

Research Article

On the Twisted q -Analogues of the Generalized Euler Numbers and Polynomials of Higher Order

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We consider the twisted q -extensions of the generalized Euler numbers and polynomials attached to χ .

1. Introduction and Preliminaries

Let p be an odd prime number. For $n \in \mathbb{Z}_+ = \mathbb{N} \cup \{0\}$, let $C_{p^n} = \{\zeta \mid \zeta^{p^n} = 1\}$ be the cyclic group of order p^n , and let $T_p = \lim_{n \rightarrow \infty} C_{p^n} = \bigcup_{n \geq 0} C_{p^n} = C_{p^\infty}$ be the space of locally constant functions in the p -adic number field \mathbb{C}_p . When one talks of q -extension, q is variously considered as an indeterminate, a complex number $q \in \mathbb{C}$, or p -adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$, one normally assumes that $|q| < 1$. If $q \in \mathbb{C}_p$, one normally assumes that $|1 - q|_p < 1$. In this paper, we use the notation

$$[x]_q = \frac{1 - q^x}{1 - q}, \quad [x]_{-q} = \frac{1 - (-q)^x}{1 + q}. \quad (1.1)$$

Let d be a fixed positive odd integer. For $N \in \mathbb{N}$, we set

$$X = X_d = \frac{\lim_{N \rightarrow \infty} \mathbb{Z}}{dp^N \mathbb{Z}}, \quad X_1 = \mathbb{Z}_p,$$

$$\begin{aligned}
 X^* &= \bigcup_{\substack{0 < a < dp \\ (a,p)=1}} (a + dp\mathbb{Z}_p), \\
 a + dp^n\mathbb{Z}_p &= \{x \in X \mid x \equiv a \pmod{dp^n}\},
 \end{aligned} \tag{1.2}$$

where $a \in \mathbb{Z}$ lies in $0 \leq a < dp^n$; compared to [1–16].

Let χ be the Dirichlet's character with an odd conductor $d \in \mathbb{N}$. Then the generalized ζ -Euler polynomials attached to χ , $E_{n,\chi,\zeta}(x)$, are defined as

$$\begin{aligned}
 F_{\chi,\zeta}(x,t) &= \frac{2 \sum_{l=0}^{d-1} (-1)^l \chi(l) \zeta^l e^{lt}}{\zeta^d e^{dt} + 1} e^{xt} \\
 &= \sum_{n=0}^{\infty} E_{n,\chi,\zeta}(x) \frac{t^n}{n!}, \quad \text{for } \zeta \in T_p.
 \end{aligned} \tag{1.3}$$

In the special case $x = 0$, $E_{n,\chi,\zeta} = E_{n,\chi,\zeta}(0)$ are called the n th ζ -Euler numbers attached to χ . For $f \in UD(\mathbb{Z}_p)$, the p -adic fermionic integral on \mathbb{Z}_p is defined by

$$\begin{aligned}
 I_{-q}(f) &= \int_{\mathbb{Z}_p} f(x) d\mu_{-q}(x) = \lim_{N \rightarrow \infty} \sum_{x=0}^{p^N-1} f(x) \mu_{-q}(x + p^N \mathbb{Z}_p) \\
 &= \lim_{N \rightarrow \infty} \sum_{x=0}^{p^N-1} f(x) (-1)^x \frac{q^x}{[p^N]_{-q}}, \quad (\text{see [7–17]}).
 \end{aligned} \tag{1.4}$$

Let $I_{-1} = \lim_{q \rightarrow 1} I_{-q}(f)$. Then, we see that

$$\int_{\mathbb{Z}_p} f(x) d\mu_{-1}(x) = \int_X f(x) d\mu_{-1}(x). \tag{1.5}$$

For $n \in \mathbb{N}$, let $f_n(x) = f(x+n)$. Then, we have

$$\int_{\mathbb{Z}_p} f(x+n) d\mu_{-1}(x) = (-1)^n \int_{\mathbb{Z}_p} f(x) d\mu_{-1}(x) + 2 \sum_{l=0}^{n-1} (-1)^{n-1-l} f(l). \tag{1.6}$$

Thus, we have

$$I_{-1}(f_n) + (-1)^{n-1} I_{-1}(f) = 2 \sum_{l=0}^{n-1} (-1)^{n-1-l} f(l), \quad (\text{see [7–17]}). \tag{1.7}$$

By (1.7), we see that

$$\int_X \chi(y) \zeta^y e^{(x+y)t} d\mu_{-1}(y) = \frac{2 \sum_{l=0}^{d-1} (-1)^l \chi(l) \zeta^l e^{lt}}{\zeta^d e^{dt} + 1} e^{xt} = \sum_{n=0}^{\infty} E_{n,\chi,\zeta}(x) \frac{t^n}{n!}. \tag{1.8}$$

From (1.8), we can derive the Witt’s formula for $E_{n,\chi,\zeta}(x)$ as follows:

$$\begin{aligned} \int_X \chi(x) x^n \zeta^x d\mu_{-1}(x) &= E_{n,\chi,\zeta}, \\ \int_X \chi(y) (y+x)^n \zeta^y d\mu_{-1}(y) &= E_{n,\chi,\zeta}(x), \quad \text{for } \zeta \in T_p, \text{ (see [5–17]).} \end{aligned} \tag{1.9}$$

The n th generalized ζ -Euler polynomials of order k , $E_{n,\chi,\zeta}^{(k)}$, are defined as

$$\left(\frac{2 \sum_{l=0}^{d-1} \zeta^l (-1)^l \chi(l) e^{lt}}{\zeta^d e^{dt} + 1} e^{xt} \right)^k = \sum_{n=0}^{\infty} E_{n,\chi,\zeta}^{(k)}(x) \frac{t^n}{n!}. \tag{1.10}$$

In the special case $x = 0$, $E_{n,\chi,\zeta}^{(k)} = E_{n,\chi,\zeta}^{(k)}(0)$ are called the n th ζ -Euler numbers of order k attached to χ .

Now, we consider the multivariate p -adic invariant integral on X as follows:

$$\begin{aligned} \int_X \cdots \int_X \left(\prod_{i=1}^k \chi(x_i) \right) e^{(x_1+\cdots+x_k+x)t} \zeta^{x_1+\cdots+x_k} d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_k) \\ = \left(\frac{2 \sum_{l=0}^{d-1} (-1)^l \chi(l) e^{lt}}{\zeta^d e^{dt} + 1} \right)^k e^{xt} = \sum_{n=0}^{\infty} E_{n,\chi,\zeta}^{(k)}(x) \frac{t^n}{n!}. \end{aligned} \tag{1.11}$$

By (1.10) and (1.11), we see the Witt’s formula for $E_{n,\chi,\zeta}^{(k)}(x)$ as follows:

$$\int_X \cdots \int_X \left(\prod_{i=1}^k \chi(x_i) \right) (x_1 + \cdots + x_k + x)^n \zeta^{x_1+\cdots+x_k} d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_k) = E_{n,\chi,\zeta}^{(k)}(x). \tag{1.12}$$

The purpose of this paper is to present a systemic study of some formulas of the twisted q -extension of the generalized Euler numbers and polynomials of order k attached to χ .

2. On the Twisted q -Extension of the Generalized Euler Polynomials

In this section, we assume that $q \in \mathbb{C}_p$ with $|1 - q|_p < 1$ and $\zeta \in T_p$. For $d \in \mathbb{N}$ with $2 \nmid d$, let χ be the Dirichlet’s character with conductor d . For $h \in \mathbb{Z}, k \in \mathbb{N}$, let us consider the twisted (h, q) -extension of the generalized Euler numbers and polynomials of order k attached to χ .

We firstly consider the twisted q -extension of the generalized Euler polynomials of higher order as follows:

$$\begin{aligned} \sum_{n=0}^{\infty} E_{n,\chi,\zeta,q}(x) \frac{t^n}{n!} &= \int_X e^{[x+y]_q t} \zeta^y \chi(y) d\mu_{-1}(y) \\ &= 2 \sum_{m=0}^{\infty} \chi(m) (-1)^m \zeta^m e^{[m+x]_q t}. \end{aligned} \quad (2.1)$$

By (2.1), we see that

$$\begin{aligned} \int_X [x+y]_q^n \chi(y) \zeta^y d\mu_{-1}(y) &= 2 \sum_{m=0}^{\infty} \chi(m) (-\zeta)^m e^{[m+x]_q t} \\ &= 2 \sum_{a=0}^{d-1} \chi(a) (-1)^a \zeta^a \frac{1}{(1-q)^n} \sum_{l=0}^n \binom{n}{l} (-1)^l \frac{\zeta^{la} q^{l(a+x)}}{1+q^{ld} \zeta^{ld}}. \end{aligned} \quad (2.2)$$

From the multivariate fermionic p -adic invariant integral on \mathbb{Z}_p , we can derive the twisted q -extension of the generalized Euler polynomials of order k attached to χ as follows:

$$\sum_{n=0}^{\infty} E_{n,\chi,\zeta,q}^{(k)}(x) \frac{t^n}{n!} = \int_X \cdots \int_X \left(\prod_{i=1}^k \chi(x_i) \right) e^{[x_1+\cdots+x_k+x]_q t} \zeta^{x_1+\cdots+x_k} d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_k). \quad (2.3)$$

Thus, we have

$$\begin{aligned} E_{n,\chi,\zeta,q}^{(k)}(x) &= \int_X \cdots \int_X \left(\prod_{i=1}^k \chi(x_i) \right) [x_1 + \cdots + x_k + x]_q^n \zeta^{x_1+\cdots+x_k} d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_k) \\ &= \sum_{a_1, \dots, a_k=0}^{d-1} \left(\prod_{i=1}^k \chi(a_i) \right) (-\zeta)^{\sum_{j=1}^k a_j} \frac{2^k}{(1-q)^n} \sum_{l=0}^n \frac{\binom{n}{l} (-1)^l q^{l(x+\sum_{j=1}^k a_j)}}{(1+q^{ld} \zeta^{ld})} \\ &= 2^k \sum_{a_1, \dots, a_k=0}^{d-1} \prod_{i=1}^k (\chi(a_i)) (-\zeta)^{\sum_{j=1}^k a_j} \sum_{m=0}^{\infty} \binom{m+k-1}{m} \times (-\zeta^d)^m [x+a_1+\cdots+a_k+md]_q^n. \end{aligned} \quad (2.4)$$

Let $F_{q,\chi,\zeta}^{(k)}(t, x) = \sum_{n=0}^{\infty} E_{n,\chi,\zeta,q}^{(k)}(x) (t^n/n!)$ be the generating function for $E_{n,\chi,\zeta,q}^{(k)}(x)$. By (2.3),

we easily see that

$$\begin{aligned}
 F_{q,\chi}^{(k)}(t, x) &= 2^k \sum_{a_1, \dots, a_k=0}^{d-1} \left(\prod_{i=1}^k \chi(a_i) \right) (-\zeta)^{\sum_{j=1}^k a_j} \sum_{m=0}^{\infty} \binom{m+k-1}{m} \times (-\zeta)^{dm} e^{[x+a_1+\dots+a_k+md]_q t} \\
 &= 2^k \sum_{n_1, \dots, n_k=0}^{\infty} (-\zeta)^{n_1+\dots+n_k} \left(\prod_{i=1}^k \chi(n_i) \right) e^{[n_1+\dots+n_k+x]_q t}.
 \end{aligned}
 \tag{2.5}$$

Therefore, we obtain the following theorem.

Theorem 2.1. For $k \in \mathbb{N}, n \geq 0$, one has

$$\begin{aligned}
 E_{n,\chi,\zeta,q}^{(k)}(x) &= 2^k \sum_{n_1, \dots, n_k=0}^{\infty} (-\zeta)^{n_1+\dots+n_k} \left(\prod_{i=1}^k \chi(n_i) \right) [n_1 + \dots + n_k + x]_q^n \\
 &= \sum_{a_1, \dots, a_r=0}^{d-1} \left(\prod_{i=1}^k \chi(a_i) \right) (-\zeta)^{\sum_{j=1}^k a_j} \frac{2^k}{(1-q)^n} \sum_{l=0}^n \frac{\binom{n}{l} (-1)^l q^{l(x+\sum_{j=1}^k a_j)}}{(1+q^{ld}\zeta^d)^n}.
 \end{aligned}
 \tag{2.6}$$

Let $h \in \mathbb{Z}, r \in \mathbb{N}$. Then we define the extension of $E_{n,\chi,\zeta,q}^{(r)}(x)$ as follows:

$$\sum_{n=0}^{\infty} E_{n,\chi,\zeta,q}^{(h,r)}(x) \frac{t^n}{n!} = \int_X \dots \int_X q^{\sum_{j=1}^r (h-j)x_j} \left(\prod_{i=1}^k \chi(x_i) \right) e^{[x+\sum_{j=1}^r x_j]_q t} \times \zeta^{x_1+\dots+x_r} d\mu_{-1}(x_1) \dots d\mu_{-1}(x_r).
 \tag{2.7}$$

Then, $E_{n,\chi,\zeta,q}^{(r)}(x)$ are called the n th generalized (h, q) -Euler polynomials of order r attached to χ . In the special case $x = 0$, $E_{n,\chi,\zeta,q}^{(r)} = E_{n,\chi,\zeta,q}^{(r)}(0)$ are called the n th generalized (h, r) -Euler numbers of order r . By (1.7), we obtain the Witt's formula for $E_{n,\chi,\zeta,q}^{(r)}(x)$ as follows:

$$\begin{aligned}
 E_{n,\chi,\zeta,q}^{(h,r)}(x) &= \int_X \dots \int_X q^{\sum_{j=1}^r (h-j)x_j} \left(\prod_{i=1}^k \chi(x_i) \right) \left[x + \sum_{j=1}^r x_j \right]_q^{\zeta^{x_1+\dots+x_r}} d\mu_{-1}(x_1) \dots d\mu_{-1}(x_r) \\
 &= \sum_{a_1, \dots, a_r=0}^{d-1} \left(\prod_{i=1}^r \chi(a_i) \right) (-\zeta)^{\sum_{j=1}^r a_j} q^{\sum_{j=1}^r a_j (h-j)} \times \frac{2^r}{(1-q)^n} \sum_{l=0}^n \frac{\binom{n}{l} (-1)^l q^{l(x+\sum_{j=1}^r a_j)}}{(-q^{d(h-r+l)} \zeta^d; q^d)_r},
 \end{aligned}
 \tag{2.8}$$

where $(a; q)_r = (1-a)(1-aq) \dots (1-aq^{r-1})$.

Let $\binom{n}{k}_q = ([n]_q [n-1]_q \cdots [n-k+1]_q) / [k]_q! = [n]_q! / ([k]_q! [n-k]_q!)$ where $[k]_q! = [k]_q [k-1]_q \cdots [2]_q [1]_q$. From (2.8), we note that

$$\begin{aligned}
 E_{n,\chi,\zeta,q}^{(h,r)}(x) &= \frac{2^r}{(1-q)^n} \sum_{a_1, \dots, a_r=0}^{d-1} \left(\prod_{i=1}^r \chi(a_i) \right) (-\zeta)^{\sum_{j=1}^r a_j} q^{\sum_{j=1}^r (h-j)a_j} \\
 &\quad \times \sum_{l=0}^n \binom{n}{l} (-1)^l q^{l(x+\sum_{j=1}^r a_j)} \sum_{m=0}^{\infty} \binom{m+r-1}{m}_{q^d} (-\zeta^d)^m q^{d(h-r)m} q^{ldm} \\
 &= 2^r \sum_{a_1, \dots, a_r=0}^{d-1} \left(\prod_{i=1}^{r-1} \chi(a_i) \right) (-\zeta)^{\sum_{j=1}^r a_j} q^{\sum_{j=1}^r (h-j)a_j} \\
 &\quad \times \sum_{m=0}^{\infty} \binom{m+r-1}{m}_{q^d} (-\zeta^d)^m q^{d(h-r)m} \frac{1}{(1-q)^n} \left(1 - q^{d(m+(x+\sum_{j=1}^r a_j)/d)} \right)^n \\
 &= 2^r [d]_q^n \sum_{m=0}^{\infty} \binom{m+r-1}{m}_{q^d} (-\zeta^d)^m q^{d(h-r)m} \sum_{a_1, \dots, a_r=0}^{d-1} \left(\prod_{i=1}^{r-1} \chi(a_i) \right) \\
 &\quad \times (-\zeta)^{\sum_{j=1}^r a_j} q^{\sum_{j=1}^r (h-j)a_j} \left[m + \frac{x + \sum_{j=1}^{d-1} a_j}{d} \right]_{q^d}^n.
 \end{aligned} \tag{2.9}$$

Let $F_{q,\chi,\zeta}^{(h,r)}(t, x) = \sum_{n=0}^{\infty} E_{n,\chi,\zeta,q}^{(h,r)}(x) (t^n/n!)$ be the generating function for $E_{n,\chi,\zeta,q}^{(h,r)}(x)$. From (2.8), we can easily derive

$$\begin{aligned}
 F_{q,\chi,\zeta}^{(h,r)}(t, x) &= 2^r \sum_{n_1, \dots, n_r=0}^{\infty} q^{\sum_{j=1}^r (h-j)n_j} (-\zeta)^{\sum_{j=1}^r n_j} \left(\prod_{j=1}^r \chi(n_j) \right) e^{[n_1+\dots+n_r+x]_q t} \\
 &= 2^r \sum_{m=0}^{\infty} \binom{m+r-1}{m}_q (-\zeta^d)^m q^{d(h-r)m} \sum_{a_1, \dots, a_r=0}^{d-1} \left(\prod_{i=1}^{r-1} \chi(a_i) \right) \\
 &\quad \times (-\zeta)^{\sum_{j=1}^r a_j} q^{\sum_{j=1}^r (h-j)a_j} e^{[md+x+\sum_{j=1}^r a_j]_q t}.
 \end{aligned} \tag{2.10}$$

By (2.10), we obtain the following theorem.

Theorem 2.2. For $h \in \mathbb{Z}$, $r \in \mathbb{N}$, one has

$$\begin{aligned}
 E_{n, \chi, \zeta, q}^{(h, r)}(x) &= 2^r \sum_{n_1, \dots, n_r=0}^{\infty} q^{\sum_{j=1}^r (h-j)n_j} (-\zeta)^{\sum_{j=1}^r n_j} \left(\prod_{j=1}^r \chi(n_j) \right) [n_1 + \dots + n_r + x]_q^n \\
 &= 2^r [d]_q^n \sum_{m=0}^{\infty} \binom{m+r-1}{m}_q (-\zeta^d)^m q^{d(h-r)m} \sum_{a_1, \dots, a_r=0}^{d-1} \left(\prod_{i=1}^{r-1} \chi(a_i) \right) \\
 &\quad \times (-\zeta)^{\sum_{j=1}^r a_j} q^{\sum_{j=1}^r (h-j)a_j} \left[m + \frac{x + \sum_{j=1}^r a_j}{d} \right]_{q^d}^n \\
 &= \sum_{a_1, \dots, a_r=0}^{d-1} \left(\prod_{j=1}^r \chi(n_j) \right) (-\zeta)^{\sum_{j=1}^r a_j} q^{\sum_{j=1}^r (h-j)a_j} \times \frac{2^r}{(1-q)^n} \sum_{l=0}^n \frac{\binom{n}{l} (-1)^l q^{l(\sum_{j=1}^r a_j)}}{(-q^{ld} \zeta^d; q^d)_r}.
 \end{aligned} \tag{2.11}$$

Let $h = r$. Then we see that

$$\begin{aligned}
 E_{n, \chi, \zeta, q}^{(r, r)}(x) &= \frac{2^r}{(1-q)^n} \sum_{a_1, \dots, a_r=0}^{d-1} \left(\prod_{i=1}^{r-1} \chi(a_i) \right) (-\zeta)^{\sum_{i=1}^r a_i} q^{\sum_{j=1}^r (h-j)a_j} \times \sum_{l=0}^n \frac{\binom{n}{l} (-1)^l q^{l(\sum_{j=1}^r a_j+x)}}{(-q^{ld} \zeta^d; q^d)_r} \\
 &= 2^r [d]_q^n \sum_{m=0}^{\infty} \binom{m+r-1}{m}_q (-\zeta)^m \sum_{a_1, \dots, a_r=0}^{d-1} \left(\prod_{i=1}^r \chi(a_i) \right) \\
 &\quad \times (-\zeta)^{\sum_{j=1}^r a_j} q^{\sum_{j=1}^r (r-j)a_j} \left[m + \frac{x + \sum_{j=1}^r a_j}{d} \right]_{q^d}^n.
 \end{aligned} \tag{2.12}$$

It is easy to see that

$$\begin{aligned}
 &\int_X \dots \int_X \left(\prod_{i=1}^k \chi(x_i) \right) q^{\sum_{j=1}^r (h-j)x_j + xm} \zeta^{x_1 + \dots + x_r} d\mu_{-1}(x_1) \dots d\mu_{-1}(x_r) \\
 &= \sum_{a_1, \dots, a_r=0}^{d-1} \left(\prod_{i=1}^r \chi(a_i) \right) q^{mx + \sum_{j=1}^r (h-j)a_j} (-\zeta)^{\sum_{j=1}^r a_j} \times \int_X \dots \int_X q^{\sum_{j=1}^r (m-j)x_j} d\mu_{-1}(x_1) \dots d\mu_{-1}(x_r) \\
 &= \frac{2^r q^{mx} \sum_{a_1, \dots, a_r=0}^{d-1} \left(\prod_{j=1}^r \chi(a_j) \right) q^{\sum_{j=1}^r (m-j)a_j} (-\zeta)^{\sum_{j=1}^r a_j}}{(-q^{d(m-r)} \zeta^d; q^d)_r}.
 \end{aligned} \tag{2.13}$$

Thus, we have

$$\begin{aligned}
& \frac{2^r q^{mx} \sum_{a_1, \dots, a_r=0}^{d-1} \left(\prod_{j=1}^r \chi(a_j) \right) q^{\sum_{j=1}^r (m-j)a_j} (-\zeta)^{\sum_{j=1}^r a_j}}{(-q^{d(m-r)} \zeta^d; q^d)_r} \\
&= \int_X \cdots \int_X \left([x + x_1 + \cdots + x_r]_q (q-1) + 1 \right)^m q^{-\sum_{j=1}^r jx_j} \zeta^{x_1 + \cdots + x_r} \\
&\quad \times \left(\prod_{j=1}^r \chi(x_j) \right) d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_r) \\
&= \sum_{l=0}^m \binom{m}{l} (q-1)^l \int_X \cdots \int_X \left(\prod_{j=1}^r \chi(x_j) \right) \\
&\quad \times [x + x_1 + \cdots + x_r]_q^l q^{-\sum_{j=1}^r jx_j} \zeta^{x_1 + \cdots + x_r} d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_r) \\
&= \sum_{l=0}^m \binom{m}{l} (q-1)^l E_{l, \chi, \zeta, q}^{(0,r)}(x).
\end{aligned} \tag{2.14}$$

By (2.14), we obtain the following theorem.

Theorem 2.3. For $d, k \in \mathbb{N}$ with $2 \nmid d$, one has

$$\frac{2^r q^{mx} \sum_{a_1, \dots, a_r=0}^{d-1} \left(\prod_{j=1}^r \chi(a_j) \right) q^{\sum_{j=1}^r (m-j)a_j} (-\zeta)^{\sum_{j=1}^r a_j}}{(-q^{d(m-r)} \zeta^d; q^d)_r} = \sum_{l=0}^m \binom{m}{l} (q-1)^l E_{l, \chi, \zeta, q}^{(0,r)}(x). \tag{2.15}$$

By (1.7), we easily see that

$$\int_X f(x+d) d\mu_{-1}(x) + \int_X f(x) d\mu_{-1}(x) = 2 \sum_{l=0}^{d-1} (-1)^l f(l). \tag{2.16}$$

Thus, we have

$$\begin{aligned}
& q^{d(h-1)} \int_X \cdots \int_X [x + d + x_1 + \cdots + x_r]_q^n q^{\sum_{j=1}^r (r-j)x_j} \zeta^{\sum_{j=1}^r x_j} \\
&\quad \times \left(\prod_{j=1}^r \chi(x_j) \right) d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_r) \\
&= - \int_X \cdots \int_X [x + x_1 + \cdots + x_r]_q^n q^{\sum_{j=1}^r (r-j)x_j} \zeta^{\sum_{j=1}^r x_j} \times \left(\prod_{j=1}^r \chi(x_j) \right) d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_r) \\
&\quad + 2 \sum_{l=0}^{d-1} \chi(l) (-\zeta)^l \int_X \cdots \int_X [x + l + x_2 + \cdots + x_r]_q^n \left(\prod_{j=1}^{r-1} \chi(x_{j+1}) \right) \\
&\quad \times q^{\sum_{j=1}^{r-1} x_{j+1} (h-1-j)} \zeta^{x_2 + x_3 + \cdots + x_r} d\mu_{-1}(x_2) \cdots d\mu_{-1}(x_r).
\end{aligned} \tag{2.17}$$

By (2.17), we obtain the following theorem.

Theorem 2.4. For $h \in \mathbb{Z}, d \in \mathbb{N}$ with $2 \nmid d$, one has

$$q^{d(h-1)} E_{n,\chi,\zeta,q}^{(h,r)}(x+d) + E_{n,\chi,\zeta,q}^{(h,r)}(x) = 2 \sum_{l=0}^{d-1} \chi(l) (-1)^l E_{n,\chi,\zeta,q}^{(h-1,r-1)}(x+l). \tag{2.18}$$

It is easy to see that

$$q^x E_{n,\chi,\zeta,q}^{(h+1,r)}(x) = (q-1) E_{n+1,\chi,\zeta,q}^{(h,r)} + E_{n,\chi,\zeta,q}^{(h,r)}(x). \tag{2.19}$$

Let $F_{q,\chi,\zeta}^{(h,1)}(t, x) = \sum_{n=0}^{\infty} E_{n,\chi,\zeta,q}^{(h,1)}(x) (t^n/n!)$. Then we note that

$$F_{q,\chi,\zeta}^{(h,1)}(t, x) = 2 \sum_{n=0}^{\infty} \chi(n) q^{(h-1)n} (-\zeta)^n e^{[n+x]_q t}. \tag{2.20}$$

From (2.20), we can derive

$$E_{n,\chi,\zeta,q}^{(h,1)}(x) = 2 \sum_{m=0}^{\infty} \chi(m) q^{(h-1)m} (-\zeta)^m [m+x]_q^n = \frac{2}{(1-q)^n} \sum_{a=0}^{d-1} \chi(a) (-\zeta)^a \sum_{l=0}^n \binom{n}{l} (-1)^l q^{l(x+a)} \frac{1}{(1+q^{ld}\zeta^d)}. \tag{2.21}$$

3. Further Remark

In this section, we assume that $q \in \mathbb{C}$ with $|q| < 1$. Let χ be the Dirichlet's character with an odd conductor $d \in \mathbb{N}$. From the Mellin transformation of $F_{q,\chi,\zeta}^{(h,r)}(t, x)$ in (2.10), we note that

$$\frac{1}{\Gamma(s)} \oint F_{q,\chi,\zeta}^{(h,r)}(-t, x) t^{s-1} dt = 2^r \sum_{m_1, \dots, m_r=0}^{\infty} \frac{q^{\sum_{j=1}^r (h-j)m_j} (-\zeta)^{m_1+\dots+m_r} \left(\prod_{j=1}^r \chi(m_j) \right)}{[m_1 + \dots + m_r + x]_q^s}, \tag{3.1}$$

where $h, s \in \mathbb{C}, x \neq 0, -1, -2, \dots$, and $r \in \mathbb{N}, \zeta = e^{2\pi i/d}$. By (3.1), we can define the Dirichlet's type multiple (h, q) - l -function as follows.

Definition 3.1. For $s \in \mathbb{C}, x \in \mathbb{R}$ with $x \neq 0, -1, -2, \dots$, one defines the Dirichlet's type multiple (h, q) - l -function related to higher order (h, q) -Euler polynomials as

$$l_q^{(h,r)}(s, x | \chi) = 2^r \sum_{m_1, \dots, m_r=0}^{\infty} \frac{q^{\sum_{j=1}^r (h-j)m_j} (-\zeta)^{m_1+\dots+m_r} \left(\prod_{i=1}^r \chi(m_i) \right)}{[m_1 + \dots + m_r + x]_q^s}, \tag{3.2}$$

where $s, h \in \mathbb{C}, x \neq 0, -1, -2, \dots, r \in \mathbb{N}$, and $\zeta = e^{2\pi i/d}$.

Note that $l_q^{(h,r)}(s, x | \chi)$ is analytic continuation in whole complex s -plane. In (2.10), we note that

$$\begin{aligned} F_{q,\chi,\zeta}^{(h,r)}(t, x) &= 2^r \sum_{n_1, \dots, n_r=0}^{\infty} q^{\sum_{j=1}^r (h-j)n_j} (-\zeta)^{n_1 + \dots + n_r} \left(\prod_{j=1}^r \chi(n_j) \right) e^{[n_1 + \dots + n_r + x]_q t} \\ &= \sum_{n=0}^{\infty} E_{n,\chi,\zeta,q}^{(h,r)}(x) \frac{t^n}{n!}. \end{aligned} \quad (3.3)$$

By Laurent series and Cauchy residue theorem in (3.1) and (3.3), we obtain the following theorem.

Theorem 3.2. *Let χ be Dirichlet's character with odd conductor $d \in \mathbb{N}$, and let $\zeta = e^{2\pi i/d}$. For $h, s \in \mathbb{C}$, $x \neq 0, -1, -2, \dots$, $r \in \mathbb{N}$, and $n \in \mathbb{Z}_+$, one has*

$$l_q^{(h,r)}(-h, x | \chi) = E_{n,\chi,\zeta,q}^{(h,r)}(x). \quad (3.4)$$

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