aligned} y^{\Delta}(t) = {}& -\sum_{j\in I(t)} A_j(t)\Bigg[x_0\mathcal{X}(\alpha_j(t),t_0) + \int_{t_0}^{\alpha_j(t)} \mathcal{X}(\alpha_j(t),\sigma(\eta))f(\eta)\Delta\eta \\ & - \int_{t_0}^{\alpha_j(t)} \mathcal{X}(\alpha_j(t),\sigma(\eta)) \sum_{i\in[1,n]_{\mathbb{N}}} A_i(\eta)\varphi(\alpha_i(\eta))\Delta\eta\Bigg] \\ & - \sum_{j\in J(t)} A_j(t)\varphi(\alpha_j(t)) + f(t) \\ = {}& -\sum_{j\in I(t)} A_j(t)y(\alpha_j(t)) - \sum_{j\in J(t)} A_j(t)y(\alpha_j(t)) + f(t), \end{aligned} \tag{2.6}$$

which proves that y satisfies (2.1) for all $t \in [t_0,\infty)_{\mathbb{T}}$ since $I(t)\cap J(t) = \emptyset$ and $I(t)\cup J(t) = [1,n]_{\mathbb{N}}$ for each $t \in [t_0,\infty)_{\mathbb{T}}$. The proof is therefore completed. □

Example 2.3. Consider the following first-order dynamic equation:

$$x^{\Delta}(t) + A(t)x(t) = 0 \quad \text{for } t \in [t_0, \infty)_{\mathbb{T}}, \tag{2.7}$$

then the fundamental solution of (2.7) can be easily computed as $\mathcal{X}(t,s) = e_{-A}(t,s)$ for $s,t \in [t_0,\infty)_{\mathbb{T}}$ provided that $-A \in \mathcal{R}([t_0,\infty)_{\mathbb{T}}, \mathbb{R})$ (see [4, Theorem 2.71]). Thus, the general solution of the initial value problem for the nonhomogeneous equation

$$\begin{gathered} x^{\Delta}(t) + A(t)x(t) = f(t) \quad \text{for } t \in [t_0, \infty)_{\mathbb{T}} \\ x(t_0) = x_0 \end{gathered} \tag{2.8}$$

can be written in the form

$$x(t) = x_0 e_{-A}(t,t_0) + \int_{t_0}^{t} e_{-A}(t,\sigma(\eta)) f(\eta)\Delta\eta \quad \text{for } t \in [t_0,\infty)_{\mathbb{T}}, \tag{2.9}$$

see [4, Theorem 2.77].

Next, we will apply the following result (see [6, page 2]).

Lemma 2.4 (see [6]). *If the delay dynamic inequality*

$$x^{\Delta}(t) + A(t)x(\alpha(t)) \le 0 \quad \textit{for } t \in [t_0, \infty)_{\mathbb{T}}, \tag{2.10}$$

where $A \in C_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}}, \mathbb{R}_0^+)$ and α is a delay function, has a solution x which satisfies $x(t) > 0$ for all $t \in [t_1,\infty)_{\mathbb{T}}$ for some fixed $t_1 \in [t_0,\infty)_{\mathbb{T}}$, then the coefficient satisfies $-A \in \mathcal{R}^+([t_2,\infty)_{\mathbb{T}}, \mathbb{R})$, where $t_2 \in [t_1,\infty)_{\mathbb{T}}$ satisfies $\alpha(t) \ge t_1$ for all $t \in [t_2,\infty)_{\mathbb{T}}$.

The following lemma plays a crucial role in our proofs.

Lemma 2.5. *Let $n \in \mathbb{N}$ and $t_0 \in \mathbb{T}$, and assume that $\alpha_i, \beta_i \in C_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}}, \mathbb{T})$, $\alpha_i(t), \beta_i(t) \le t$ for all $t \in [t_0,\infty)_{\mathbb{T}}$, $K_i \in C_{\mathrm{rd}}(\mathbb{T}\times\mathbb{T}, \mathbb{R}_0^+)$ for all $i \in [1,n]_{\mathbb{N}}$, and two functions $f, g \in C_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}}, \mathbb{R})$ satisfy*

$$f(t) = \sum_{i\in[1,n]_{\mathbb{N}}} \int_{\alpha_i(t)}^{t} K_i(t,\eta) f(\beta_i(\eta))\Delta\eta + g(t) \quad \forall t \in [t_0,\infty)_{\mathbb{T}}. \tag{2.11}$$

Then, nonnegativity of g on $[t_0,\infty)_{\mathbb{T}}$ implies the same for f.

Proof. Assume for the sake of contradiction that g is nonnegative but f becomes negative at some points in $[t_0,\infty)_{\mathbb{T}}$. Set

$$t_1 := \sup\{t \in [t_0,\infty)_{\mathbb{T}} : f(\eta) \ge 0 \ \forall \eta \in [t_0, t]_{\mathbb{T}}\}. \tag{2.12}$$

We first prove that t_1 cannot be right scattered. Suppose the contrary that t_1 is right scattered; that is, $\sigma(t_1) > t_1$, then we must have $f(t) \ge 0$ for all $t \in [t_0,t_1]_{\mathbb{T}}$ and $f^{\sigma}(t_1) < 0$; otherwise, this contradicts the fact that t_1 is maximal. It follows from (2.11) that after we have applied the formula for Δ-integrals, we have

$$f(\sigma(t_1)) = \sum_{i\in[1,n]_{\mathbb{N}}} \int_{\alpha_i(\sigma(t_1))}^{t_1} K_i(\sigma(t_1),\eta) f(\beta_i(\eta))\Delta\eta + \sum_{i\in[1,n]_{\mathbb{N}}} \mu(t_1) K_i(\sigma(t_1),t_1) f(\beta_i(t_1)) + g(\sigma(t_1)) \geq 0. \tag{2.13}$$

This is a contradiction, and therefore t_1 is right-dense. Note that every right-neighborhood of t_1 contains some points for which f becomes negative; therefore, $\inf_{\eta\in[t_1,t)_{\mathbb{T}}}\{f(\eta)\} < 0$ for all $t \in (t_1,\infty)_{\mathbb{T}}$. It is well known that rd-continuous functions (more truly regulated functions) are bounded on compact subsets of time scales. Pick $t_3 \in (t_1,\infty)_{\mathbb{T}}$, then for each $i \in [0,n]_{\mathbb{N}}$, we may find $M_i \in \mathbb{R}^+$ such that $K_i(t,s) \leq M_i$ for all $t \in [t_1,t_3]_{\mathbb{T}}$ and all $s \in [\alpha_i(t),t]_{\mathbb{T}}$. Set $M := \sum_{i\in[1,n]_{\mathbb{N}}} M_i$. Moreover, since t_1 is right-dense and f is rd-continuous, we have $\lim_{t\to t_1^+} f(t) = f(t_1)$; hence, we may find $t_2 \in (t_1,t_3]_{\mathbb{T}}$ with $t_2 - t_1 \leq 1/(3M)$ such that $\inf_{\eta\in[t_1,t_2)_{\mathbb{T}}} f(\eta) \geq 2f(t_2)$ and $f(t_2) < 0$. Note that $\inf_{\eta\in[t_0,t_2)_{\mathbb{T}}} f(\eta) = \inf_{\eta\in[t_1,t_2)_{\mathbb{T}}} f(\eta)$ since $f \geq 0$ on $[t_0,t_1]_{\mathbb{T}}$. Then, we get

$$\begin{aligned} f(t_2) &\geq \sum_{i\in[1,n]_{\mathbb{N}}} \int_{t_1}^{t_2} K_i(t,\eta) f(\beta_i(\eta))\Delta\eta + g(t_2) \geq \left(\sum_{i\in[1,n]_{\mathbb{N}}} \int_{t_1}^{t_2} M_i \Delta\eta\right) \inf_{\eta\in[t_0,t_2)_{\mathbb{T}}} \{f(\eta)\} \\ &\geq M(t_2 - t_1) \inf_{\eta\in[t_0,t_2)_{\mathbb{T}}} f(\eta) \geq \frac{2}{3} f(t_2), \end{aligned} \tag{2.14}$$

which yields the contradiction $1 \leq 2/3$ by canceling the negative terms $f(t_2)$ on both sides of the inequality. This completes the proof. □

The following lemma will be applied in the sequel.

Lemma 2.6 (see [6, Lemma 2]). *Assume that $A \in C_{\mathrm{rd}}(\mathbb{T},\mathbb{R}_0^+)$ satisfies $-A \in \mathcal{R}^+(\mathbb{T},\mathbb{R})$, then one has*

$$1 - \int_s^t A(\eta)\Delta\eta \leq e_{-A}(t,s) \leq \exp\left\{-\int_s^t A(\eta)\Delta\eta\right\} \quad \forall s,t \in \mathbb{T} \text{ with } t \geq s. \tag{2.15}$$

3. Main Nonoscillation Results

Consider the delay dynamic equation

$$x^{\Delta}(t) + \sum_{i\in[1,n]_{\mathbb{N}}} A_i(t) x(\alpha_i(t)) = 0 \quad \text{for } t \in [t_0,\infty)_{\mathbb{T}} \tag{3.1}$$

and the corresponding inequalities

$$x^{\Delta}(t)+\sum_{i\in[1,n]_{\mathbb{N}}}A_i(t)x(\alpha_i(t))\leq 0\quad\text{for } t\in[t_0,\infty)_{\mathbb{T}},\tag{3.2}$$

$$x^{\Delta}(t)+\sum_{i\in[1,n]_{\mathbb{N}}}A_i(t)x(\alpha_i(t))\geq 0\quad\text{for } t\in[t_0,\infty)_{\mathbb{T}}\tag{3.3}$$

under the same assumptions which were formulated for (2.1). We now prove the following result, which plays a major role throughout the paper.

Theorem 3.1. *Suppose that for all $i\in[1,n]_{\mathbb{N}}$, $\alpha_i\in C_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}},\mathbb{T})$ is a delay function and $A_i\in C_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}},\mathbb{R}^+)$. Then, the following conditions are equivalent.*

(i) *Equation* (3.1) *has an eventually positive solution.*

(ii) *Inequality* (3.2) *has an eventually positive solution and/or* (3.3) *has an eventually negative solution.*

(iii) *There exist a sufficiently large $t_1\in[t_0,\infty)_{\mathbb{T}}$ and $\Lambda\in C_{\mathrm{rd}}([t_1,\infty)_{\mathbb{T}},\mathbb{R}_0^+)$ such that $-\Lambda\in\mathcal{R}^+([t_1,\infty)_{\mathbb{T}},\mathbb{R})$ and for all $t\in[t_1,\infty)_{\mathbb{T}}$*

$$\Lambda(t)\geq\sum_{i\in[1,n]_{\mathbb{N}}}A_i(t)e_{\ominus-\Lambda}(t,\alpha_i(t)).\tag{3.4}$$

(iv) *The fundamental solution $\mathcal{X}$ is eventually positive; that is, there exists a sufficiently large $t_1\in[t_0,\infty)_{\mathbb{T}}$ such that $\mathcal{X}(\cdot,s)>0$ holds on $[s,\infty)_{\mathbb{T}}$ for any $s\in[t_1,\infty)_{\mathbb{T}}$; moreover, if* (3.4) *holds for all $t\in[t_1,\infty)_{\mathbb{T}}$ for some fixed $t_1\in[t_0,\infty)_{\mathbb{T}}$, then $\mathcal{X}(\cdot,s)>0$ holds on $[s,\infty)_{\mathbb{T}}$ for any $s\in[t_1,\infty)_{\mathbb{T}}$.*

Proof. Let us prove the implications as follows: (i)⇒(ii)⇒(iii)⇒(iv)⇒(i).

(i)⇒(ii) This part is trivial, since any eventually positive solution of (3.1) satisfies (3.2) too, which indicates that its negative satisfies (3.3).

(ii)⇒(iii) Let x be an eventually positive solution of (3.2), the case where x is an eventually negative solution to (3.3) is equivalent, and thus we omit it. Let us assume that there exists $t_1\in[t_0,\infty)_{\mathbb{T}}$ such that $x(t)>0$ and $x(\alpha_i(t))>0$ for all $t\in[t_1,\infty)_{\mathbb{T}}$ and all $i\in[1,n]_{\mathbb{N}}$. It follows from (3.2) that $x^{\Delta}\leq 0$ holds on $[t_1,\infty)_{\mathbb{T}}$, that is, x is nonincreasing on $[t_1,\infty)_{\mathbb{T}}$. Set

$$\Lambda(t):=-\frac{x^{\Delta}(t)}{x(t)}\quad\text{for } t\in[t_1,\infty)_{\mathbb{T}}.\tag{3.5}$$

Evidently $\Lambda\in C_{\mathrm{rd}}([t_1,\infty)_{\mathbb{T}},\mathbb{R}_0^+)$. From (3.5), we see that Λ satisfies the ordinary dynamic equation

$$x^{\Delta}(t)+\Lambda(t)x(t)=0\quad\forall t\in[t_1,\infty)_{\mathbb{T}}.\tag{3.6}$$

From Lemma 2.4, we deduce that $-\Lambda \in \mathcal{R}^+([t_1,\infty)_{\mathbb{T}},\mathbb{R})$. Since $x^{\Delta} = -\Lambda x$ on $[t_1,\infty)_{\mathbb{T}}$, then by [4, Theorem 2.35] and (3.6), we have

$$x(t) = x(t_1)e_{-\Lambda}(t,t_1) \quad \forall\, t \in [t_1,\infty)_{\mathbb{T}}. \tag{3.7}$$

Hence, using (3.7) in (3.2), for all $t \in [t_1,\infty)_{\mathbb{T}}$, we obtain

$$-\Lambda(t)x(t_1)e_{-\Lambda}(t,t_1) + \sum_{i\in[1,n]_{\mathbb{N}}} A_i(t)x(t_1)e_{-\Lambda}(\alpha_i(t),t_1) \le 0. \tag{3.8}$$

Since $x(t_1) > 0$, then by [4, Theorem 2.36] we have

$$\Lambda(t) \ge \sum_{i\in[1,n]_{\mathbb{N}}} A_i(t)\frac{e_{-\Lambda}(\alpha_i(t),t_1)}{e_{-\Lambda}(t,t_1)} = \sum_{i\in[1,n]_{\mathbb{N}}} A_i(t)e_{\ominus-\Lambda}(t,\alpha_i(t)). \tag{3.9}$$

$\Rightarrow$(iv) Let $\Lambda \in \mathrm{C}_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}},\mathbb{R}_0^+)$ satisfy $-\Lambda \in \mathcal{R}^+([t_1,\infty)_{\mathbb{T}},\mathbb{R})$ and (3.4) on $[t_1,\infty)_{\mathbb{T}}$, where $t_1 \in [t_0,\infty)_{\mathbb{T}}$ is such that $\alpha_{\min}(t) \ge t_0$ for all $t \in [t_1,\infty)_{\mathbb{T}}$. Now, consider the initial value problem

$$\begin{gathered} x^{\Delta}(t) + \sum_{i\in[1,n]_{\mathbb{N}}} A_i(t)x(\alpha_i(t)) = f(t) \quad \text{for } t \in [t_1,\infty)_{\mathbb{T}} \\ x(t) \equiv 0 \quad \text{for } t \in [t_0,t_1]_{\mathbb{T}}. \end{gathered} \tag{3.10}$$

Let x be a solution of (3.10), and set $g(t) := x^{\Delta}(t) + \Lambda(t)x(t)$ for $t \in [t_1,\infty)_{\mathbb{T}}$, then we see that x also satisfies the following auxiliary equation

$$\begin{gathered} x^{\Delta}(t) + \Lambda(t)x(t) = g(t) \quad \text{for } t \in [t_1,\infty)_{\mathbb{T}} \\ x(t_1) = 0, \end{gathered} \tag{3.11}$$

which has the unique solution

$$x(t) = \int_{t_1}^{t} e_{-\Lambda}(t,\sigma(\eta))g(\eta)\Delta\eta \quad \text{for } t \in [t_1,\infty)_{\mathbb{T}} \tag{3.12}$$

(see Example 2.3). Substituting (3.12) in (3.10), for all $t \in [t_1,\infty)_{\mathbb{T}}$, we obtain

$$\begin{aligned} f(t) = {} & -\Lambda(t)\int_{t_1}^{t} e_{-\Lambda}(t,\sigma(\eta))g(\eta)\Delta\eta + e_{-\Lambda}(\sigma(t),\sigma(t))g(t) \\ & + \sum_{i\in[1,n]_{\mathbb{N}}} A_i(t)e_{\ominus-\Lambda}(t,\alpha_i(t))\int_{t_1}^{\alpha_i(t)} e_{-\Lambda}(t,\sigma(\eta))g(\eta)\Delta\eta, \end{aligned} \tag{3.13}$$

which can be rewritten as

$$f(t) = -\Lambda(t)\int_{t_1}^{t} e_{-\Lambda}(t,\sigma(\eta))g(\eta)\Delta\eta + g(t)$$
$$+ \sum_{i\in[1,n]_{\mathbb{N}}} A_i(t)e_{\ominus-\Lambda}(t,\alpha_i(t))\int_{t_1}^{t} e_{-\Lambda}(t,\sigma(\eta))g(\eta)\Delta\eta \tag{3.14}$$
$$- \sum_{i\in[1,n]_{\mathbb{N}}} A_i(t)e_{\ominus-\Lambda}(t,\alpha_i(t))\int_{\alpha_i(t)}^{t} e_{-\Lambda}(t,\sigma(\eta))g(\eta)\Delta\eta.$$

Hence, we get

$$g(t) = \sum_{i\in[0,n]_{\mathbb{N}}} \Upsilon_i(t)\int_{\alpha_i(t)}^{t} e_{-\Lambda}(t,\sigma(\eta))g(\eta)\Delta\eta + f(t) \tag{3.15}$$

for all $t \in [t_1,\infty)_{\mathbb{T}}$, where $\alpha_0(t) := t_1$ for $t \in [t_1,\infty)_{\mathbb{T}}$,

$$\begin{aligned}
\Upsilon_i(t) &:= A_i(t)e_{\ominus-\Lambda}(t,\alpha_i(t)) \geq 0 \quad \text{for } t \in [t_1,\infty)_{\mathbb{T}}, \quad i \in [1,n]_{\mathbb{N}} \\
\Upsilon_0(t) &:= \Lambda(t) - \sum_{i\in[1,n]_{\mathbb{N}}} A_i(t)e_{\ominus-\Lambda}(t,\alpha_i(t)) \geq 0 \quad \text{for } t \in [t_1,\infty)_{\mathbb{T}}.
\end{aligned} \tag{3.16}$$

Applying Lemma 2.5 to (3.15), we learn that nonnegativity of f on $[t_1,\infty)_{\mathbb{T}}$ implies nonnegativity of g on $[t_1,\infty)_{\mathbb{T}}$, and nonnegativity of g on $[t_1,\infty)_{\mathbb{T}}$ implies the same for x on $[t_1,\infty)_{\mathbb{T}}$ by (3.12). On the other hand, by Lemma 2.2, x has the following representation:

$$x(t) = \int_{t_1}^{t} \mathcal{X}(t,\sigma(\eta))f(\eta)\Delta\eta \quad \text{for } t \in [t_1,\infty)_{\mathbb{T}}. \tag{3.17}$$

Since x is eventually nonnegative for any eventually nonnegative function f, we infer that the kernel $\mathcal{X}$ of the integral on the right-hand side of (3.17) is eventually nonnegative. Indeed, assume the contrary that $x \geq 0$ on $[t_1,\infty)_{\mathbb{T}}$ but $\mathcal{X}$ is not nonnegative, then we may pick $t_2 \in [t_1,\infty)_{\mathbb{T}}$ and find $s \in [t_1,t_2)_{\mathbb{T}}$ such that $\mathcal{X}(t_2,\sigma(s)) < 0$. Then, letting $f(t) := -\min\{\mathcal{X}(t_2,\sigma(t)),0\} \geq 0$ for $t \in [t_1,\infty)_{\mathbb{T}}$, we are led to the contradiction $x(t_2) < 0$, where x is defined by (3.17). To prove eventual positivity of $\mathcal{X}$, set

$$x(t) := \begin{cases} \mathcal{X}(t,s) - e_{-\Lambda}(t,s) & \text{for } t \in [t_1,\infty)_{\mathbb{T}}, \\ 0 & \text{for } t \in [t_0,t_1]_{\mathbb{T}}, \end{cases} \tag{3.18}$$

where $s \in [t_1,\infty)_{\mathbb{T}}$ is an arbitrarily fixed number, and substitute (3.18) into (3.10), to see that x satisfies (3.10) with a nonnegative forcing term f. Hence, as is proven previously, we infer that x is nonnegative on $[s,\infty)_{\mathbb{T}}$. Consequently, we have $\mathcal{X}(\cdot,s) \geq e_{-\Lambda}(\cdot,s) > 0$ on $[s,\infty)_{\mathbb{T}}$ for any $s \in [t_1,\infty)_{\mathbb{T}}$ (see [4, Theorem 2.48]).

(iv)⇒(i) Clearly, $\mathcal{X}(\cdot,t_0)$ is an eventually positive solution of (3.1).

The proof is therefore completed. □

Remark 3.2. Note that Theorem 3.1 for (1.6) includes Theorem C, by letting $\Lambda(t) := \lambda A(t)$ for $t \in [t_1,\infty)_{\mathbb{T}}$, where $\lambda \in \mathbb{R}^+$ satisfies $-\lambda A \in \mathcal{R}^+([t_1,\infty)_{\mathbb{T}},\mathbb{R})$. And Theorem 3.1 reduces to Theorem D, by letting $\Lambda(t) := \lambda \sum_{i\in[1,n]_{\mathbb{T}}} A_i(t)$ for $t \in [t_1,\infty)_{\mathbb{T}}$, where $\lambda \in \mathbb{R}^+$ satisfies $-\lambda \sum_{i\in[1,n]_{\mathbb{T}}} A_i \in \mathcal{R}^+([t_1,\infty)_{\mathbb{T}},\mathbb{R})$.

Corollary 3.3. *If* $\Lambda \in C_{\mathrm{rd}}([t_{-1},\infty)_{\mathbb{T}},\mathbb{R}_0^+)$, $-\Lambda \in \mathcal{R}^+([t_0,\infty)_{\mathbb{T}},\mathbb{R})$ *satisfies* (3.4) *on* $[t_0,\infty)_{\mathbb{T}}$ *and* $x_0 \geq 0$, *then*

$$x(t) := \begin{cases} x_0 e_{-\Lambda}(t,t_0) & \text{for } t \in [t_0,\infty)_{\mathbb{T}}, \\ x_0 & \text{for } t \in [t_{-1},t_0]_{\mathbb{T}} \end{cases} \tag{3.19}$$

is a positive solution of (3.2), *and* $-x$ *is a negative solution to* (3.3).

The following three examples are special cases of the above result, and the first two of them are corollaries for the cases $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = h\mathbb{Z}$, which are well known in literature, and the third one, for $\mathbb{T} = \overline{q^{\mathbb{Z}}}$ with $q > 1$, has not been stated thus far yet.

Example 3.4 (see [2, Theorem 1] and [8, Section 3]). Let $\mathbb{T} = \mathbb{R}$, and suppose that there exist $\lambda \in \mathbb{R}_0^+$ and $t_1 \in [t_0,\infty)$ such that

$$\lambda \geq \sum_{i\in[1,n]_{\mathbb{N}}} A_i(t) e^{\lambda(t-\alpha_i(t))} \quad \forall t \in [t_1,\infty). \tag{3.20}$$

Then, the delay-differential equation (1.2) has an eventually positive solution, and the fundamental solution $\mathcal{X}$ satisfies $\mathcal{X}(\cdot,s) > 0$ on $[s,\infty)_{\mathbb{T}}$ for any $s \in [t_1,\infty)_{\mathbb{T}}$ because we may let $\Lambda(t) :\equiv \lambda$ for $t \in [t_0,\infty)$.

Example 3.5 (see [3, Theorem 2.1] and [8, Section 7.8]). Let $h \in (0,\infty)$, $\mathbb{T} = h\mathbb{Z}$, and suppose that there exist $\lambda \in (0,1]$ and $t_1 \in [t_0,\infty)_{h\mathbb{Z}}$ such that

$$1-\lambda \geq h \sum_{i\in[1,n]_{\mathbb{N}}} A_i(t) \lambda^{-(t-\alpha_i(t))} \quad \forall t \in [t_1,\infty)_{h\mathbb{Z}}. \tag{3.21}$$

Then, the following delay h-difference equation:

$$\Delta_h x(t) + \sum_{i\in[1,n]_{\mathbb{N}}} A_i(t) x(\alpha_i(t)) = 0 \quad \text{for } t \in [t_0,\infty)_{h\mathbb{Z}}, \tag{3.22}$$

where Δ_h is defined by

$$\Delta_h x(t) := \frac{x(t+h)-x(t)}{h} \quad \text{for } t \in [t_0,\infty)_{h\mathbb{Z}}, \tag{3.23}$$

has an eventually positive solution, and the fundamental solution $\mathcal{X}$ satisfies $\mathcal{X}(\cdot, s) > 0$ on $[s, \infty)_{h\mathbb{Z}} \subset [t_1, \infty)_{h\mathbb{Z}}$ because we may let $\Lambda(t) :\equiv (1-\lambda)/h$ for $t \in [t_0, \infty)_{h\mathbb{Z}}$. Notice that if for all $t \in [t_1, \infty)_{h\mathbb{Z}}$ and all $i \in [1, n]_{\mathbb{N}}$, $A_i(t)$ and $t - \alpha_i(t)$ are constants, then (3.21) reduces to an algebraic inequality.

Example 3.6. Let $\mathbb{T} = \overline{q^{\mathbb{Z}}}$ for $q \in (1, \infty)$, and suppose that there exist $\lambda \in (0, 1]$ and $t_1 \in [t_0, \infty)_{\overline{q^{\mathbb{Z}}}}$, where $t_0 \in q^{\mathbb{Z}}$, such that

$$1 - \lambda \geq (q-1)t \sum_{i \in [1,n]_{\mathbb{N}}} A_i(t) \lambda^{-\log_q(t/\alpha_i(t))} \quad \forall\, t \in [t_1, \infty)_{\overline{q^{\mathbb{Z}}}}. \tag{3.24}$$

Then, the following delay q-difference equation:

$$D_q x(t) + \sum_{i \in [1,n]_{\mathbb{N}}} A_i(t) x(\alpha_i(t)) = 0 \quad \text{for } t \in [t_0, \infty)_{\overline{q^{\mathbb{Z}}}}, \tag{3.25}$$

where the q-difference operator D_q is defined by

$$D_q x(t) := \begin{cases} \dfrac{x(qt) - x(t)}{(q-1)t}, & t > 0, \\ \lim\limits_{\substack{s \in q^{\mathbb{Z}} \\ s \to 0^+}} \dfrac{x(s) - x(0)}{s}, & t = 0 \end{cases} \tag{3.26}$$

has an eventually positive solution, and the fundamental solution $\mathcal{X}$ satisfies $\mathcal{X}(\cdot, s) > 0$ on $[s, \infty)_{\overline{q^{\mathbb{Z}}}} \subset [t_1, \infty)_{\overline{q^{\mathbb{Z}}}}$ because we may let $\Lambda(t) := (1-\lambda)/((q-1)t)$ for $t \in [t_0, \infty)_{\overline{q^{\mathbb{Z}}}}$. Notice that if for all $t \in [t_1, \infty)_{h\mathbb{Z}}$ and all $i \in [1, n]_{\mathbb{N}}$, $tA_i(t)$ and $t/\alpha_i(t)$ are constants, then (3.24) becomes an algebraic inequality.

4. Comparison Theorems

In this section, we state comparison results on oscillation and nonoscillation of delay dynamic equations. To this end, consider (3.1) together with the following equation:

$$x^{\Delta}(t) + \sum_{i \in [1,n]_{\mathbb{N}}} B_i(t) x(\beta_i(t)) = 0 \quad \text{for } t \in [t_0, \infty)_{\mathbb{T}}, \tag{4.1}$$

where $n \in \mathbb{N}$, $B_i \in \mathrm{C_{rd}}([t_0, \infty)_{\mathbb{T}}, \mathbb{R})$ and $\beta_i \in \mathrm{C_{rd}}([t_0, \infty)_{\mathbb{T}}, \mathbb{T})$ is a delay function for all $i \in [1, n]_{\mathbb{N}}$. Let $\mathcal{Y}$ be the fundamental solution of (4.1).

Theorem 4.1. *Suppose that $B_i \in \mathrm{C_{rd}}([t_0, \infty)_{\mathbb{T}}, \mathbb{R}_0^+)$, $A_i \geq B_i$ and $\alpha_i \leq \beta_i$ on $[t_1, \infty)_{\mathbb{T}}$ for all $i \in [1, n]_{\mathbb{N}}$ and some fixed $t_1 \in [t_0, \infty)_{\mathbb{T}}$. If the fundamental solution $\mathcal{X}$ of* (3.1) *is eventually positive, then the fundamental solution $\mathcal{Y}$ of* (4.1) *is also eventually positive.*

Proof. By Theorem 3.1, there exist a sufficiently large $t_1 \in [t_0, \infty)_{\mathbb{T}}$ and $\Lambda \in \mathrm{C_{rd}}([t_1, \infty)_{\mathbb{T}}, \mathbb{R}_0^+)$ with $-\Lambda \in \mathcal{R}^+([t_1, \infty)_{\mathbb{T}}, \mathbb{R})$ such that (3.4) holds on $[t_1, \infty)_{\mathbb{T}}$. Note that $\Lambda \in \mathrm{C_{rd}}([t_1, \infty)_{\mathbb{T}}, \mathbb{R}_0^+)$

and $-\Lambda \in \mathcal{R}^{+}([t_1,\infty)_{\mathbb{T}},\mathbb{R})$ imply that $e_{-\Lambda}(t,s)$ is nondecreasing in s, hence $e_{\ominus-\Lambda}=1/e_{-\Lambda}(t,s)$ is nonincreasing in s (see [4, Theorem 2.36]). Without loss of generality, we may suppose that $A_i \geq B_i$ and $\alpha_i \leq \beta_i$ hold on $[t_1,\infty)_{\mathbb{T}}$ for all $i \in [1,n]_{\mathbb{N}}$. Then, we have

$$\Lambda(t) \geq \sum_{i\in[1,n]_{\mathbb{N}}} A_i(t)e_{\ominus-\Lambda}(t,\alpha_i(t)) \geq \sum_{i\in[1,n]_{\mathbb{N}}} B_i(t)e_{\ominus-\Lambda}(t,\beta_i(t)) \tag{4.2}$$

for all $t \in [t_1,\infty)_{\mathbb{T}}$. Thus, by Theorem 3.1 we have $\mathcal{Y}(\cdot,s)>0$ on $[s,\infty)_{\mathbb{T}}$ for any $s \in [t_1,\infty)_{\mathbb{T}}$, and equivalently, (4.1) has an eventually positive solution, which completes the proof. □

The following result is an immediate consequence of Theorem 4.1.

Corollary 4.2. *Assume that all the conditions of Theorem 4.1 hold. If* (4.1) *is oscillatory, then so is* (3.1)*.*

For the following result, we do not need the coefficient B_i to be nonnegative for all $i \in [1,n]_{\mathbb{N}}$; consider (3.1) together with the following equation:

$$x^{\Delta}(t) + \sum_{i\in[1,n]_{\mathbb{N}}} B_i(t)x(\alpha_i(t)) = 0 \quad \text{for } t \in [t_0,\infty)_{\mathbb{T}}, \tag{4.3}$$

where for all $i \in [1,n]_{\mathbb{N}}$, $B_i \in \mathrm{C}_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}},\mathbb{R})$ and α_i is the same delay function as in (3.1). Let $\mathcal{X}$ and $\mathcal{Y}$ be the fundamental solutions of (3.1) and (4.3), respectively.

Theorem 4.3. *Suppose that $A_i \in \mathrm{C}_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}},\mathbb{R}_0^{+})$, $A_i \geq B_i$ on $[t_1,\infty)_{\mathbb{T}}$ for all $i \in [1,n]_{\mathbb{N}}$ and some fixed $t_1 \in [t_0,\infty)_{\mathbb{T}}$, and that $\mathcal{X}(\cdot,s)>0$ on $[s,\infty)_{\mathbb{T}}$ for any $s \in [t_1,\infty)_{\mathbb{T}}$. Then, $\mathcal{Y}(\cdot,s) \geq \mathcal{X}(\cdot,s)$ holds on $[s,\infty)_{\mathbb{T}}$ for any $s \in [t_1,\infty)_{\mathbb{T}}$.*

Proof. From (4.3), any fixed $s \in [t_1,\infty)_{\mathbb{T}}$ and all $t \in [s,\infty)_{\mathbb{T}}$, we obtain

$$\mathcal{Y}^{\Delta}(t,s) + \sum_{i\in[1,n]_{\mathbb{N}}} A_i(t)\mathcal{Y}(\alpha_i(t),s) = \sum_{i\in[1,n]_{\mathbb{N}}} [A_i(t)-B_i(t)]\mathcal{Y}(\alpha_i(t),s). \tag{4.4}$$

It follows from the solution representation formula (2.3) that

$$\mathcal{Y}(t,s) = \mathcal{X}(t,s) + \sum_{i\in[1,n]_{\mathbb{N}}} \int_s^t \mathcal{X}(t,\sigma(\eta))[A_i(\eta)-B_i(\eta)]\mathcal{Y}(\alpha_i(\eta),s)\Delta\eta \tag{4.5}$$

for all $t \in [s,\infty)_{\mathbb{T}}$. Lemma 2.5 implies nonnegativity of $\mathcal{Y}(\cdot,s)$ since $\mathcal{X}(\cdot,s)>0$ on $[s,\infty)_{\mathbb{T}} \subset [t_1,\infty)_{\mathbb{T}}$ and the kernels of the integrals in (4.5) are nonnegative. Then dropping the nonnegative integrals on the right-hand side of (4.5), we get $\mathcal{Y}(t,s) \geq \mathcal{X}(t,s)$ for all $t \in [s,\infty)_{\mathbb{T}}$. The proof is hence completed. □

Corollary 4.4. *Suppose that the delay differential inequality*

$$x^{\Delta}(t) + \sum_{i\in[1,n]_{\mathbb{N}}} A_i^{+}(t)x(\alpha_i(t)) \leq 0 \quad \text{for } t \in [t_0, \infty)_{\mathbb{T}}, \tag{4.6}$$

where $A_i^{+}(t) := \max\{A_i(t), 0\}$ *for* $t \in [t_0,\infty)_{\mathbb{T}}$ *and* A_i, α_i *are same as in* (3.1) *for all* $i \in [1,n]_{\mathbb{N}}$, *has an eventually positive solution, then so does* (3.1).

Proof. By Theorem 3.1, we know that the fundamental solution of the corresponding differential equation

$$x^{\Delta}(t) + \sum_{i\in[1,n]_{\mathbb{N}}} A_i^{+}(t)x(\alpha_i(t)) = 0 \quad \text{for } t \in [t_0, \infty)_{\mathbb{T}} \tag{4.7}$$

is eventually positive, applying Theorem 4.3, we learn that the fundamental solution of (3.1) is also eventually positive since $A_i^{+} \geq A_i$ holds on $[t_0,\infty)_{\mathbb{T}}$ for all $i \in [1,n]_{\mathbb{N}}$. The proof is hence completed. □

We now compare two solutions of (2.1) and the following initial value problem:

$$\begin{gathered} x^{\Delta}(t) + \sum_{i\in[1,n]_{\mathbb{N}}} B_i(t)x(\alpha_i(t)) = g(t) \quad \text{for } t \in [t_0, \infty)_{\mathbb{T}} \\ x(t_0) = x_0, \quad x(t) = \varphi(t) \quad \text{for } t \in [t_{-1}, t_0)_{\mathbb{T}}, \end{gathered} \tag{4.8}$$

where $n \in \mathbb{N}$, x_0, φ and α_i for all $i \in [1,n]_{\mathbb{N}}$ are the same as in (2.1) and $B_i, g \in \mathrm{C}_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}}, \mathbb{R})$ for all $i \in [1,n]_{\mathbb{N}}$.

Theorem 4.5. *Suppose that* $A_i \geq B_i$ *for all* $i \in [1,n]_{\mathbb{N}}$ *and* $g \geq f$ *on* $[t_0,\infty)_{\mathbb{T}}$, *and* $\mathcal{X}(\cdot,s) > 0$ *on* $[s,\infty)_{\mathbb{T}}$ *for any* $s \in [t_0,\infty)_{\mathbb{T}}$. *Let* x *be a solution of* (2.1) *with* $x > 0$ *on* $[t_0,\infty)_{\mathbb{T}}$, *then* $y \geq x$ *holds on* $[t_0,\infty)_{\mathbb{T}}$, *where* y *is a solution of* (4.8).

Proof. By Theorems 3.1 and 4.3, we have $\mathcal{Y}(\cdot,s) \geq \mathcal{X}(\cdot,s) > 0$ on $[s,\infty)_{\mathbb{T}}$ for any $s \in [t_0,\infty)_{\mathbb{T}}$. Rearranging (2.1), we have

$$x^{\Delta}(t) + \sum_{i\in[1,n]_{\mathbb{N}}} B_i(t)x(\alpha_i(t)) = f(t) - \sum_{i\in[1,n]_{\mathbb{N}}} [A_i(t) - B_i(t)]x(\alpha_i(t)) \tag{4.9}$$

for all $t \in [t_0,\infty)_{\mathbb{T}}$. In view of the solution representation formula (2.3), for all $t \in [t_0,\infty)_{\mathbb{T}}$, we have

$$\begin{aligned}
x(t) &= x_0\mathcal{Y}(t,t_0) + \int_{t_0}^{t} \mathcal{Y}(t,\sigma(\eta))\left[f(\eta) - \sum_{i\in[1,n]_{\mathbb{N}}} [A_i(\eta) - B_i(\eta)]\chi_{[t_0,\infty)_{\mathbb{T}}}(\alpha_i(\eta))x(\alpha_i(\eta))\right]\Delta\eta \\
&\quad - \sum_{i\in[1,n]_{\mathbb{N}}} \int_{t_0}^{t} \mathcal{Y}(t,\sigma(\eta))A_i(\eta)\varphi(\alpha_i(\eta))\Delta\eta \\
&\leq x_0\mathcal{Y}(t,t_0) + \int_{t_0}^{t} \mathcal{Y}(t,\sigma(\eta))g(\eta)\Delta\eta - \sum_{i\in[1,n]_{\mathbb{N}}} \int_{t_0}^{t} \mathcal{Y}(t,\sigma(\eta))B_i(\eta)\varphi(\alpha_i(\eta))\Delta\eta = y(t),
\end{aligned} \tag{4.10}$$

which implies $y \geq x$ on $[t_0,\infty)_{\mathbb{T}}$. Therefore, the proof is completed. □

As an application of Theorem 4.5, we give a simple example on a nonstandard time scale below.

Example 4.6. Let $\mathbb{T} = \sqrt[3]{\mathbb{N}} := \{\sqrt[3]{n} : n \in \mathbb{N}\}$, and consider the following initial value problems:

$$\begin{aligned}
x^{\Delta}(t) + \frac{2}{t^3}x\left(\sqrt[3]{t^3-2}\right) &= \frac{3}{2t^3} \quad \text{for } t \in \left[\sqrt[3]{3},\infty\right)_{\sqrt[3]{\mathbb{N}}} \\
x(t) &\equiv 1 \quad \text{for } t \in \left[1,\sqrt[3]{3}\right]_{\sqrt[3]{\mathbb{N}}},
\end{aligned} \tag{4.11}$$

where

$$x^{\Delta}(t) = \frac{x\left(\sqrt[3]{t^3+1}\right) - x(t)}{\sqrt[3]{t^3+1} - t} \quad \text{for } t \in \sqrt[3]{\mathbb{N}} \tag{4.12}$$

and

$$\begin{aligned}
y^{\Delta}(t) + \frac{1}{t^3}y\left(\sqrt[3]{t^3-2}\right) &= \frac{3}{t^3} \quad \text{for } t \in \left[\sqrt[3]{3},\infty\right)_{\sqrt[3]{\mathbb{N}}}, \\
y(t) &\equiv 1 \quad \text{for } t \in \left[1,\sqrt[3]{3}\right]_{\sqrt[3]{\mathbb{N}}}.
\end{aligned} \tag{4.13}$$

Denoting by x and y the solutions of (4.11) and (4.13), respectively. Then, $y \geq x$ on $[\sqrt[3]{3},\infty)_{\sqrt[3]{\mathbb{N}}}$ by Theorem 4.5. For the graph of 30 iterates, see Figure 1.

Corollary 4.7. *Suppose that $A_i \in C_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}},\mathbb{R}_0^+)$ for all $i \in [1,n]_{\mathbb{N}}$ and $\mathcal{X}(\cdot,s) > 0$ on $[s,\infty)_{\mathbb{T}}$ for any $s \in [t_0,\infty)_{\mathbb{T}}$. Let x,y,z be solutions of* (3.1), (3.2) *and* (3.3), *respectively. If $y > 0$ on $[t_0,\infty)_{\mathbb{T}}$ and $x \equiv y \equiv z$ on $[t_{-1},t_0]_{\mathbb{T}}$, then one has $y \leq x \leq z$ on $[t_0,\infty)_{\mathbb{T}}$.*

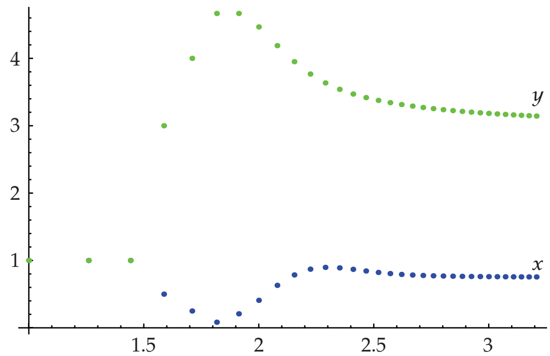

Figure 1: The graph of 30 iterates for the solutions of (4.11) and (4.13) illustrates the result of Theorem 4.5, here $y(t) > x(t)$ for all $t \in (\sqrt[3]{3}, \infty)_{\sqrt[3]{\mathbb{N}}}$.

Corollary 4.8. *Let x be a solution of* (3.1)*, and $\mathcal{Y}(\cdot, s) > 0$ on $[s, \infty)_{\mathbb{T}}$ for any $s \in [t_0, \infty)_{\mathbb{T}}$ be the fundamental solution of*

$$x^{\Delta}(t) + \sum_{i \in [1,n]_{\mathbb{N}}} A_i^{+}(t) x(\alpha_i(t)) = 0 \quad \text{for } t \in [t_0, \infty)_{\mathbb{T}}, \tag{4.14}$$

and $y > 0$ on $[t_0, \infty)_{\mathbb{T}}$ be a solution of this equation. If $x \equiv y$ holds on $[t_{-1}, t_0]_{\mathbb{T}}$, then $x \geq y$ holds on $[t_0, \infty)_{\mathbb{T}}$.

Theorem 4.9. *Suppose that there exist $t_1 \in [t_0, \infty)_{\mathbb{T}}$ and $\Lambda \in C_{\mathrm{rd}}([t_1, \infty)_{\mathbb{T}}, \mathbb{R}_0^{+})$ such that $-\Lambda \in \mathcal{R}^{+}([t_1, \infty)_{\mathbb{T}}, \mathbb{R})$ and for all $t \in [t_1, \infty)_{\mathbb{T}}$*

$$\sum_{i \in [1,n]_{\mathbb{N}}} A_i^{+}(t) \leq \Lambda(t) \left(1 - \int_{\alpha_{\min}(t)}^{t} \Lambda(\eta) \Delta\eta \right). \tag{4.15}$$

Then, (3.1) *has an eventually positive solution.*

Proof. By Corollary 4.4, it suffices to prove that (4.6) has an eventually positive solution. For this purpose, by Theorem 3.1, it is enough to demonstrate that Λ satisfies

$$\Lambda(t) \geq \sum_{i \in [1,n]_{\mathbb{N}}} A_i^{+}(t) e_{\ominus - \Lambda}(t, \alpha_i(t)) \quad \forall t \in [t_1, \infty)_{\mathbb{T}}. \tag{4.16}$$

Note that $\Lambda \in C_{\mathrm{rd}}([t_1, \infty)_{\mathbb{T}}, \mathbb{R}_0^{+})$ and $-\Lambda \in \mathcal{R}^{+}([t_1, \infty)_{\mathbb{T}}, \mathbb{R})$ imply that $e_{-\Lambda}(t, s)$ is nondecreasing in s, hence $e_{\ominus - \Lambda} = 1/e_{-\Lambda}(t, s)$ is nonincreasing in s (see [4, Theorem 2.36]). From (4.15) and

Lemma 2.6, for all $t \in [t_1,\infty)_{\mathbb{T}}$, we have

$$\begin{aligned}\Lambda(t) &\geq \sum_{i\in[1,n]_{\mathbb{N}}} \frac{A_i^+(t)}{1-\int_{\alpha_{\min}(t)}^{t}\Lambda(\eta)\Delta\eta} \geq \sum_{i\in[1,n]_{\mathbb{N}}} \frac{A_i^+(t)}{e_{-\Lambda}(t,\alpha_{\min}(t))} \\ &= \sum_{i\in[1,n]_{\mathbb{N}}} A_i^+(t)e_{\ominus-\Lambda}(t,\alpha_{\min}(t)) \geq \sum_{i\in[1,n]_{\mathbb{N}}} A_i^+(t)e_{\ominus-\Lambda}(t,\alpha_i(t)),\end{aligned} \tag{4.17}$$

which implies that (4.16) holds. The proof is therefore completed. □

Corollary 4.10. *Suppose that there exist $M,\lambda\in\mathbb{R}_0^+$ with $\lambda(1-M\lambda)\geq 1$ and $t_1\in[t_0,\infty)_{\mathbb{T}}$ such that $-\lambda\sum_{i\in[1,n]_{\mathbb{N}}} A_i^+ \in \mathcal{R}^+([t_1,\infty)_{\mathbb{T}},\mathbb{R})$ and*

$$\int_{\alpha_{\min}(t)}^{t} \sum_{i\in[1,n]_{\mathbb{N}}} A_i^+(\eta)\Delta\eta \leq M \quad \forall t\in[t_2,\infty)_{\mathbb{T}}, \tag{4.18}$$

where $t_2\in[t_1,\infty)_{\mathbb{T}}$ satisfies $\alpha_{\min}(t)\geq t_1$ for all $t\in[t_2,\infty)_{\mathbb{T}}$. Then, (3.1) *has an eventually positive solution.*

Proof. In this present case, we may let $\Lambda(t) := \lambda\sum_{i\in[1,n]_{\mathbb{N}}} A_i^+(t)$ for $t\in[t_1,\infty)_{\mathbb{T}}$ to obtain (4.15). □

Remark 4.11. Particularly, letting $\lambda = 2$ and $M = 1/4$ in Corollary 4.10, we learn that (3.1) admits a nonoscillatory solution if $-2\sum_{i\in[1,n]_{\mathbb{N}}} A_i^+ \in \mathcal{R}^+([t_1,\infty)_{\mathbb{T}},\mathbb{R})$ and

$$\int_{\alpha_{\min}(t)}^{t} \sum_{i\in[1,n]_{\mathbb{N}}} A_i^+(\eta)\Delta\eta \leq \frac{1}{4} \quad \forall t\in[t_2,\infty)_{\mathbb{T}}. \tag{4.19}$$

It is a well-known fact that the constant $1/4$ above is the best possible for difference equations since the difference equation

$$\Delta x(t) + ax(t-1) = 0 \quad \text{for } t\in\mathbb{N}, \tag{4.20}$$

where $a\in\mathbb{R}^+$, is nonoscillatory if and only if $a\leq 1/4$ (see [3, 12]).

The following example illustrates Corollary 4.10 for the nonstandard time scale $\mathbb{T} = \overline{q^{\mathbb{Z}}}$.

Example 4.12. Let $a_i\in\mathbb{R}^+$, $p_i\in\mathbb{N}$ for $i\in[1,n]_{\mathbb{N}}$ and $q\in(1,\infty)$. We consider the following q-difference equation

$$D_q x(t) + \sum_{i\in[1,n]_{\mathbb{N}}} \frac{a_i}{t} x\left(\frac{t}{q^{p_i}}\right) = 0 \quad \text{for } t\in[1,\infty)_{\overline{q^{\mathbb{Z}}}}, \tag{4.21}$$

where the q-difference operator D_q is defined by (3.26). For simplicity of notation, we let

$$p := \max_{i\in[1,n]_{\mathbb{N}}} \{p_i\} \geq 1, \qquad a := \sum_{i\in[1,n]_{\mathbb{N}}} a_i > 0. \tag{4.22}$$

Then, we have

$$\int_{t/q^p}^{t} \sum_{i\in[1,n]_{\mathbb{N}}} \left[\frac{a_i}{\eta}\right]^{+} \Delta\eta = a\int_{t/q^p}^{t} \frac{1}{\eta}\Delta\eta \equiv ap(q-1) \quad \forall t\in[1,\infty)_{\overline{q^{\mathbb{Z}}}}. \tag{4.23}$$

Letting $\lambda = 1/(2ap(q-1))$, we can compute that

$$1-(q-1)t\lambda\sum_{i\in[1,n]_{\mathbb{N}}} \left[\frac{a_i}{t}\right]^{+} = 1-\frac{a(q-1)}{2ap(q-1)} = \frac{2p-1}{2p} > 0 \quad \forall t\in[1,\infty)_{\overline{q^{\mathbb{Z}}}}, \tag{4.24}$$

which implies that the regressivity condition in Corollary 4.10 holds. So that (4.21) has an eventually positive solution if

$$\lambda(1-M\lambda) = \frac{1}{4a\alpha(q-1)} \geq 1, \tag{4.25}$$

where $M = ap(q-1)$, or equivalently $a\alpha(q-1)\leq 1/4$.

Theorem 4.13. *Suppose that $A_i \in C_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}},\mathbb{R}_0^{+})$ for all $i\in[1,n]_{\mathbb{N}}$ and (4.15) is true on $[t_0,\infty)_{\mathbb{T}}$. If $x_0>0$ and $x_0\geq\varphi\geq 0$ on $[t_{-1},t_0)_{\mathbb{T}}$, then for the solution x of*

$$\begin{gathered} x^{\Delta}(t) + \sum_{i\in[1,n]_{\mathbb{N}}} A_i(t)x(\alpha_i(t)) = 0 \quad \text{for } t\in[t_0,\infty)_{\mathbb{T}} \\ x(t_0)=x_0, \quad x(t)=\varphi(t) \quad \text{for } t\in[t_{-1},t_0)_{\mathbb{T}}, \end{gathered} \tag{4.26}$$

we have $x>0$ on $[t_0,\infty)_{\mathbb{T}}$.

Proof. As in the proof of Theorem 4.9, we deduce that there exists Λ satisfying (3.4). Hence, $\mathcal{X}(\cdot,s)>0$ on $[s,\infty)_{\mathbb{T}}$ for any $s\in[t_0,\infty)_{\mathbb{T}}$. By the solution representation formula (2.3), we get

$$x(t) = x_0\mathcal{X}(t,t_0) - \sum_{i\in[1,n]_{\mathbb{N}}}\int_{t_0}^{t}\mathcal{X}(t,\sigma(\eta))A_i(\eta)\varphi(\alpha_i(\eta))\Delta\eta \tag{4.27}$$

for all $t\in[t_0,\infty)_{\mathbb{T}}$. Let

$$y(t) := \begin{cases} x_0 e_{-\Lambda}(t,t_0) & \text{for } t\in[t_0,\infty)_{\mathbb{T}}, \\ x_0 & \text{for } t\in[t_{-1},t_0]_{\mathbb{T}}. \end{cases} \tag{4.28}$$

By Corollary 3.3, we have $g(t) := y^{\Delta}(t) + \sum_{i\in[1,n]_{\mathbb{N}}} A_i(t)y(\alpha_i(t)) \le 0$ for all $t \in [t_0,\infty)_{\mathbb{T}}$. Then, y solves

$$\begin{gathered} x^{\Delta}(t) + \sum_{i\in[1,n]_{\mathbb{N}}} A_i(t)x(\alpha_i(t)) = g(t) \quad \text{for } t \in [t_0,\infty)_{\mathbb{T}} \\ x(t) \equiv x_0 \quad \text{for } t \in [t_{-1},t_0]_{\mathbb{T}}. \end{gathered} \tag{4.29}$$

By Corollary 4.7, we know that y given by

$$y(t) = x_0\mathcal{X}(t,t_0) + \int_{t_0}^{t} \mathcal{X}(t,\sigma(\eta))g(\eta)\Delta - x_0 \sum_{i\in[1,n]_{\mathbb{N}}} \int_{t_0}^{t} \mathcal{X}(t,\sigma(\eta))A_i(\eta)\chi_{[t_{-1},t_0)_{\mathbb{T}}}(\alpha_i(\eta))\Delta\eta \tag{4.30}$$

cannot exceed the solution x of (4.26) which has representation (4.27). Thus, $x \ge y > 0$ on $[t_0,\infty)_{\mathbb{T}}$ because of $x_0 \ge \varphi$ on $[t_{-1},t_0)_{\mathbb{T}}$, and $g \le 0$ on $[t_0,\infty)_{\mathbb{T}}$, which completes the proof. □

Theorem 4.14. *Suppose that $A_i \in C_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}},\mathbb{R}_0^+)$ for all $i \in [1,n]_{\mathbb{N}}$, $\mathcal{X}(\cdot,s) > 0$ on $[s,\infty)_{\mathbb{T}}$ for any $s \in [t_0,\infty)_{\mathbb{T}}$, and the solution y of the initial value problem*

$$\begin{gathered} y^{\Delta}(t) + \sum_{i\in[1,n]_{\mathbb{N}}} A_i(t)y(\alpha_i(t)) = 0 \quad \text{for } t \in [t_0,\infty)_{\mathbb{T}} \\ y(t) \equiv y_0 \quad \text{for } t \in [t_{-1},t_0]_{\mathbb{T}} \end{gathered} \tag{4.31}$$

is positive. If $x_0 \ge y_0 > 0$ and $y_0 \ge \varphi \ge 0$ on $[t_{-1},t_0)_{\mathbb{T}}$, then the solution x of (4.26) *is positive on $[t_0,\infty)_{\mathbb{T}}$.*

Proof. Solution representation formula (2.3) implies for a solution of (4.31) that

$$\begin{aligned} y(t) &= y_0\left[\mathcal{X}(t,t_0) - \int_{t_0}^{t} \mathcal{X}(t,\sigma(\eta)) \sum_{i\in[1,n]_{\mathbb{N}}} A_i(\eta)\chi_{[t_{-1},t_0)_{\mathbb{T}}}(\alpha_i(\eta))\Delta\eta\right] \\ &\le x_0\mathcal{X}(t,t_0) - \int_{t_0}^{t} \mathcal{X}(t,\sigma(\eta)) \sum_{i\in[1,n]_{\mathbb{N}}} A_i(\eta)\varphi(\alpha_i(\eta))\Delta\eta = x(t) \end{aligned} \tag{4.32}$$

for all $t \in [t_0,\infty)_{\mathbb{T}}$ since $x_0 \ge y_0$ and $x_0 \ge \varphi \ge 0$ on $[t_{-1},t_0)_{\mathbb{T}}$. Hence, $x \ge y > 0$ holds on $[t_0,\infty)_{\mathbb{T}}$. Thus, the proof is completed. □

Theorem 4.15. *Suppose that* $A_i \in C_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}},\mathbb{R}_0^+)$, $i \in [1,n]_{\mathbb{N}}$, (3.4) *has a solution* $\Lambda \in C_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}},\mathbb{R}_0^+)$ *with* $-\Lambda \in \mathcal{R}^+([t_0,\infty)_{\mathbb{T}},\mathbb{R})$, x *is a solution of* (4.26) *and* y *is a positive solution of the following initial value problem*

$$\begin{gathered} y^{\Delta}(t)+\sum_{i\in[1,n]_{\mathbb{N}}} A_i(t)y(\alpha_i(t))=0 \quad \textit{for } t\in[t_0,\infty)_{\mathbb{T}} \\ y(t_0)=y_0, \quad x(t)=\varphi(t) \quad \textit{for } t\in[t_{-1},t_0)_{\mathbb{T}}. \end{gathered} \tag{4.33}$$

If $x_0 \geq y_0 \geq 0$ *and* $\psi \geq \varphi \geq 0$ *on* $[t_{-1},t_0)_{\mathbb{T}}$, *then we have* $x \geq y$ *on* $[t_0,\infty)_{\mathbb{T}}$.

Proof. The proof is similar to that of Theorem 4.13. □

We give the following example as an application of Theorem 4.15.

Example 4.16. Let $\mathbb{T}=\mathbb{N}^3 := \{n^3 : n\in\mathbb{N}\}$, and consider the following initial value problems:

$$\begin{gathered} x^{\Delta}(t)+\frac{1}{t}x\left(\left(\sqrt[3]{t}-3\right)^3\right)=0 \quad \text{for } t\in[64,\infty)_{\mathbb{N}^3} \\ x(t)=3^{4-\log_3(t)} \quad \text{for } t\in[1,64]_{\mathbb{N}^3}, \end{gathered} \tag{4.34}$$

where

$$x^{\Delta}(t)=\frac{x\left(\left(\sqrt[3]{t}+1\right)^3\right)-x(t)}{\left(\sqrt[3]{t}+1\right)^3-t} \qquad \text{for } t\in\mathbb{N}^3 \tag{4.35}$$

and

$$\begin{gathered} y^{\Delta}(t)+\frac{1}{t}y\left(\left(\sqrt[3]{t}-3\right)^3\right)= \quad \text{for } t\in[64,\infty)_{\mathbb{N}^3}, \\ y(t)=1-2^{-\log_3(t)} \quad \text{for } t\in[1,64]_{\mathbb{N}^3}. \end{gathered} \tag{4.36}$$

If x and y are the unique solutions of (4.34) and (4.36), respectively, then we have the graph of 7 iterates, see Figure 2, where $x > y$ by Theorem 4.15.

5. Discussion

In this paper, we have extended to equations on time scales most results obtained in [2, 3]: nonoscillation criteria, comparison theorems, and efficient nonoscillation conditions. However, there are some relevant problems that have not been considered.

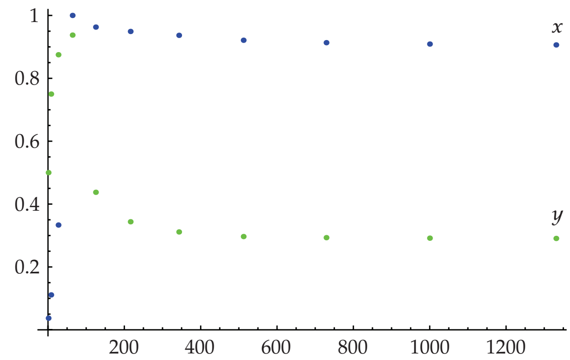

Figure 2: The graph of 7 iterates for the solutions of (4.34) and (4.36) illustrates the result of Theorem 4.15, here $x(t) > y(t)$ for all $t \in [64, \infty)_{\mathbb{N}^3}$.

(P1) In [2], it was demonstrated that equations with positive coefficients has slowly oscillating solutions only if it is oscillatory. The notion of slowly oscillating solutions can be easily extended to equations on time scales in such a way that it generalizes the one discussed in [2].

Definition 5.1. A solution x of (3.1) is said to be *slowly oscillating* if it is oscillating and for every $t_1 \in [t_0, \infty)_{\mathbb{T}}$ there exist $t_2, t_3 \in [t_1, \infty)_{\mathbb{T}}$ with $t_3 > t_2$ and $\alpha_{\min}(t) \geq t_2$ for all $t \in [t_3, \infty)_{\mathbb{T}}$ such that $x > 0$ on $(t_2, t_3)_{\mathbb{T}}$ and $x(t_4) < 0$ for some $t_4 \in (t_3, \infty)_{\mathbb{T}}$.

Is the following proposition valid?

Proposition 5.2. *Suppose that for all $i \in [1, n]_{\mathbb{N}}$, $\alpha_i \in C_{\mathrm{rd}}([t_0, \infty)_{\mathbb{T}}, \mathbb{T})$ is a delay function and $A_i \in C_{\mathrm{rd}}([t_0, \infty)_{\mathbb{T}}, \mathbb{R}^+)$. If (3.1) is nonoscillatory, then the equation has no slowly oscillating solutions.*

(P2) In Section 4, oscillation properties of equations with different coefficients, delays and initial functions were compared, as well as two solutions of equations with the same delays and initial conditions. Can any relation be deduced between nonoscillation properties of the same equation on different time scales?

(P3) The results of the present paper involve nonoscillation conditions for equations with positive and negative coefficients: if the relevant equation with positive coefficients only is nonoscillatory, so is the equation with coefficients of both signs. Is it possible to obtain efficient nonoscillation conditions for equations with positive and negative coefficients when the relevant equation with positive coefficients only is oscillatory?

We will only comment affirmatively on the proof of the proposition in Problem (P1). Really, let us assume the contrary that (3.1) is nonoscillatory but x is a slowly oscillating solution of this equation. By Theorem 3.1, the fundamental solution $\mathcal{X}(\cdot, s)$ of (3.1) is positive on $(s, \infty)_{\mathbb{T}} \subset [t_1, \infty)_{\mathbb{T}}$ for some $t_1 \in [t_0, \infty)_{\mathbb{T}}$. There exist $t_2 \in [t_1, \infty)_{\mathbb{T}}$ and $t_3 \in (t_2, \infty)_{\mathbb{T}}$ with $\alpha_{\min}(t) \geq t_2$ for all $t \in [t_3, \infty)_{\mathbb{T}}$ such that $x > 0$ on $(t_2, t_3)_{\mathbb{T}}$ and $x \not\geq 0$ on $(t_3, \infty)_{\mathbb{T}}$. Therefore, we have

$$A_i(t)\chi_{[t_2,t_3)_{\mathbb{T}}}(\alpha_i(t))x(\alpha_i(t)) \geq 0, \qquad A_i(t)\chi_{[t_2,t_3)_{\mathbb{T}}}(\alpha_i(t))x(\alpha_i(t)) \not\equiv 0 \tag{5.1}$$

for all $t \in [t_3, \infty)_{\mathbb{T}}$ and all $i \in [1, n]_{\mathbb{N}}$. It follows from Lemma 2.2 that

$$\begin{aligned} x(t) &= x(t_3)\mathcal{X}(t,t_3) - \int_{t_3}^{t} \mathcal{X}(t,\sigma(\eta)) \sum_{i\in[1,n]_{\mathbb{N}}} A_i(\eta)\chi_{[t_2,t_3)_{\mathbb{T}}}(\alpha_i(\eta))x(\alpha_i(\eta))\Delta\eta \\ &\leq -\int_{t_3}^{t} \mathcal{X}(t,\sigma(\eta)) \sum_{i\in[1,n]_{\mathbb{N}}} A_i(\eta)\chi_{[t_2,t_3)_{\mathbb{T}}}(\alpha_i(\eta))x(\alpha_i(\eta))\Delta\eta \end{aligned} \tag{5.2}$$

for all $t \in [t_3, \infty)_{\mathbb{T}}$. Since the integrand is nonnegative and not identically zero by (5.1), we learn that the right-hand side of (5.2) is negative on $(t_3, \infty)_{\mathbb{T}}$; that is, $x < 0$ on $(t_3, \infty)_{\mathbb{T}}$. Hence, x is nonoscillatory, which is the contradiction justifying the proposition.

Thus, under the assumptions of Proposition 5.2 existence of a slowly oscillating solution of (3.1) implies oscillation of all solutions.

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