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Research Article

Existence and Uniqueness of Positive Solutions for Discrete Fourth-Order Lidstone Problem with a Parameter

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This work presents sufficient conditions for the existence and uniqueness of positive solutions for a discrete fourth-order beam equation under Lidstone boundary conditions with a parameter; the iterative sequences yielding approximate solutions are also given. The main tool used is monotone iterative technique.

1. Introduction

In this paper, we are interested in the existence, uniqueness, and iteration of positive solutions for the following nonlinear discrete fourth-order beam equation under Lidstone boundary conditions with explicit parameter β given by

$$\Delta^4 y(t-2) - \beta\Delta^2 y(t-1) = h(t)\left[f_1(y(t)) + f_2(y(t))\right], \quad t \in [a+1, b-1]_{\mathbb{Z}}, \tag{1.1}$$

$$y(a) = 0 = \Delta^2 y(a-1), \qquad y(b) = 0 = \Delta^2 y(b-1), \tag{1.2}$$

where Δ is the usual forward difference operator given by $\Delta y(t) = y(t+1) - y(t)$, $\Delta^n y(t) = \Delta^{n-1}(\Delta y(t))$, $[c,d]_{\mathbb{Z}} := \{c, c+1, \ldots, d-1, d\}$, and $\beta > 0$ is a real parameter.

In recent years, the theory of nonlinear difference equations has been widely applied to many fields such as economics, neural network, ecology, and cybernetics, for details, see

[1–7] and references therein. Especially, there was much attention focused on the existence and multiplicity of positive solutions of fourth-order problem, for example, [8–10], and in particular the discrete problem with Lidstone boundary conditions [11–17]. However, very little work has been done on the uniqueness and iteration of positive solutions of discrete fourth-order equation under Lidstone boundary conditions. We would like to mention some results of Anderson and Minhós [11] and He and Su [12], which motivated us to consider the BVP (1.1) and (1.2).

In [11], Anderson and Minhós studied the following nonlinear discrete fourth-order equation with explicit parameters β and λ given by

$$\Delta^4 y(t-2) - \beta\Delta^2 y(t-1) = \lambda f(t, y(t)), \quad t \in [a+1, b-1]_{\mathbb{Z}}, \tag{1.3}$$

with Lidstone boundary conditions (1.2), where $\beta > 0$ and $\lambda > 0$ are real parameters. The authors obtained the following result.

Theorem 1.1 (see [11]). *Assume that the following condition is satisfied*

- (A_1) $f(t,y) = g(t)w(y)$, *where* $g : [a+1, b-1]_{\mathbb{Z}} \to [0,\infty)$ *with* $\sum_{z=a+1}^{b-1} g(z) > 0$, $w : [0,\infty) \to (0,\infty)$ *is continuous and nondecreasing, and there exists* $\theta \in (0,1)$ *such that* $w(\kappa y) \geq \kappa^{\theta} w(y)$ *for* $\kappa \in (0,1)$ *and* $y \in [0,\infty)$,

then, for any $\lambda \in (0,+\infty)$, *the BVP* (1.3) *and* (1.2) *has a unique positive solution* y_λ. *Furthermore, such a solution* y_λ *satisfies the following properties:*

- (i) $\lim_{\lambda\to 0^+} \|y_\lambda\| = 0$ *and* $\lim_{\lambda\to\infty} \|y_\lambda\| = \infty$;
- (ii) y_λ *is nondecreasing in* λ;
- (iii) y_λ *is continuous in* λ, *that is, if* $\lambda \to \lambda_0$, *then* $\|y_\lambda - y_{\lambda_0}\| \to 0$.

Very recently, in [12], He and Su investigated the existence, multiplicity, and nonexistence of nontrivial solutions to the following discrete nonlinear fourth-order boundary value problem

$$\begin{gathered} \Delta^4 u(t-2) + \eta\Delta^2 u(t-1) - \xi u(t) = \lambda f(t, u(t)), \quad t \in \mathbb{Z}[a+1, b+1], \\ u(a) = 0 = \Delta^2 u(a-1), \qquad u(b+2) = 0 = \Delta^2 u(b+1), \end{gathered} \tag{1.4}$$

where Δ denotes the forward difference operator defined by $\Delta u(t) = u(t+1) - u(t)$, $\Delta^n u(t) = \Delta(\Delta^{n-1} u(t))$, $\mathbb{Z}[a+1, b+1]$ is the discrete interval given by $\{a+1, a+2, \ldots, b+1\}$ with a and b $(a < b)$ integers, η, ξ, λ are real parameters and satisfy

$$\eta < 8\sin^2\frac{\pi}{2(b-a+2)}, \quad \eta^2 + 4\xi \geq 0, \quad \xi + 4\eta\sin^2\frac{\pi}{2(b-a+2)} < 16\sin^4\frac{\pi}{2(b-a+2)}, \quad \lambda > 0. \tag{1.5}$$

For the function f, the authors imposed the following assumption:

- (B_1) $f(t,x) = g(t)h(x)$, where $g : \mathbb{Z}[a+1, b+1] \to [0,\infty)$ with $\sum_{t=a+1}^{b+1} g(t) > 0$, $h : \mathbb{R} \to (0,\infty)$ is continuous and nondecreasing, and there exists $\theta \in (0,1)$ such that $h(\mu x) \geq \mu^{\theta} h(x)$ for $\mu \in (0,1)$ and $x \in [0,\infty)$.

Their main result is the following theorem.

Theorem 1.2 (see [12]). *Assume that* (B_1) *holds. Then for any* $\lambda \in (0,+\infty)$*, the BVP* (1.4) *has a unique positive solution* u_λ*. Furthermore, such a solution* u_λ *satisfies the properties (i)–(iii) stated in Theorem 1.1.*

The aim of this work is to relax the assumptions (A_1) and (B_1) on the nonlinear term, without demanding the existence of upper and lower solutions, we present conditions for the BVP (1.1) and (1.2) to have a unique solution and then study the convergence of the iterative sequence. The ideas come from Zhai et al. [18, 19] and Liang [20].

Let $\mathbb{B}$ denote the Banach space of real-valued functions on $[a-1,b+1]_{\mathbb{Z}}$, with the supremum norm

$$\|y\| = \sup_{t\in[a-1,b+1]_{\mathbb{Z}}} |y(t)|. \tag{1.6}$$

Throughout this paper, we need the following hypotheses:

(H_1) $f_i : [0,+\infty) \to [0,+\infty)$ are continuous and $f_i(y) > 0$ for $y > 0$ $(i = 1,2)$;

(H_2) $h : [a+1,b-1]_{\mathbb{Z}} \to [0,+\infty)$ with $\sum_{z=a+1}^{b-1} h(z) > 0$;

(H_3) $f_1 : [0,+\infty) \to [0,+\infty)$ is nondecreasing, $f_2 : [0,+\infty) \to [0,+\infty)$ is nonincreasing, and there exist $\varphi(\tau)$, $\psi(\tau)$ on interval $[a+1,b-1]_{\mathbb{Z}}$ with $\varphi : [a+1,b-1]_{\mathbb{Z}} \to (0,1)$, for all $e_0 \in (0,1)$, there exists $\tau_0 \in [a+1,b-1]_{\mathbb{Z}}$ such that $\varphi(\tau_0) = e_0$, and $\psi(\tau) > \varphi(\tau)$, for all $\tau \in [a+1,b-1]_{\mathbb{Z}}$ which satisfy

$$f_1(\varphi(\tau)y) \ge \psi(\tau) f_1(y), \quad f_2\left(\frac{1}{\varphi(\tau)}y\right) \ge \psi(\tau) f_2(y), \quad \forall \tau \in [a+1,b-1]_{\mathbb{Z}},\ y \ge 0. \tag{1.7}$$

2. Two Lemmas

To prove the main results in this paper, we will employ two lemmas. These lemmas are based on the linear discrete fourth-order equation

$$\Delta^4 y(t-2) - \beta\Delta^2 y(t-1) = u(t), \quad t \in [a+1,b-1]_{\mathbb{Z}}, \tag{2.1}$$

with Lidstone boundary conditions (1.2).

Lemma 2.1 (see [11]). *Let* $u : [a+1,b-1]_{\mathbb{Z}} \to \mathbb{R}$ *be a function. Then the nonhomogeneous discrete fourth-order Lidstone boundary value problem* (2.1), (1.2) *has solution*

$$y(t) = \sum_{s=a}^{b} \sum_{z=a+1}^{b-1} G_2(t,s) G_1(s,z) u(z), \quad t \in [a-1,b+1]_{\mathbb{Z}}, \tag{2.2}$$

where $G_2(t,s)$ *given by*

$$G_2(t,s) = \frac{1}{\ell(1,0)\ell(b,a)} \begin{cases} \ell(t,a)\ell(b,s): & t \le s, \\ \ell(s,a)\ell(b,t): & s \le t, \end{cases} \quad (t,s) \in [a-1,b+1]_{\mathbb{Z}} \times [a,b]_{\mathbb{Z}} \tag{2.3}$$

with $\ell(t,s) = \mu^{t-s} - \mu^{s-t}$ *for* $\mu = (\beta + 2 + \sqrt{\beta(\beta+4)})/2$, *is the Green's function for the second-order discrete boundary value problem*

$$\begin{gathered} -\left(\Delta^2 y(t-1) - \beta y(t)\right) = 0, \quad t \in [a,b]_{\mathbb{Z}}, \\ y(a) = 0 = y(b), \end{gathered} \tag{2.4}$$

and $G_1(s,z)$ *given by*

$$G_1(s,z) = \frac{1}{b-a} \begin{cases} (s-a)(b-z): & s \le z, \\ (z-a)(b-s): & z \le s, \end{cases} \quad (s,z) \in [a,b]_{\mathbb{Z}} \times [a+1,b-1]_{\mathbb{Z}} \tag{2.5}$$

is the Green's function for the second-order discrete boundary value problem

$$\begin{gathered} -\Delta^2 x(s-1) = 0, \quad s \in [a+1,b-1]_{\mathbb{Z}}, \\ x(a) = 0 = x(b). \end{gathered} \tag{2.6}$$

Lemma 2.2 (see [11]). *Let*

$$m := \frac{\ell(1,0)\ell(b,a+1)}{(b-a)\ell^2(b,a)}, \qquad M := \frac{(b-a)\ell^2(b/2,a/2)}{4\ell(1,0)\ell(b,a)}. \tag{2.7}$$

Then, for $(t,s,z) \in [a+1,b-1]_{\mathbb{Z}}^3$, *one has*

$$m \le G_2(t,s)G_1(s,z) \le M. \tag{2.8}$$

3. Main Results

Theorem 3.1. *Assume that* (H_1)–(H_3) *hold. Then, the BVP* (1.1) *and* (1.2) *has a unique solution* $y^*(t)$ *in* D, *where*

$$D = \left\{ y \in \mathbb{B} \mid y(a) = 0 = y(b),\ y(t) > 0,\ t \in [a+1,b-1]_{\mathbb{Z}} \right\}. \tag{3.1}$$

Moreover, for any $x_0, y_0 \in D$, constructing successively the sequences

$$x_{n+1}(t) = \sum_{s=a}^{b} \sum_{z=a+1}^{b-1} G_2(t,s)G_1(s,z)h(z)\left[f_1(x_n(z)) + f_2(y_n(z))\right], \quad t \in [a-1,b+1]_{\mathbb{Z}},\ n = 0,1,2,\ldots,$$
$$y_{n+1}(t) = \sum_{s=a}^{b} \sum_{z=a+1}^{b-1} G_2(t,s)G_1(s,z)h(z)\left[f_1(y_n(z)) + f_2(x_n(z))\right], \quad t \in [a-1,b+1]_{\mathbb{Z}},\ n = 0,1,2,\ldots, \tag{3.2}$$

One has $x_n(t)$, $y_n(t)$ converge uniformly to $y^*(t)$ in $[a-1,b+1]_{\mathbb{Z}}$.

Proof. First, we show that the BVP (1.1) and (1.2) has a solution.

It is easy to see that the BVP (1.1) and (1.2) has a solution $y = y(t)$ if and only if y is a fixed point of the operator equation

$$A(y_1,y_2)(t) = \sum_{s=a}^{b} \sum_{z=a+1}^{b-1} G_2(t,s)G_1(s,z)h(z)\left[f_1(y_1(z)) + f_2(y_2(z))\right], \quad t \in [a-1,b+1]_{\mathbb{Z}}. \tag{3.3}$$

In view of (H_3) and (3.3), $A(y_1,y_2)$ is nondecreasing in y_1 and nonincreasing in y_2. Moreover, for any $\tau \in [a+1,b-1]_{\mathbb{Z}}$, we have

$$\begin{aligned} A\left(\varphi(\tau)y_1, \frac{1}{\varphi(\tau)}y_2\right)(t) &= \sum_{s=a+1}^{b-1} \sum_{z=a+1}^{b-1} G_2(t,s)G_1(s,z)h(z)\left[f_1(\varphi(\tau)y_1(z)) + f_2\left(\frac{1}{\varphi(\tau)}y_2(z)\right)\right] \\ &\geq \varphi(\tau) \sum_{s=a+1}^{b-1} \sum_{z=a+1}^{b-1} G_2(t,s)G_1(s,z)h(z)\left[f_1(y_1(z)) + f_2(y_2(z))\right] \\ &= \varphi(\tau)A(y_1,y_2)(t) \end{aligned} \tag{3.4}$$

for $t \in [a,b]_{\mathbb{Z}}$ and $y_1, y_2 \in D$.

Let

$$L = (b-a-1) \sum_{z=a+1}^{b-1} h(z), \tag{3.5}$$

condition (H_2) implies $L > 0$. Since $f_i(y) > 0$ for $y > 0$ $(i = 1, 2)$, by Lemma 2.2, we have

$$\begin{aligned} A(L, L) &= \sum_{s=a+1}^{b-1} \sum_{z=a+1}^{b-1} G_2(t, s) G_1(s, z) h(z) \left[f_1(L) + f_2(L) \right] \\ &\geq m \left[f_1(L) + f_2(L) \right] \sum_{s=a+1}^{b-1} \sum_{z=a+1}^{b-1} h(z) \\ &= m \left[f_1(L) + f_2(L) \right] L \end{aligned} \tag{3.6}$$

for m in (2.1) and L in (3.5).

Moreover, we obtain

$$A(L, L) \leq M \left[f_1(L) + f_2(L) \right] L \tag{3.7}$$

for M in (2.1).

Thus

$$m \left[f_1(L) + f_2(L) \right] L \leq A(L, L) \leq M \left[f_1(L) + f_2(L) \right] L. \tag{3.8}$$

Therefore, we can choose a sufficiently small number $e_1 \in (0, 1)$ such that

$$e_1 L \leq A(L, L) \leq \frac{L}{e_1}, \tag{3.9}$$

which together with (H_3) implies that there exists $\tau_1 \in [a + 1, b - 1]_{\mathbb{Z}}$ such that $\varphi(\tau_1) = e_1$, so

$$\varphi(\tau_1) L \leq A(L, L) \leq \frac{L}{\varphi(\tau_1)}. \tag{3.10}$$

Since $\psi(\tau_1)/\varphi(\tau_1) > 1$, we can take a sufficiently large positive integer k such that

$$\left[\frac{\psi(\tau_1)}{\varphi(\tau_1)} \right]^k \geq \frac{1}{\varphi(\tau_1)}. \tag{3.11}$$

It is clear that

$$\left[\frac{\varphi(\tau_1)}{\psi(\tau_1)} \right]^k \leq \varphi(\tau_1). \tag{3.12}$$

We define

$$u_0(t) = \begin{cases} -[\varphi(\tau_1)]^k L: & t = a-1,\ b+1, \\ 0: & t = a, b, \\ [\varphi(\tau_1)]^k L: & t \in [a+1, b-1]_{\mathbb{Z}}, \end{cases}$$
$$v_0(t) = \begin{cases} -\dfrac{L}{[\varphi(\tau_1)]^k}: & t = a-1,\ b+1, \\ 0: & t = a, b, \\ \dfrac{L}{[\varphi(\tau_1)]^k}: & t \in [a+1, b-1]_{\mathbb{Z}}. \end{cases} \tag{3.13}$$

Evidently, for $t \in [a,b]_{\mathbb{Z}}$, $u_0 \le v_0$. Take any $\lambda \in (0, [\varphi(\tau_1)]^{2k}]$, then $\lambda \in (0,1)$ and $u_0 \ge \lambda v_0$.

By the mixed monotonicity of A, we have $A(u_0, v_0) \le A(v_0, u_0)$. In addition, combining (H_3) with (3.10) and (3.11), we get

$$\begin{aligned} A(u_0, v_0) &= A\left([\varphi(\tau_1)]^k L, \frac{1}{[\varphi(\tau_1)]^k} L \right) \\ &= A\left(\varphi(\tau_1)[\varphi(\tau_1)]^{k-1} L, \frac{1}{\varphi(\tau_1)[\varphi(\tau_1)]^{k-1}} L \right) \\ &\ge \varphi(\tau_1) A\left([\varphi(\tau_1)]^{k-1} L, \frac{1}{[\varphi(\tau_1)]^{k-1}} L \right) \ge \cdots \\ &\ge [\varphi(\tau_1)]^k A(L, L) \ge [\varphi(\tau_1)]^k \varphi(\tau_1) L \\ &\ge [\varphi(\tau_1)]^k L = u_0. \end{aligned} \tag{3.14}$$

From (H_3), we have

$$\begin{aligned} A(y_1, y_2) &= A\left(\varphi(s) \frac{y_1}{\varphi(s)}, \frac{1}{\varphi(s)} \varphi(s) y_2 \right) \\ &\ge \varphi(s) A\left(\frac{y_1}{\varphi(s)}, \varphi(s) y_2 \right), \quad \forall s \in [a+1, b-1]_{\mathbb{Z}},\ y_1, y_2 \ge 0, \end{aligned} \tag{3.15}$$

and hence

$$A\left(\frac{y_1}{\varphi(s)}, \varphi(s) y_2 \right) \le \frac{1}{\varphi(s)} A(y_1, y_2), \quad \forall s \in [a+1, b-1]_{\mathbb{Z}},\ y_1, y_2 \ge 0. \tag{3.16}$$

Thus, we have

$$\begin{aligned}
A(v_0,u_0) &= A\left(\frac{L}{[\varphi(\tau_1)]^k}, [\varphi(\tau_1)]^k L\right)\\
&= A\left(\frac{L}{\varphi(\tau_1)[\varphi(\tau_1)]^{k-1}}, \varphi(\tau_1)[\varphi(\tau_1)]^{k-1} L\right)\\
&\le \frac{1}{\varphi(\tau_1)} A\left(\frac{L}{[\varphi(\tau_1)]^{k-1}}, [\varphi(\tau_1)]^{k-1} L\right) \le \cdots\\
&\le \frac{1}{[\varphi(\tau_1)]^k} A(L,L) \le \frac{1}{[\varphi(\tau_1)]^k}\frac{L}{\varphi(\tau_1)}.
\end{aligned} \tag{3.17}$$

In accordance with (3.12), we can see that

$$A(v_0,u_0) \le \frac{L}{[\varphi(\tau_1)]^k} = v_0. \tag{3.18}$$

Construct successively the sequences

$$u_n = A(u_{n-1}, v_{n-1}), \quad v_n = A(v_{n-1}, u_{n-1}), \quad n = 1,2,\ldots. \tag{3.19}$$

By the mixed monotonicity of A, we have $u_1 = A(u_0,v_0) \le A(v_0,u_0) = v_1$. By induction, we obtain $u_n \le v_n,\ n = 1,2,\ldots$. It follows from (3.14), (3.18), and the mixed monotonicity of A that

$$u_0 \le u_1 \le \cdots \le u_n \le \cdots \le v_n \le \cdots \le v_1 \le v_0. \tag{3.20}$$

Note that $u_0 \ge \lambda v_0$, so we can get $u_n(t) \ge u_0(t) \ge \lambda v_0(t) \ge \lambda v_n(t),\ t \in [a,b]_{\mathbb{Z}},\ n = 1,2,\ldots$. Let

$$\lambda_n = \sup\{\lambda > 0 \mid u_n(t) \ge \lambda v_n(t),\ t \in [a,b]_{\mathbb{Z}}\}, \quad n = 1,2,\ldots. \tag{3.21}$$

Thus, we have

$$u_n(t) \ge \lambda_n v_n(t), \quad t \in [a,b]_{\mathbb{Z}},\ n = 1,2,\ldots, \tag{3.22}$$

and then

$$u_{n+1}(t) \ge u_n(t) \ge \lambda_n v_n(t) \ge \lambda_n v_{n+1}(t), \quad t \in [a,b]_{\mathbb{Z}},\ n = 1,2,\ldots. \tag{3.23}$$

Therefore, $\lambda_{n+1} \ge \lambda_n$, that is, $\{\lambda_n\}$ is increasing with $\{\lambda_n\} \subset (0,1]$. Set $\widetilde{\lambda} = \lim_{n\to\infty}\lambda_n$. We can show that $\widetilde{\lambda} = 1$. In fact, if $0 < \widetilde{\lambda} < 1$, by (H_3), there exists $\tau_2 \in [a+1,b-1]_{\mathbb{Z}}$ such that $\varphi(\tau_2) = \widetilde{\lambda}$. Consider the following two cases.

(i) There exists an integer N such that $\lambda_N = \tilde{\lambda}$. In this case, we have $\lambda_n = \tilde{\lambda}$ for all $n \geq N$ holds. Hence, for $n \geq N$, it follows from (3.4) and the mixed monotonicity of A that

$$u_{n+1} = A(u_{\mathrm{n}}, v_n) \geq A\left(\tilde{\lambda} v_n, \frac{1}{\tilde{\lambda}} u_n\right) = A\left(\varphi(\tau_2) v_n, \frac{1}{\varphi(\tau_2)} u_n\right) \geq \psi(\tau_2) A(v_n, u_n) = \psi(\tau_2) v_{n+1}. \tag{3.24}$$

By the definition of λ_n, we have

$$\lambda_{n+1} = \tilde{\lambda} \geq \psi(\tau_2) > \varphi(\tau_2) = \tilde{\lambda}. \tag{3.25}$$

This is a contradiction.

(ii) For all integer n, $\lambda_n < \tilde{\lambda}$. In this case, we have $0 < \lambda_n / \tilde{\lambda} < 1$. In accordance with (H_3), there exists $\theta_n \in [a+1, b-1]_{\mathbb{Z}}$ such that $\varphi(\theta_n) = \lambda_n / \tilde{\lambda}$. Hence, combining (3.4) with the mixed monotonicity of A, we have

$$\begin{aligned}
u_{n+1} &= A(u_n, v_n) \geq A\left(\lambda_n v_n, \frac{1}{\lambda_n} u_n\right) \\
&= A\left(\frac{\lambda_n}{\tilde{\lambda}} \tilde{\lambda} v_n, \frac{u_n}{\left(\lambda_n / \tilde{\lambda}\right) \tilde{\lambda}}\right) = \mathrm{A}\left(\varphi(\theta_n) \varphi(\tau_2) v_n, \frac{u_n}{\varphi(\theta_n) \varphi(\tau_2)}\right) \\
&\geq \psi(\theta_n) A\left(\varphi(\tau_2) v_n, \frac{u_n}{\varphi(\tau_2)}\right) \geq \psi(\theta_n) \psi(\tau_2) A(v_n, u_n) \\
&= \psi(\theta_n) \psi(\tau_2) v_{n+1}.
\end{aligned} \tag{3.26}$$

By the definition of λ_n, we have

$$\lambda_{n+1} \geq \psi(\theta_n) \psi(\tau_2) > \varphi(\theta_n) \psi(\tau_2) = \frac{\lambda_n}{\tilde{\lambda}} \psi(\tau_2). \tag{3.27}$$

Let $n \to \infty$, we have $\tilde{\lambda} \geq (\tilde{\lambda}/\tilde{\lambda}) \psi(\tau_2) > (\tilde{\lambda}/\tilde{\lambda}) \varphi(\tau_2) = \varphi(\tau_2) = \tilde{\lambda}$, and this is also a contradiction. Hence, $\lim_{n \to \infty} \lambda_n = 1$.

Thus, combining (3.20) with (3.22), we have

$$0 \leq u_{n+l}(t) - u_n(t) \leq v_n(t) - u_n(t) \leq v_n(t) - \lambda_n v_n(t) = (1 - \lambda_n) v_n(t) \leq (1 - \lambda_n) v_0(t) \tag{3.28}$$

for $t \in [a, b]_{\mathbb{Z}}$, where l is a nonnegative integer. Thus,

$$\| u_{n+l} - u_n \| \leq \| v_n - u_n \| \leq (1 - \lambda_n) v_0. \tag{3.29}$$

Therefore, there exists a function $y^* \in D$ such that

$$\lim_{n \to \infty} u_n(t) = \lim_{n \to \infty} v_n(t) = y^*(t) \quad \text{for } t \in [a-1, b+1]_{\mathbb{Z}}. \tag{3.30}$$

By the mixed monotonicity of A and (3.20), we have

$$u_{n+1}(t) = A(u_n(t), v_n(t)) \leq A\big(y^*(t), y^*(t)\big) \leq A(v_n(t), u_n(t)) = v_{n+1}(t). \tag{3.31}$$

Let $n \to \infty$ and we get $A(y^*(t), y^*(t)) = y^*(t)$, $t \in [a-1, b+1]_{\mathbb{Z}}$. That is, y^* is a nontrivial solution of the BVP (1.1) and (1.2).

Next, we show the uniqueness of solutions of the BVP (1.1) and (1.2). Assume, to the contrary, that there exist two nontrivial solutions y_1 and y_2 of the BVP (1.1) and (1.2) such that $A(y_1(t), y_1(t)) = y_1(t)$ and $A(y_2(t), y_2(t)) = y_2(t)$ for $t \in [a-1, b+1]_{\mathbb{Z}}$. According to (3.9), we can know that there exists $0 < \eta \leq 1$ such that $\eta y_2(t) \leq y_1(t) \leq (1/\eta) y_2(t)$ for $t \in [a, b]_{\mathbb{Z}}$. Let

$$\eta_0 = \sup\left\{0 < \eta \leq 1 \mid \eta y_2 \leq y_1 \leq \frac{1}{\eta} y_2\right\}. \tag{3.32}$$

Then $0 < \eta_0 \leq 1$ and $\eta_0 y_2(t) \leq y_1(t) \leq (1/\eta_0) y_2(t)$ for $t \in [a, b]_{\mathbb{Z}}$.

We now show that $\eta_0 = 1$. In fact, if $0 < \eta_0 < 1$, then, in view of (H_3), there exists $\overline{\tau} \in [a+1, b-1]_{\mathbb{Z}}$ such that $\varphi(\overline{\tau}) = \eta_0$. Furthermore, we have

$$y_1 = A(y_1, y_1) \geq A\left(\eta_0 y_2, \frac{1}{\eta_0} y_2\right) = A\left(\varphi(\overline{\tau}) y_2, \frac{1}{\varphi(\overline{\tau})} y_2\right) \geq \psi(\overline{\tau}) A(y_2, y_2) = \psi(\overline{\tau}) y_2, \tag{3.33}$$

$$y_1 = A(y_1, y_1) \leq A\left(\frac{y_2}{\eta_0}, \eta_0 y_2\right) = A\left(\frac{y_2}{\varphi(\overline{\tau})}, \varphi(\overline{\tau}) y_2\right) \leq \frac{1}{\psi(\overline{\tau})} A(y_2, y_2) = \frac{1}{\psi(\overline{\tau})} y_2. \tag{3.34}$$

In (3.34), we used the relation formula (3.16). Since $\psi(\overline{\tau}) > \varphi(\overline{\tau}) = \eta_0$, this contradicts the definition of η_0. Hence $\eta_0 = 1$. Therefore, the BVP (1.1) and (1.2) has a unique solution.

Finally, we show that "moreover" part of the theorem. For any initial $x_0, y_0 \in D$, in accordance with (3.9), we can choose a sufficiently small number $e_2 \in (0, 1)$ such that

$$e_2 L \leq x_0 \leq \frac{1}{e_2} L, \qquad e_2 L \leq y_0 \leq \frac{1}{e_2} L. \tag{3.35}$$

It follows from (H_3) that there exists $\tau_3 \in [a+1, b-1]_{\mathbb{Z}}$ such that $\varphi(\tau_3) = e_2$, and hence

$$\varphi(\tau_3) L \leq x_0 \leq \frac{L}{\varphi(\tau_3)}, \qquad \varphi(\tau_3) L \leq y_0 \leq \frac{L}{\varphi(\tau_3)}. \tag{3.36}$$

Thus, we can choose a sufficiently large positive integer k such that

$$\left[\frac{\psi(\tau_3)}{\varphi(\tau_3)}\right]^k \geq \frac{1}{\varphi(\tau_3)}. \tag{3.37}$$

Define

$$\widehat{u}_0 = \big[\varphi(\tau_3)\big]^k L, \qquad \widehat{v}_0 = \frac{L}{\big[\varphi(\tau_3)\big]^k}. \tag{3.38}$$

Obviously, $\widehat{u}_0 < x_0,\ y_0 < \widehat{v}_0$. Let

$$\widehat{u}_n = A(\widehat{u}_{n-1}, \widehat{v}_{n-1}), \quad \widehat{v}_n = A(\widehat{v}_{n-1}, \widehat{u}_{n-1}), \quad n = 1, 2, \ldots,$$

$$\begin{aligned} x_n(t) &= A(x_{n-1}, y_{n-1})(t) = \sum_{s=a}^{b} \sum_{z=a+1}^{b-1} G_2(t,s) G_1(s,z) h(z) \left[f_1(x_{n-1}(z)) + f_2(y_{n-1}(z)) \right], \\ y_n(t) &= A(y_{n-1}, x_{n-1})(t) = \sum_{s=a}^{b} \sum_{z=a+1}^{b-1} G_2(t,s) G_1(s,z) h(z) \left[f_1(y_{n-1}(z)) + f_2(x_{n-1}(z)) \right] \end{aligned} \tag{3.39}$$

for $t \in [a-1, b+1]_{\mathbb{Z}},\ n = 1, 2, \ldots$. By induction, we get $\widehat{u}_n \le x_n \le \widehat{v}_n$, $\widehat{u}_n \le y_n \le \widehat{v}_n$, $n = 1, 2, \ldots$.

Similarly to the above proof, it follows that there exists $\widehat{y} \in D$ such that

$$\lim_{n\to\infty} \widehat{u}_n = \lim_{n\to\infty} \widehat{v}_n = \widehat{y}, \quad A(\widehat{y}, \widehat{y}) = \widehat{y}. \tag{3.40}$$

By the uniqueness of fixed points A in D, we get $\widehat{y} = y^*$. Therefore, we have

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = y^*. \tag{3.41}$$

This completes the proof of the theorem. □

Remark 3.2. From the proof of Theorem 3.1, we easily know that assume $\overline{y} = A(\overline{y}, \overline{x})$, $\overline{x} = A(\overline{x}, \overline{y})$, thus, let $y_0 = \overline{y}$, $x_0 = \overline{x}$, we have

$$y_n = \overline{y}, \quad x_n = \overline{x}, \quad n = 1, 2, \ldots. \tag{3.42}$$

Therefore $\overline{y} = \overline{x} = y^*$.

Theorem 3.3. *Assume that (H_2) holds, and the following conditions are satisfied:*

(C_1) $f : [0, +\infty) \to [0, +\infty)$ *is continuous and* $f(y) > 0$ *for* $y > 0$;

(C_2) $f : [0, +\infty) \to [0, +\infty)$ *is nondecreasing;*

$$\sum_{s=a}^{b} \sum_{z=a+1}^{b-1} G_2(t,s) G_1(s,z) h(z) f(\varphi(\tau) y(z)) \ge \psi(\tau, y) \sum_{s=a}^{b} \sum_{z=a+1}^{b-1} G_2(t,s) G_1(s,z) h(z) f(y(z)), \tag{3.43}$$

for all $\tau \in [a+1, b-1]_{\mathbb{Z}},\ y \in [0, +\infty)$, *where* $\varphi : [a+1, b-1]_{\mathbb{Z}} \to (0,1)$, *for all* $e_0 \in (0,1)$, *there exists* $\tau_0 \in [a+1, b-1]_{\mathbb{Z}}$ *such that* $\varphi(\tau_0) = e_0$, *and* $\psi : [a+1, b-1]_{\mathbb{Z}} \times [0, +\infty) \to (0, +\infty)$, *with* $\psi(\tau, y) > \varphi(\tau)$, *for all* $\tau \in [a+1, b-1]_{\mathbb{Z}},\ y \in [0, +\infty)$;

(C_3) *for fixed* $\tau \in [a+1, b-1]_{\mathbb{Z}}$, *one has*

(i) $\psi(\tau, y)$ *is nonincreasing with respect to* y, *and there exists* $\tau_4 \in [a+1, b-1]_{\mathbb{Z}}$ *such that*

$$mf(L) \geq \varphi(\tau_4), \qquad \frac{\psi(\tau_4, L/\varphi(\tau_4))}{\varphi(\tau_4)} \geq Mf(L) \tag{3.44}$$

or

(ii) $\psi(\tau, y)$ *is nondecreasing with respect to* y, *and there exists* $\tau_5 \in [a+1, b-1]_{\mathbb{Z}}$ *such that*

$$mf(L) \geq \frac{\varphi(\tau_5)}{\psi(\tau_5, L)}, \qquad \frac{1}{\varphi(\tau_5)} \geq Mf(L), \tag{3.45}$$

where m, M *are defined in* (2.1), L *is defined in* (3.5). *Then, the BVP*

$$\begin{gathered} \Delta^4 y(t-2) - \beta\Delta^2 y(t-1) = h(t) f(y(t)), \quad t \in [a+1, b-1]_{\mathbb{Z}}, \\ y(a) = 0 = \Delta^2 y(a-1), \qquad y(b) = 0 = \Delta^2 y(b-1) \end{gathered} \tag{3.46}$$

has a unique solution y^*.

Proof. For convenience, we still define the operator equation A by

$$Ay(t) = \sum_{s=a}^{b} \sum_{z=a+1}^{b-1} G_2(t,s) G_1(s,z) h(z) f(y(z)), \quad t \in [a-1, b+1]_{\mathbb{Z}}. \tag{3.47}$$

In the following, we consider the following two cases.

(i) For fixed $\tau \in [a+1, b-1]_{\mathbb{Z}}$, $\psi(\tau, y)$ is nonincreasing with respect to y.

According to condition (C_3) and Lemma 2.2, we can know that there exists $\tau_4 \in [a+1, b-1]_{\mathbb{Z}}$ such that

$$\varphi(\tau_4) L \leq A(L) \leq \frac{\psi(\tau_4, L/\varphi(\tau_4))}{\varphi(\tau_4)} L. \tag{3.48}$$

Since $\psi(\tau_4, L)/\varphi(\tau_4) > 1$, we can find a sufficiently large positive integer k such that

$$\left[\frac{\psi(\tau_4, L)}{\varphi(\tau_4)}\right]^k \geq \frac{1}{\varphi(\tau_4)}. \tag{3.49}$$

For $t \in [a+1, b-1]_{\mathbb{Z}}$, we still define

$$u_0(t) = [\varphi(\tau_4)]^k L, \qquad v_0(t) = \frac{L}{[\varphi(\tau_4)]^k}, \tag{3.50}$$
$$u_n(t) = Au_{n-1}(t), \quad v_n(t) = Av_{n-1}(t), \quad n = 1, 2, \ldots.$$

By the proof of Theorem 3.1, it is sufficient to show that

$$u_0 \leq u_1 \leq v_1 \leq v_0. \tag{3.51}$$

Obviously, $u_0 \leq v_0$ and $u_1 \leq v_1$.

In this case, it follows from conditions (C_2), (C_3), and (3.49) that

$$\begin{aligned}
u_1 = Au_0 &= A\left([\varphi(\tau_4)]^k L\right) \\
&= A\left(\varphi(\tau_4)[\varphi(\tau_4)]^{k-1} L\right) \\
&\geq \psi\left(\tau_4, [\varphi(\tau_4)]^{k-1} L\right) A\left([\varphi(\tau_4)]^{k-1} L\right) \\
&= \psi\left(\tau_4, [\varphi(\tau_4)]^{k-1} L\right) A\left(\varphi(\tau_4)[\varphi(\tau_4)]^{k-2} L\right) \\
&\geq \psi\left(\tau_4, [\varphi(\tau_4)]^{k-1} L\right) \psi\left(\tau_4, [\varphi(\tau_4)]^{k-2} L\right) A\left([\varphi(\tau_4)]^{k-2} L\right) \\
&\geq \cdots \\
&\geq \psi\left(\tau_4, [\varphi(\tau_4)]^{k-1} L\right) \psi\left(\tau_4, [\varphi(\tau_4)]^{k-2} L\right) \cdots \psi(\tau_4, L) A(L) \\
&\geq [\psi(\tau_4, L)]^k \varphi(\tau_4) L \\
&\geq [\varphi(\tau_4)]^k L = u_0.
\end{aligned} \tag{3.52}$$

In accordance with (3.16), we have

$$A\left(\frac{y}{\varphi(s)}\right) \leq \frac{1}{\psi(s, y/\varphi(s))} Ay, \tag{3.53}$$

which together with condition (C_2) and (3.48) implies that

$$
\begin{aligned}
v_1 &= Av_0 = A\left(\frac{L}{[\varphi(\tau_4)]^k}\right)\\
&= A\left(\frac{L}{\varphi(\tau_4)[\varphi(\tau_4)]^{k-1}}\right)\\
&\le \frac{1}{\psi\left(\tau_4, L/[\varphi(\tau_4)]^k\right)}A\left(\frac{L}{[\varphi(\tau_4)]^{k-1}}\right)\\
&= \frac{1}{\psi\left(\tau_4, L/[\varphi(\tau_4)]^k\right)}A\left(\frac{L}{\varphi(\tau_4)[\varphi(\tau_4)]^{k-2}}\right)\\
&\le \frac{1}{\psi\left(\tau_4, L/[\varphi(\tau_4)]^k\right)}\frac{1}{\psi\left(\tau_4, L/[\varphi(\tau_4)]^{k-1}\right)}A\left(\frac{L}{[\varphi(\tau_4)]^{k-2}}\right)\\
&\le \frac{1}{\psi\left(\tau_4, L/[\varphi(\tau_4)]^k\right)}\frac{1}{\psi\left(\tau_4, L/[\varphi(\tau_4)]^{k-1}\right)}\cdots\frac{1}{\psi(\tau_4, L/\varphi(\tau_4))}A(L)\\
&\le \frac{1}{[\varphi(\tau_4)]^{k-1}}\frac{1}{\psi(\tau_4, L/\varphi(\tau_4))}A(L)\\
&\le \frac{L}{[\varphi(\tau_4)]^k} = v_0.
\end{aligned}
\tag{3.54}
$$

(ii) For fixed $\tau \in [a+1, b-1]_{\mathbb{Z}}$, $\psi(\tau, y)$ is nondecreasing with respect to y.

In this case, by condition (C_3) and Lemma 2.2, we can know that there exists $\tau_5 \in [a+1, b-1]_{\mathbb{Z}}$ such that

$$\frac{\varphi(\tau_5)L}{\psi(\tau_5, L)} \le A(L) \le \frac{L}{\varphi(\tau_5)}. \tag{3.55}$$

Since $0 < \varphi(\tau_5)/\psi(\tau_5, L/\varphi(\tau_5)) < 1$, we can take a sufficiently large positive integer k such that

$$\left[\frac{\varphi(\tau_5)}{\psi(\tau_5, L/\varphi(\tau_5))}\right]^k \le \varphi(\tau_5). \tag{3.56}$$

For $t \in [a+1, b-1]_{\mathbb{Z}}$, we still define

$$\begin{gathered} u_0(t) = [\varphi(\tau_5)]^k L, \qquad v_0(t) = \frac{L}{[\varphi(\tau_5)]^k}, \\ u_n(t) = Au_{n-1}(t), \quad v_n(t) = Av_{n-1}(t), \quad n = 1, 2, \ldots. \end{gathered} \tag{3.57}$$

We continue to prove that

$$u_1 \geq u_0, \quad v_1 \leq v_0. \tag{3.58}$$

By (3.52), combining (3.55) with the monotonicity of ψ, we have

$$\begin{aligned} u_1 &= Au_0 = A\left([\varphi(\tau_5)]^k L\right) \\ &\geq \psi\left(\tau_5, [\varphi(\tau_5)]^{k-1} L\right)\psi\left(\tau_5, [\varphi(\tau_5)]^{k-2} L\right)\cdots\psi(\tau_5, L)A(L) \\ &\geq [\varphi(\tau_5)]^{k-1}\psi(\tau_5, L)A(L) \\ &\geq [\varphi(\tau_5)]^k L = u_0. \end{aligned} \tag{3.59}$$

In accordance with (3.54), combining the monotonicity of ψ and (3.55), we get

$$\begin{aligned} v_1 &= Av_0 = A\left(\frac{L}{[\varphi(\tau_5)]^k}\right) \\ &\leq \frac{1}{\psi\left(\tau_5, L/[\varphi(\tau_5)]^k\right)}\frac{1}{\psi\left(\tau_5, L/[\varphi(\tau_5)]^{k-1}\right)}\cdots\frac{1}{\psi(\tau_5, L/\varphi(\tau_5))}A(L) \\ &\leq \frac{1}{[\psi(\tau_5, L/\varphi(\tau_5))]^k}\frac{L}{\varphi(\tau_5)}. \end{aligned} \tag{3.60}$$

An application of (3.56) yields

$$v_1 \leq \frac{1}{[\varphi(\tau_5)]^k} L = v_0. \tag{3.61}$$

Therefore, we obtain

$$u_0 \leq u_1 \leq v_1 \leq v_0. \tag{3.62}$$

For $t = a-1,\ b+1$, the proof is similar and hence omitted. This completes the proof of the theorem. □

Remark 3.4. In Theorem 3.1, the more general conditions are imposed on the nonlinear term than Theorem 1.1. In particular, in Theorem 3.3, $\varphi(\tau, y)$ contains the variable y; therefore, the more comprehensive functions can be incorporated.

4. An Example

Example 4.1. Consider the following discrete fourth-order Lidstone problem:

$$\begin{gathered} \Delta^4 y(t-2) - \Delta^2 y(t-1) = t\left[1 + y^{1/4}(t) + 2 + \frac{1}{y^{1/4}(t)}\right], \quad t \in [2+1, 7-1]_{\mathbb{Z}}, \\ y(2) = 0 = \Delta^2 y(1), \qquad y(7) = 0 = \Delta^2 y(6). \end{gathered} \tag{4.1}$$

We claim that the BVP (4.1) and (1.2) has a unique solution $y^*(t)$ in D, where

$$D = \left\{ y \in \mathbb{B} \mid y(2) = 0 = y(7),\ y(t) > 0,\ t \in [3,6]_{\mathbb{Z}} \right\}. \tag{4.2}$$

Moreover, for any $x_0, y_0 \in D$, constructing successively the sequences

$$\begin{gathered} x_{n+1}(t) = \sum_{s=2}^{7} \sum_{z=3}^{6} G_2(t,s) G_1(s,z) z \left[1 + x_n^{1/4}(z) + 2 + \frac{1}{y_n^{1/4}(z)}\right], \quad t \in [1,8]_{\mathbb{Z}},\ n = 0,1,2,\ldots, \\ y_{n+1}(t) = \sum_{s=2}^{7} \sum_{z=3}^{6} G_2(t,s) G_1(s,z) z \left[1 + y_n^{1/4}(z) + 2 + \frac{1}{x_n^{1/4}(z)}\right], \quad t \in [2,8]_{\mathbb{Z}},\ n = 0,1,2,\ldots, \end{gathered} \tag{4.3}$$

we have $x_n(t), y_n(t)$ converge uniformly to $y^*(t)$ in $[2,8]_{\mathbb{Z}}$.

In fact, we choose $f_1(y) = 1 + y^{1/4}$, $f_2(y) = 2 + 1/y^{1/4}$, $h(z) = z$, thus $f_i(y) > 0$ for $y > 0$ $(i = 1,2)$, $\sum_{z=3}^{6} h(z) = \sum_{z=3}^{6} z = 18 > 0$. It is easy to check that f_1 is nondecreasing on $[0, +\infty)$, f_2 is nonincreasing on $[0, +\infty)$. In addition, we set

$$\tau = \begin{cases} 3, & \varphi(\tau) \in \left(0, \dfrac{1}{4}\right], \\ 4, & \varphi(\tau) \in \left(\dfrac{1}{4}, \dfrac{1}{2}\right], \\ 5, & \varphi(\tau) \in \left(\dfrac{1}{2}, \dfrac{3}{4}\right], \\ 6, & \varphi(\tau) \in \left(\dfrac{3}{4}, 1\right), \end{cases} \tag{4.4}$$

$\psi(\tau) = [\varphi(\tau)]^{1/2}$. It is easy to see that

$$\begin{aligned} f_1(\varphi(\tau)y) &= 1 + (\varphi(\tau)y)^{1/4} \geq \psi(\tau)\left(1 + y^{1/4}\right) = \psi(\tau) f_1(y), \quad \forall \tau \in [3,6]_{\mathbb{Z}},\ y \geq 0, \\ f_2\left(\frac{y}{\varphi(\tau)}\right) &= 2 + \frac{1}{(y/\varphi(\tau))^{1/4}} \geq \psi(\tau)\left(2 + \frac{1}{y^{1/4}}\right), \quad \forall \tau \in [3,6]_{\mathbb{Z}},\ y \geq 0. \end{aligned} \tag{4.5}$$

The conclusion then follows from Theorem 3.1.

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