Research Article

New Existence Results for Nonlinear Fractional Differential Equations with Three-Point Integral Boundary Conditions

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This paper studies a boundary value problem of nonlinear fractional differential equations of order $q \in (1, 2]$ with three-point integral boundary conditions. Some new existence and uniqueness results are obtained by using standard fixed point theorems and Leray-Schauder degree theory. Our results are new in the sense that the nonlocal parameter in three-point integral boundary conditions appears in the integral part of the conditions in contrast to the available literature on three-point boundary value problems which deals with the three-point boundary conditions restrictions on the solution or gradient of the solution of the problem. Some illustrative examples are also discussed.

1. Introduction

In recent years, boundary value problems for nonlinear fractional differential equations have been addressed by several researchers. Fractional derivatives provide an excellent tool for the description of memory and hereditary properties of various materials and processes; see [1]. These characteristics of the fractional derivatives make the fractional-order models more realistic and practical than the classical integer-order models. As a matter of fact, fractional differential equations arise in many engineering and scientific disciplines such as physics, chemistry, biology, economics, control theory, signal and image processing, biophysics, blood flow phenomena, aerodynamics, and fitting of experimental data, [1–4]. For some recent development on the topic, see [5–21] and the references therein.

We discuss the existence and uniqueness of solutions for a boundary value problem of nonlinear fractional differential equations of order $q \in (1, 2]$ with three-point integral boundary conditions given by

$$
{}^{c}D^{q}x(t) = f(t, x(t)), \quad 0 < t < 1, \ 1 < q \leq 2,
$$

$$
x(0) = 0, \quad x(1) = \alpha \int_{0}^{\eta} x(s)ds, \quad 0 < \eta < 1, \tag{1.1}
$$

where ${}^{c}D^{q}$ denotes the Caputo fractional derivative of order q, $f : [0, 1] \times X \to X$ is continuous, and $\alpha \in \mathbb{R}$ is such that $\alpha \neq 2/\eta^2$. Here, $(X, \| \cdot \|)$ is a Banach space and $\mathcal{C} = C([0, 1], X)$ denotes the Banach space of all continuous functions from $[0, 1] \to X$ endowed with a topology of uniform convergence with the norm denoted by $\| \cdot \|$.

Note that the three-point boundary condition in (1.1) corresponds to the area under the curve of solutions $x(t)$ from $t = 0$ to $t = \eta$.

2. Preliminaries

Let us recall some basic definitions of fractional calculus [2, 4].

Definition 2.1. For a continuous function $g : [0, \infty) \to \mathbb{R}$, the Caputo derivative of fractional order q is defined as

$$
{}^{c}D^{q}g(t) = \frac{1}{\Gamma(n-q)} \int_{0}^{t} (t-s)^{n-q-1} g^{(n)}(s)ds, \quad n-1 < q < n, \ n = [q] + 1, \tag{2.1}
$$

where $[q]$ denotes the integer part of the real number q.

Definition 2.2. The Riemann-Liouville fractional integral of order q is defined as

$$
I^{q}g(t) = \frac{1}{\Gamma(q)} \int_{0}^{t} \frac{g(s)}{(t-s)^{1-q}} ds, \quad q > 0, \tag{2.2}
$$

provided the integral exists.

Definition 2.3. The Riemann-Liouville fractional derivative of order q for a continuous function $g(t)$ is defined by

$$
D^{q}g(t) = \frac{1}{\Gamma(n-q)} \left(\frac{d}{dt} \right)^{n} \int_{0}^{t} \frac{g(s)}{(t-s)^{q-n+1}} ds, \quad n = [q] + 1, \tag{2.3}
$$

provided the right-hand side is pointwise defined on $(0, \infty)$.

Lemma 2.4 (see [2]). *For $q > 0$, the general solution of the fractional differential equation $^cD^qx(t) = 0$ is given by*

$$x(t) = c_0 + c_1t + c_2t^2 + \cdots + c_{n-1}t^{n-1}, \tag{2.4}$$

where $c_i \in \mathbb{R}$, $i = 0, 1, 2, \ldots, n - 1$ $(n = [q] + 1)$.

In view of Lemma 2.4, it follows that

$$I^q\,{}^cD^qx(t) = x(t) + c_0 + c_1t + c_2t^2 + \cdots + c_{n-1}t^{n-1}, \tag{2.5}$$

for some $c_i \in \mathbb{R}$, $i = 0, 1, 2, \ldots, n - 1$ $(n = [q] + 1)$.

Lemma 2.5. *A unique solution of the boundary value problem* (1.1) *is given by*

$$
\begin{aligned}
x(t) = {} & \frac{1}{\Gamma(q)} \int_0^t (t - s)^{q-1} f(s, x(s)) ds \\
& - \frac{2t}{(2 - \alpha\eta^2)\Gamma(q)} \int_0^1 (1 - s)^{q-1} f(s, x(s)) ds \\
& + \frac{2\alpha t}{(2 - \alpha\eta^2)\Gamma(q)} \int_0^\eta \left(\int_0^s (s - m)^{q-1} f(m, x(m)) dm \right) ds.
\end{aligned}
\tag{2.6}
$$

Proof. For some constants $c_0, c_1 \in X$, we have

$$x(t) = I^q f(t, x(t)) - c_0 - c_1t = \int_0^t \frac{(t - s)^{q-1}}{\Gamma(q)} f(s, x(s)) ds - c_0 - c_1t. \tag{2.7}$$

From $x(0) = 0$, we have $c_0 = 0$. Applying the second boundary condition for (1.1), we find that

$$
\begin{aligned}
\alpha \int_0^\eta x(s) ds &= \alpha \int_0^\eta \left(\int_0^s \frac{(s - m)^{q-1}}{\Gamma(q)} f(m, x(m)) dm - c_1 s \right) ds \\
&= \alpha \int_0^\eta \left(\int_0^s \frac{(s - m)^{q-1}}{\Gamma(q)} f(m, x(m)) dm \right) ds - \alpha c_1 \frac{\eta^2}{2}, \\
x(1) &= \int_0^1 \frac{(1 - s)^{q-1}}{\Gamma(q)} f(s, x(s)) ds - c_1,
\end{aligned}
\tag{2.8}
$$

which imply that

$$c_1 = \frac{2}{2 - \alpha\eta^2} \left[\int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} f(s, x(s)) ds - \alpha \int_0^\eta \left(\int_0^s \frac{(s-m)^{q-1}}{\Gamma(q)} f(m, x(m)) dm \right) ds \right]. \quad (2.9)$$

Substituting the values of c_0 and c_1 in (2.7), we obtain the solution (2.6).   □

In view of Lemma 2.5, we define an operator $\mathbf{F} : \mathcal{C} \to \mathcal{C}$ by

$$(\mathbf{F}x)(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, x(s)) ds$$

$$- \frac{2t}{(2 - \alpha\eta^2)\Gamma(q)} \int_0^1 (1-s)^{q-1} f(s, x(s)) ds \quad (2.10)$$

$$+ \frac{2\alpha t}{(2 - \alpha\eta^2)\Gamma(q)} \int_0^\eta \left(\int_0^s (s-m)^{q-1} f(m, x(m)) dm \right) ds, \quad t \in [0, 1].$$

To prove the main results, we need the following assumptions:

(A$_1$) $\|f(t, x) - f(t, y)\| \le L\|x - y\|$, for all $t \in [0, 1]$, $L > 0$, $x, y \in X$;

(A$_2$) $\|f(t, x)\| \le \mu(t)$, for all $(t, x) \in [0, 1] \times X$, and $\mu \in L^1([0, 1], R^+)$.

For convenience, let us set

$$\Lambda = \frac{1}{\Gamma(q+1)} \left(1 + \frac{2[(q+1) + |\alpha|\eta^{q+1}]}{|2 - \alpha\eta^2|(q+1)} \right). \quad (2.11)$$

3. Existence Results in a Banach Space

Theorem 3.1. *Assume that $f : [0, 1] \times X \to X$ is a jointly continuous function and satisfies the assumption (A$_1$) with $L < 1/\Lambda$, where Λ is given by (2.11). Then the boundary value problem (1.1) has a unique solution.*

Proof. Setting $\sup_{t\in[0,1]}|f(t,0)| = M$ and choosing $r \geq \Lambda M/(1 - L\Lambda)$, we show that $\mathbf{F}B_r \subset B_r$, where $B_r = \{x \in C : \|x\| \leq r\}$. For $x \in B_r$, we have

$$
\begin{aligned}
\|(\mathbf{F}x)(t)\| &\leq \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}\|f(s,x(s))\|ds \\[2mm]
&\quad + \left|\frac{2t}{(2-\alpha\eta^2)\Gamma(q)}\right|\int_0^1 (1-s)^{q-1}\|f(s,x(s))\|ds \\[2mm]
&\quad + \left|\frac{2\alpha t}{(2-\alpha\eta^2)\Gamma(q)}\right|\int_0^\eta\left(\int_0^s (s-m)^{q-1}\|f(m,x(m))\|dm\right)ds \\[2mm]
&\leq \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}(\|f(s,x(s))-f(s,0)\| + \|f(s,0)\|)ds \\[2mm]
&\quad + \left|\frac{2t}{(2-\alpha\eta^2)\Gamma(q)}\right|\int_0^1 (1-s)^{q-1}(\|f(s,x(s))-f(s,0)\| + \|f(s,0)\|)ds \\[2mm]
&\quad + \left|\frac{2\alpha t}{(2-\alpha\eta^2)\Gamma(q)}\right|\int_0^\eta\left(\int_0^s (s-m)^{q-1}(\|f(m,x(m))-f(m,0)\| + \|f(m,0)\|)dm\right)ds \\[2mm]
&\leq (Lr + M)\left[\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}ds + \left|\frac{2t}{(2-\alpha\eta^2)\Gamma(q)}\right|\int_0^1 (1-s)^{q-1}ds\right. \\[2mm]
&\quad \left. + \left|\frac{2\alpha t}{(2-\alpha\eta^2)\Gamma(q)}\right|\int_0^\eta\left(\int_0^s (s-m)^{q-1}dm\right)ds\right] \\[2mm]
&\leq \frac{(Lr + M)}{\Gamma(q+1)}\left(1 + \frac{2[(q+1) + |\alpha|\eta^{q+1}]}{|2-\alpha\eta^2|(q+1)}\right) \\[2mm]
&= (Lr + M)\Lambda \leq r.
\end{aligned}
$$

$$(3.1)$$

Now, for $x, y \in C$ and for each $t \in [0,1]$, we obtain

$$
\begin{aligned}
\|(\mathbf{F}x)(t) - (\mathbf{F}y)(t)\| &\leq \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}\|f(s,x(s)) - f(s,y(s))\|ds \\[2mm]
&\quad + \left|\frac{2t}{(2-\alpha\eta^2)\Gamma(q)}\right|\int_0^1 (1-s)^{q-1}\|f(s,x(s)) - f(s,y(s))\|ds \\[2mm]
&\quad + \left|\frac{2\alpha t}{(2-\alpha\eta^2)\Gamma(q)}\right|\int_0^\eta\left(\int_0^s (s-m)^{q-1}\|f(m,x(m)) - f(m,y(m))\|dm\right)ds
\end{aligned}
$$

$$
\leq L \|x - y\| \left[\frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} ds + \left| \frac{2t}{(2-\alpha\eta^2)\Gamma(q)} \right| \left| \int_0^1 (1-s)^{q-1} ds \right. \right.
$$

$$
\left. \left. + \left| \frac{2\alpha t}{(2-\alpha\eta^2)\Gamma(q)} \right| \left| \int_0^\eta \left(\int_0^s (s-m)^{q-1} dm \right) ds \right. \right]
$$

$$
\leq \frac{L}{\Gamma(q+1)} \left(1 + \frac{2[(q+1) + |\alpha|\eta^{q+1}]}{|2-\alpha\eta^2|(q+1)} \right) \|x - y\| = L\Lambda \|x - y\|,
$$

$$(3.2)$$

where Λ is given by (2.11). Observe that Λ depends only on the parameters involved in the problem. As $L < 1/\Lambda$, therefore **F** is a contraction. Thus, the conclusion of the theorem follows by the contraction mapping principle (Banach fixed point theorem). $\qquad\square$

Now, we prove the existence of solutions of (1.1) by applying Krasnoselskii's fixed point theorem [22].

Theorem 3.2 (Krasnoselskii's fixed point theorem). *Let M be a closed convex and nonempty subset of a Banach space X. Let A, B be the operators such that (i) $Ax + By \in M$ whenever $x, y \in M$; (ii) A is compact and continuous; (iii) B is a contraction mapping. Then there exists $z \in M$ such that $z = Az + Bz$.*

Theorem 3.3. *Let $f : [0,1] \times X \to X$ be a jointly continuous function mapping bounded subsets of $[0,1] \times X$ into relatively compact subsets of X, and the assumptions (A_1) and (A_2) hold with*

$$
\frac{L}{\Gamma(q+1)} \left(\frac{2[(q+1) + |\alpha|\eta^{q+1}]}{|2-\alpha\eta^2|(q+1)} \right) < 1.
$$

$$(3.3)$$

Then the boundary value problem (1.1) has at least one solution on $[0,1]$.

Proof. Letting $\sup_{t\in[0,1]} |\mu(t)| = \|\mu\|$, we fix

$$
\bar{r} \geq \frac{\|\mu\|}{\Gamma(q+1)} \left(1 + \frac{2[(q+1) + |\alpha|\eta^{q+1}]}{|2-\alpha\eta^2|(q+1)} \right),
$$

$$(3.4)$$

and consider $B_{\bar{r}} = \{x \in C : \|x\| \leq \bar{r}\}$. We define the operators P and Q on $B_{\bar{r}}$ as

$$
(Px)(t) = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s, u(s)) ds,
$$

$$
(Qx)(t) = -\frac{2t}{(2-\alpha\eta^2)\Gamma(q)} \int_0^1 (1-s)^{q-1} f(s, x(s)) ds
$$

$$(3.5)$$

$$
+ \frac{2\alpha t}{(2-\alpha\eta^2)\Gamma(q)} \int_0^\eta \left(\int_0^s (s-m)^{q-1} f(m, x(m)) dm \right) ds.
$$

For $x, y \in B_{\bar{r}}$, we find that

$$\|\mathcal{P}x + Qy\| \le \frac{\|\mu\|}{\Gamma(q+1)}\left(1 + \frac{2[(q+1) + |\alpha|\eta^{q+1}]}{|2 - \alpha\eta^2|(q+1)}\right) \le \bar{r}. \tag{3.6}$$

Thus, $\mathcal{P}x + Qy \in B_{\bar{r}}$. It follows from the assumption (A_1) together with (3.3) that Q is a contraction mapping. Continuity of f implies that the operator $\mathcal{P}$ is continuous. Also, $\mathcal{P}$ is uniformly bounded on $B_{\bar{r}}$ as

$$\|\mathcal{P}x\| \le \frac{\|\mu\|}{\Gamma(q+1)}. \tag{3.7}$$

Now we prove the compactness of the operator $\mathcal{P}$.

In view of (A_1), we define $\sup_{(t,x)\in[0,1]\times B_{\bar{r}}}|f(t,x)| = \bar{f}$, and consequently we have

$$\|(\mathcal{P}x)(t_1) - (\mathcal{P}x)(t_2)\|$$

$$= \left\|\frac{1}{\Gamma(q)}\int_0^{t_1}\left[(t_2-s)^{q-1} - (t_1-s)^{q-1}\right]f(s,x(s))ds + \int_{t_1}^{t_2}(t_2-s)^{q-1}f(s,x(s))ds\right\| \tag{3.8}$$

$$\le \frac{\bar{f}}{\Gamma(q+1)}\left|2(t_2-t_1)^q + t_1^q - t_2^q\right|,$$

which is independent of x. Thus, $\mathcal{P}$ is equicontinuous. Using the fact that f maps bounded subsets into relatively compact subsets, we have that $\mathcal{P}(\mathcal{A})(t)$ is relatively compact in X for every t, where $\mathcal{A}$ is a bounded subset of C. So $\mathcal{P}$ is relatively compact on $B_{\bar{r}}$. Hence, by the Arzelá-Ascoli Theorem, $\mathcal{P}$ is compact on $B_{\bar{r}}$. Thus all the assumptions of Theorem 3.2 are satisfied. So the conclusion of Theorem 3.2 implies that the boundary value problem (1.1) has at least one solution on $[0,1]$. □

4. Existence of Solution via Leray-Schauder Degree Theory

Theorem 4.1. *Let $f : [0,1] \times \mathbb{R} \to \mathbb{R}$. Assume that there exist constants $0 \le \kappa < 1/\Lambda$, where Λ is given by (2.11) and $M > 0$ such that $|f(t,x)| \le \kappa|x| + M$ for all $t \in [0,1], x \in C[0,1]$. Then the boundary value problem (1.1) has at least one solution.*

Proof. Let us define an operator $F : C[0,1] \to C[0,1]$ as

$$x = Fx, \tag{4.1}$$

where

$$(Fx)(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, x(s)) ds$$

$$- \frac{2t}{(2-\alpha\eta^2)\Gamma(q)} \int_0^1 (1-s)^{q-1} f(s, x(s)) ds \tag{4.2}$$

$$+ \frac{2\alpha t}{(2-\alpha\eta^2)\Gamma(q)} \int_0^\eta \left(\int_0^s (s-m)^{q-1} f(m, x(m)) dm \right) ds.$$

In view of the fixed point problem (4.1), we just need to prove the existence of at least one solution $x \in C[0,1]$ satisfying (4.1). Define a suitable ball $B_R \subset C[0,1]$ with radius $R > 0$ as

$$B_R = \left\{ x \in C[0,1] : \max_{t \in [0,1]} |x(t)| < R \right\}, \tag{4.3}$$

where R will be fixed later. Then, it is sufficient to show that $F : \overline{B}_R \to C[0,1]$ satisfies

$$x \neq \lambda F x, \quad \forall x \in \partial B_R, \ \forall \lambda \in [0,1]. \tag{4.4}$$

Let us set

$$H(\lambda, x) = \lambda F x, \quad x \in C(\mathbb{R}), \ \lambda \in [0,1]. \tag{4.5}$$

Then, by the Arzelá-Ascoli Theorem, $h_\lambda(x) = x - H(\lambda, x) = x - \lambda F x$ is completely continuous. If (4.4) is true, then the following Leray-Schauder degrees are well defined and by the homotopy invariance of topological degree, it follows that

$$\deg(h_\lambda, B_R, 0) = \deg(I - \lambda F, B_R, 0) = \deg(h_1, B_R, 0)$$

$$= \deg(h_0, B_R, 0) = \deg(I, B_R, 0) = 1 \neq 0, \quad 0 \in B_r, \tag{4.6}$$

where I denotes the unit operator. By the nonzero property of Leray-Schauder degree, $h_1(t) = x - \lambda F x = 0$ for at least one $x \in B_R$. In order to prove (4.4), we assume that $x = \lambda F x$ for some

$\lambda \in [0,1]$ and for all $t \in [0,1]$ so that

$$|x(t)| = |\lambda(\mathcal{F}x)(t)|$$

$$\le \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \|f(s,x(s))\| ds$$

$$+ \left| \frac{2t}{(2-\alpha\eta^2)\Gamma(q)} \right| \int_0^1 (1-s)^{q-1} \|f(s,x(s))\| ds$$

$$+ \left| \frac{2\alpha t}{(2-\alpha\eta^2)\Gamma(q)} \right| \int_0^\eta \left(\int_0^s (s-m)^{q-1} \|f(m,x(m))\| dm \right) ds$$

$$\le (\kappa|x| + M) \left[\frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} ds + \left| \frac{2t}{(2-\alpha\eta^2)\Gamma(q)} \right| \int_0^1 (1-s)^{q-1} ds \right.$$

$$\left. + \left| \frac{2\alpha t}{(2-\alpha\eta^2)\Gamma(q)} \right| \int_0^\eta \left(\int_0^s (s-m)^{q-1} dm \right) ds \right]$$

$$\le \frac{\kappa|x| + M}{\Gamma(q+1)} \left(1 + \frac{2[(q+1) + |\alpha|\eta^{q+1}]}{|2-\alpha\eta^2|(q+1)} \right)$$

$$= (\kappa|x| + M)\Lambda,$$

(4.7)

which, on taking norm $(\sup_{t\in[0,1]} |x(t)| = \|x\|)$ and solving for $\|x\|$, yields

$$\|x\| \le \frac{M\Lambda}{1 - \kappa\Lambda}.$$

(4.8)

Letting $R = M\Lambda/(1 - \kappa\Lambda) + 1$, (4.4) holds. This completes the proof. □

5. Examples

Example 5.1. Consider the following three-point integral fractional boundary value problem:

$${}^cD^{3/2}x(t) = \frac{1}{(t+9)^2} \frac{\|x\|}{1 + \|x\|}, \quad t \in [0,1],$$

$$x(0) = 0, \quad x(1) = \int_0^{3/4} x(s) ds.$$

(5.1)

Here, $q = 3/2$, $\alpha = 1$, $\eta = 3/4$, and $f(t, x) = (1/(t+9)^2)(\|x\|/(1 + \|x\|))$. As $\|f(t, x) - f(t, y)\| \le (1/81)\|x - y\|$, therefore, (A$_1$) is satisfied with $L_1 = 1/81$. Further,

$$L\Lambda = L\Lambda = \frac{L}{\Gamma(q+1)}\left(1 + \frac{2[(q+1) + |\alpha|\eta^{q+1}]}{|2 - \alpha\eta^2|(q+1)}\right) = \frac{4}{27945\sqrt{\pi}}\left(275 + 18\sqrt{3}\right) < 1. \qquad (5.2)$$

Thus, by the conclusion of **Theorem** 3.1, the boundary value problem (5.1) has a unique solution on $[0, 1]$.

Example 5.2. Consider the following boundary value problem:

$$^cD^{3/2}x(t) = \frac{1}{(4\pi)}\sin(2\pi x) + \frac{|x|}{1 + |x|}, \quad t \in [0, 1], \ 1 < q \le 2,$$

$$x(0) = 0, \quad x(1) = \int_0^{1/2} x(s)ds. \qquad (5.3)$$

Here, $q = 3/2$, $\alpha = 1$, $\eta = 1/2$, and

$$|f(t, x)| = \left|\frac{1}{(4\pi)}\sin(2\pi x) + \frac{|x|}{1 + |x|}\right| \le \frac{1}{2}|x| + 1. \qquad (5.4)$$

Clearly $M = 1$ and

$$\kappa = \frac{1}{2} < \frac{1}{\Lambda} = \frac{105\sqrt{2\pi}}{4\left(75\sqrt{2} + 4\right)} = 0.5978138748. \qquad (5.5)$$

Thus, all the conditions of **Theorem** 4.1 are satisfied and consequently the problem (5.3) has at least one solution.

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