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Research Article

Strong Stability and Asymptotical Almost Periodicity of Volterra Equations in Banach Spaces

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We study strong stability and asymptotical almost periodicity of solutions to abstract Volterra equations in Banach spaces. Relevant criteria are established, and examples are given to illustrate our results.

1. Introduction

Owing to the *memory behavior* (cf., e.g., [1, 2]) of materials, many practical problems in engineering related to *viscoelasticity* or *thermoviscoelasticity* can be reduced to the following Volterra equation:

$$\begin{gathered} u'(t) = Au(t) + \int_0^t a(t-s)Au(s)\mathrm{d}s, \quad t \geq 0, \\ u(0) = x \end{gathered} \tag{1.1}$$

in a Banach space X, with A being the infinitesimal generator of a C_0-semigroup $T(t)$ defined on X, and $a(\cdot) \in L^p(\mathbb{R}_+, \mathbb{C})$ a scalar function ($\mathbb{R}_+ := [0, \infty)$ and $1 \leq p < \infty$), which is often called *kernel function* or *memory kernel* (cf., e.g., [1]). It is known that the above equation is well-posed. This implies the existence of the *resolvent* operator $S(t)$, and the mild solution is then given by

$$u(t) = S(t)x, \quad t \geq 0, \tag{1.2}$$

which is actually a classical solution if $x \in D(A)$. In the present paper, we investigate strong stability and asymptotical almost periodicity of the solutions. For more information and related topics about the two concepts, we refer to the monographs [3, 4]. In particular, their connections with the vector-valued Laplace transform and theorems of Widder type can be found in [4–6]. Recall the following.

Definition 1.1. Let X be a Banach space and $f : \mathbb{R}_+ \to X$ a bounded uniformly continuous function.

(i) f is called *almost periodic* if it can be uniformly approximated by linear combinations of $e^{ibt}x$ $(b \in \mathbb{R},\ x \in X)$. Denote by $AP(\mathbb{R}_+, X)$ the space of all almost periodic functions on $\mathbb{R}_+$.

(ii) f is called *asymptotically almost periodic* if $f = f_1 + f_2$ with $\lim_{t\to\infty} f_1(t) = 0$ and $f_2 \in AP(\mathbb{R}_+, X)$. Denote by $APP(\mathbb{R}_+, X)$ the space of all asymptotically almost periodic functions on $\mathbb{R}_+$.

(iii) We call (1.1) or $S(t)$ *strongly stable* if, for each $x \in D(A)$, $\lim_{t\to\infty} S(t)x = 0$. We call (1.1) or $S(t)$ asymptotically almost periodic if for each $x \in D(A)$, $S(\cdot)x \in APP(\mathbb{R}_+, X)$.

The following two results on C_0-semigroup will be used in our investigation, among which the first is due to Ingham (see, e.g., [7, Section 1] and the second is known as *Countable Spectrum Theorem* [3, Theorem 5.5.6]. As usual, the letter i denotes the imaginary unit and $i\mathbb{R}$ the imaginary axis.

Lemma 1.2. *Suppose that A generates a bounded C_0-semigroup $T(t)$ on a Banach space X. If $\sigma(A) \cap i\mathbb{R} = \emptyset$, then*

$$\left\| T(t)A^{-1} \right\| \longrightarrow 0, \quad t \longrightarrow 0. \tag{1.3}$$

Lemma 1.3. *Let $T(t)$ be a bounded C_0-semigroup on a reflexive Banach space X with generator A. If $\sigma(A) \cap i\mathbb{R}$ is countable, then $T(t)$ is asymptotically almost periodic.*

2. Results and Proofs

Asymptotic behaviors of solutions to the special case of $a(t) \equiv 0$ have been studied systematically, see, for example, [3, Chapter 4] and [8, Chapter V]. The following example shows that asymptotic behaviors of solutions to (1.1) are more complicated even in the finite-dimensional case.

Example 2.1. Let $X = \mathbb{C}$, $A = -2I$, $a(t) = -e^{-t}$ in (1.1). Then taking Laplace transform we can calculate

$$u(t) = \frac{1}{3}\left(1 + 2e^{-3t}\right)x. \tag{2.1}$$

It is clear that the following assertions hold.

(a) The corresponding semigroup $T(t) = e^{-2t}$ is exponentially stable.
(b) Each solution with initial value $x \in D(A),\ x \neq 0$ is *not* strongly stable and hence not exponentially stable.
(c) Each solution with $x \in D(A)$ is asymptotically almost periodic.

It is well known that the *semigroup approach* is useful in the study of (1.1). More information can be found in the book [8, Chapter VI.7] or the papers [9–11].

Let $\mathcal{X} := X \times L^p(\mathbb{R}_+, X)$ be the product Banach space with the norm

$$\left\| \begin{pmatrix} x \\ f \end{pmatrix} \right\|^2 := \|x\|^2 + \|f\|^2_{L^p(\mathbb{R}_+,X)} \tag{2.2}$$

for each $x \in X$ and $f \in L^p(\mathbb{R}_+, X)$. Then the operator matrix

$$\mathcal{A} := \begin{pmatrix} A & \delta_0 \\ B & \dfrac{\mathrm{d}}{\mathrm{d}s} \end{pmatrix}, \quad D(\mathcal{A}) := D(A) \times W^{1,p}(\mathbb{R}_+, X) \tag{2.3}$$

generates a C_0-semigroup on $\mathcal{X}$. Here, $W^{1,p}(\mathbb{R}_+, X)$ is the vector-valued Sobolev space and δ_0 the Dirac distribution, that is, $\delta_0(f) = f(0)$ for each $f \in W^{1,p}(\mathbb{R}_+, X)$; the operator B is given by

$$Bx := a(\cdot)Ax \quad \text{for each } x \in D(A). \tag{2.4}$$

Denote by $\mathcal{S}(t)$ the C_0-semigroup generated by $\mathcal{A}$. It follows that, for each $x \in D(A)$, the first coordinate of

$$\begin{pmatrix} u(t) \\ F(t,\cdot) \end{pmatrix} := \mathcal{S}(t) \begin{pmatrix} x \\ 0 \end{pmatrix} \tag{2.5}$$

is the unique solution of (1.1).

Theorem 2.2. *Let A be the generator of a C_0-semigroup $T(t)$ on the Banach space X and $a(\cdot) \in L^p(\mathbb{R}_+, \mathbb{C})$ with $1 \leq p < \infty$. Assume that*

(i) *M is a left-shift invariant closed subspace of $L^p(\mathbb{R}_+, X)$ such that $a(\cdot)Ax \in M$ for all $x \in D(A)$;*

(ii) *$\mathbb{C}_+ \subset \rho(\mathcal{A}|_{\mathfrak{D}})$ and*

$$\|R(\lambda, \mathcal{A}|_{\mathfrak{D}})\| \leq \frac{K}{|\lambda|}, \quad \lambda \in \mathbb{C}_+ \tag{2.6}$$

for some constant $K > 0$. *Here,* $\mathfrak{D} := D(A) \times \{f \in W^{1,p}(\mathbb{R}_+, X) \cap M : f' \in M\} := D(A) \times M_1$.

Then

(a) (1.1) *is strongly stable if* $i\mathbb{R} \subset \rho(\mathcal{A}|_{\mathfrak{D}})$*;*

(b) *if* X *is reflexive and* $1 < p < \infty$*, then every solution to* (1.1) *is asymptotically almost periodic provided* $\sigma(\mathcal{A}|_{\mathfrak{D}}) \cap i\mathbb{R}$ *is countable.*

Proof. Since the first coordinate of (2.5) is the unique solution of (1.1), it is easy to see that the strong stability and asymptotic almost periodicity of (1.1) follows from the strong stability and asymptotic almost periodicity of $\mathcal{S}(t)$, respectively.

Moreover, from [9, Proposition 2.8]) we know that if M is a closed subspace of $L^p(\mathbb{R}_+, X)$ such that M is $S_l(t)$-invariant and $a(\cdot)Ax \in M$ for all $x \in D(A)$, then $\mathcal{A}|_{\mathfrak{D}}$ (the restriction of $\mathcal{A}$ to $\mathfrak{D}$) generates the C_0-semigroup

$$\widetilde{\mathcal{S}}(t) := \mathcal{S}(t)|_{\mathcal{M}}, \tag{2.7}$$

which is defined on the Banach space

$$\mathcal{M} := X \times M. \tag{2.8}$$

Thus, by assumptions (i), (ii) and the well-known Hille-Yosida theorem for C_0-semigroups, we know that $\widetilde{\mathcal{S}}(t)$ is bounded. Hence, in view of Lemma 1.2, we get

$$\left\| \widetilde{\mathcal{S}}(t)\mathcal{A}|_{\mathfrak{D}}^{-1} \right\| \longrightarrow 0, \quad t \longrightarrow \infty. \tag{2.9}$$

Clearly

$$\mathcal{A}|_{\mathfrak{D}}^{-1} \begin{pmatrix} Ax \\ a(\cdot)Ax \end{pmatrix} = \begin{pmatrix} x \\ 0 \end{pmatrix} \tag{2.10}$$

for each $x \in D(A)$. So, combining (2.5) with (2.9), we have

$$\begin{aligned} \|u(t)\|^2 &\le \left\| \begin{pmatrix} u(t) \\ F(t,\cdot) \end{pmatrix} \right\|^2 \\ &= \left\| \widetilde{\mathcal{S}}(t) \begin{pmatrix} x \\ 0 \end{pmatrix} \right\|^2 \\ &= \left\| \widetilde{\mathcal{S}}(t)\mathcal{A}|_{\mathfrak{D}}^{-1} \begin{pmatrix} Ax \\ a(\cdot)Ax \end{pmatrix} \right\|^2 \\ &\le \left\| \widetilde{\mathcal{S}}(t)\mathcal{A}|_{\mathfrak{D}}^{-1} \right\|^2 \cdot \left[1 + \|a(\cdot)\|_{L^p}^2 \right] \cdot \|Ax\|^2 \\ &\longrightarrow 0, \quad t \longrightarrow \infty. \end{aligned} \tag{2.11}$$

This means that (a) holds.

On the other hand, we note that, to get (b), it is sufficient to show that $\widetilde{\mathcal{S}}(t)$ is asymptotically almost periodic. Actually, if X is reflexive and $1 < p < \infty$, then it is not hard to verify that $L^p(\mathbb{R}_+, X)$ is reflexive. Hence, $X \times L^p(\mathbb{R}_+, X)$ is reflexive. By assumption (i), $\mathcal{M}$ is a closed subspace of $X \times L^p(\mathbb{R}_+, X)$. Thus, Pettis's theorem shows that $\mathcal{M}$ is also reflexive. Hence, in view of Lemma 1.3, we get (b). This completes the proof. □

Corollary 2.3. *Let A be the generator of a C_0-semigroup $T(t)$ on the Banach space X and $a(t) = \alpha e^{-\beta t}$ $(\beta > 0,\ \alpha \neq 0)$. Assume that*

(i) *for each $\lambda \in \mathbb{C}_+$, λ, $\lambda(\lambda+\beta)/(\lambda+\alpha+\beta) \in \rho(A)$,*

(ii) *there exists a constant $C > 0$ satisfying*

$$\|H(\lambda)\|^2 + \frac{|\alpha|^2}{\beta \cdot |\lambda+\alpha+\beta|^2} \cdot \|I - \lambda H(\lambda)\|^2 \leq \frac{C}{|\lambda|^2}, \quad \lambda \in \mathbb{C}_+ \tag{2.12}$$

with

$$H(\lambda) := \frac{\lambda+\beta}{\lambda+\alpha+\beta}\left[\frac{\lambda(\lambda+\beta)}{\lambda+\alpha+\beta} - A\right]^{-1}. \tag{2.13}$$

Then

(a) *if* $\operatorname{Re}(\alpha+\beta) \neq 0$ *and*

$$\frac{\lambda(-1+i\beta)}{\alpha+\beta+i\lambda} \in \rho(A) \tag{2.14}$$

for each $\lambda \in \mathbb{R}$, then (1.1) *is strongly stable;*

(b) *if X is reflexive and $1 < p < \infty$, then* (1.1) *is asymptotically almost periodic provided*

$$\left\{\lambda \in \mathbb{R} : i\lambda(i\lambda+\beta)(i\lambda+\alpha+\beta)^{-1} \in \sigma(A)\right\} \tag{2.15}$$

is countable.

Proof. As in [9, Section 3], we take

$$M := \left\{e^{-\beta s}x : x \in X\right\} \subset L^p(\mathbb{R}_+, X). \tag{2.16}$$

In view of the discussion in [8, Lemma VI.7.23], we can infer that

$$\begin{gathered}\mathbb{C}_+ \subset \rho(\mathcal{A}|_{\mathfrak{D}}), \quad \text{if } \mathbb{C}_+ \subset \rho(A), \\ \frac{\lambda(\lambda+\beta)}{\lambda+\alpha+\beta} \in \rho(A), \quad \text{for each } \lambda \in \mathbb{C}_+.\end{gathered} \tag{2.17}$$

Moreover, we have

$$R(\lambda,\mathscr{A})=\begin{pmatrix}[I-\widehat{a}(\lambda)R(\lambda,A)A]^{-1} & 0\\ R\left(\lambda,\dfrac{\mathrm{d}}{\mathrm{d}s}\right)B[I-\widehat{a}(\lambda)R(\lambda,A)A]^{-1} & I\end{pmatrix}\begin{pmatrix}R(\lambda,A) & R(\lambda,A)\delta_0R\left(\lambda,\dfrac{\mathrm{d}}{\mathrm{d}s}\right)\\ 0 & R\left(\lambda,\dfrac{\mathrm{d}}{\mathrm{d}s}\right)\end{pmatrix}$$
$$=\begin{pmatrix}H(\lambda) & H(\lambda)\delta_0R\left(\lambda,\dfrac{\mathrm{d}}{\mathrm{d}s}\right)\\ R\left(\lambda,\dfrac{\mathrm{d}}{\mathrm{d}s}\right)BH(\lambda) & R\left(\lambda,\dfrac{\mathrm{d}}{\mathrm{d}s}\right)BH(\lambda)\delta_0R\left(\lambda,\dfrac{\mathrm{d}}{\mathrm{d}s}\right)+R\left(\lambda,\dfrac{\mathrm{d}}{\mathrm{d}s}\right)\end{pmatrix}. \tag{2.18}$$

Hence,

$$\|R(\lambda,\mathscr{A}|_{\mathfrak{D}})\|^2\leq\left[\|H(\lambda)\|^2+\frac{|\alpha|^2}{\beta\cdot|\lambda+\alpha+\beta|^2}\cdot\|I-\lambda H(\lambda)\|^2\right]\times\left(\frac{2\beta}{|\lambda+\beta|^2}+1\right)+\frac{1}{|\lambda+\beta|^2} \tag{2.19}$$

with $H(\lambda)$ being defined as in (2.13). Thus, it is clear that $\widetilde{\mathcal{S}}(t)$ is bounded if (2.12) is satisfied.

Next, for $\lambda\in\mathbb{R}$, we consider the eigenequation

$$(i\lambda-\mathscr{A}|_{\mathfrak{D}})\begin{pmatrix}x\\ f\end{pmatrix}=\begin{pmatrix}y\\ g\end{pmatrix}. \tag{2.20}$$

Writing $f=e^{-\beta s}f_0$ and $g=e^{-\beta s}g_0$, we see easily that (2.20) is equivalent to

$$\begin{gathered}(i\lambda-A)x-f_0=y,\\ -\alpha Ax+(i\lambda+\beta)f_0=g_0.\end{gathered} \tag{2.21}$$

Thus, if $\operatorname{Re}(\alpha+\beta)\neq 0$ and

$$\frac{\lambda(-1+i\beta)}{\alpha+\beta+i\lambda}\in\rho(A), \tag{2.22}$$

then by (2.21) we obtain

$$x = (\alpha+\beta+i\lambda)^{-1}\left[\frac{\lambda(-1+i\beta)}{\alpha+\beta+i\lambda} - A\right]^{-1}[(i\lambda+\beta)y+g_0],$$
$$f_0 = (\alpha+\beta+i\lambda)^{-1}(i\lambda - A)\left[\frac{\lambda(-1+i\beta)}{\alpha+\beta+i\lambda} - A\right]^{-1}[(i\lambda+\beta)y+g_0] - y. \tag{2.23}$$

By the closed graph theorem, the operator

$$(i\lambda - A)\left[\frac{\lambda(-1+i\beta)}{\alpha+\beta+i\lambda} - A\right]^{-1} \tag{2.24}$$

in the second equality of (2.23) is bounded. Hence, noting that

$$\begin{aligned}\|(i\lambda+\beta)y+g_0\|^2 &\le 2\left[\|(i\lambda+\beta)y\|^2 + \|g_0\|^2\right]\\ &= 2\left[|i\lambda+\beta|^2\cdot\|y\|^2 + 2\beta\|g\|^2\right],\end{aligned} \tag{2.25}$$

we have

$$i\lambda \in \rho(\mathcal{A}|_{\mathfrak{D}}) \quad \text{for each } \lambda \in \mathbb{R}. \tag{2.26}$$

Consequently, in view of (a) of Theorem 2.2, we know that (1.1) is strongly stable if (2.14) holds.

Furthermore, by [9, Lemma 3.3], we have

$$\sigma(\mathcal{A}|_{\mathfrak{D}}) \subset \left\{\lambda\in\mathbb{C} : \lambda(\lambda+\beta)(\lambda+\alpha+\beta)^{-1}\in\sigma(A)\right\} \cup \{-(\alpha+\beta)\}. \tag{2.27}$$

Combining this with (b) of Theorem 2.2, we conclude that (1.1) is asymptotically almost periodic if X is reflexive, $1 < p < \infty$, and the set in (2.15) is countable. □

Theorem 2.4. *Let A be the generator of a C_0-semigroup $T(t)$ on the Banach space X and $a(\cdot) \in L^p(\mathbb{R}_+,\mathbb{C})$ with $1 \le p < \infty$. Assume that*

(i) *for all $\lambda \in \mathbb{C}_+$,*

$$\hat{a}(\lambda) \ne -1, \qquad \lambda[1+\hat{a}(\lambda)]^{-1} \in \rho(A),$$
$$\sup_{\lambda>0,\, n=0,1,2,\dots}\left\|\frac{\lambda^{n+1}[\lambda H(\lambda)-1]^{(n)}(\lambda)}{n!}\right\| < \infty, \tag{2.28}$$

$\|\lambda H(\lambda)\|$, $\|\lambda^2 H'(\lambda)\|$ *is bounded on $\mathbb{C}_+$, where $H(\lambda) := [\lambda - (1+\hat{a}(\lambda))A]^{-1}$,*

(ii) *$q(\lambda)$ is analytic on $\mathbb{C}_+$ and $\|\lambda q(\lambda)\|$, $\|\lambda^2 q'(\lambda)\|$ are bounded on $\mathbb{C}_+$, where*

$$q(\lambda) := \frac{H(\lambda)}{\lambda - \alpha - i\eta}, \tag{2.29}$$

for each $i\eta \in iE$ (iE is the set of half-line spectrum of $H(\lambda)$) and $\alpha > 0$,

(iii) *for each $x \in X$ and $i\eta \in iE$, the limit*

$$\lim_{\alpha \to 0+} \alpha e^{(\alpha + i\eta)t} H(\alpha + i\eta)x \tag{2.30}$$

exists uniformly for $t \geq 0$.

Then every solution to (1.1) *is asymptotically almost periodic. Moreover, if for each $x \in X$ and $i\eta \in iE$ the limit in* (2.30) *equals 0 uniformly for $t \geq 0$, then* (1.1) *is strongly stable.*

Proof. Take $x \in D(A)$. Then the solution $S(t)x$ to (1.1) is Lipschitz continuous and hence uniformly continuous. Actually, by assumption (i), we know that

$$\int_0^\infty e^{-\lambda t} S(t)x\mathrm{d}t = [\lambda - (1 + \widehat{a}(\lambda))A]^{-1}x, \quad \operatorname{Re}\lambda > 0, \tag{2.31}$$

and that

$$r(\lambda) := [\lambda H(\lambda) - 1]x \tag{2.32}$$

is analytic on $\mathbb{C}_+$. Thus, $r \in C^\infty((0,\infty), X)$ and

$$H(\lambda)x - \frac{1}{\lambda}x = \frac{r(\lambda)}{\lambda} = \int_0^\infty e^{-\lambda t}[S(t)x - x]\mathrm{d}t, \quad \lambda > 0. \tag{2.33}$$

On the other hand, if (2.28) holds, then there exists $K > 0$ such that

$$\begin{aligned} \sup_{\lambda>0}\left\|\frac{\lambda^{n+1} r^{(n)}(\lambda)}{n!}\right\| &\leq \sup_{\lambda>0}\left\|\frac{\lambda^{n+1}[\lambda H(\lambda) - 1]^{(n)}(\lambda)}{n!}\right\| \|x\| \\ &\leq K\|x\|, \quad n = 0, 1, 2, \ldots. \end{aligned} \tag{2.34}$$

Hence, from [4, Chapter 1] (or [5]) and the uniqueness of the Laplace transform, it follows that

$$F(t) := S(t)x - x \tag{2.35}$$

satisfies

$$\frac{r(\lambda)}{\lambda} = \int_0^\infty e^{-\lambda t} F(t)\mathrm{d}t, \quad \lambda > 0, \tag{2.36}$$

and that

$$\|F(t+h)-F(t)\| = \|S(t+h)x - S(t)x\| \le Kh\|x\|, \quad t \ge 0,\ h \ge 0. \tag{2.37}$$

Moreover, by [3, Corollary 2.5.2], the assumption (i) implies the boundedness of $S(t)$. Therefore,

$$f(t) := S(t)x \tag{2.38}$$

is bounded and uniformly continuous on $[0,\infty)$. In addition, the half-line spectrum set of $f(t)$ is just the following set:

$$\{i\eta \in i\mathbb{R} : H(\lambda) \text{ cannot be analytically extened to an eighborhood of } i\eta\}. \tag{2.39}$$

Write $\tau = \alpha + i\eta$. Then

$$\begin{aligned} \int_0^\infty e^{-\tau s} f(t+s)\mathrm{d}s &= e^{\tau t}\int_t^\infty e^{-\tau s} f(s)\mathrm{d}s \\ &= e^{\tau t}\left[\hat{f}(\tau) - \int_0^t e^{-\tau s} f(s)\mathrm{d}s\right] \\ &= e^{\tau t}H(\tau)x - \left(e^{\tau\cdot} * f\right)(t), \end{aligned} \tag{2.40}$$

$$q(\lambda) = \frac{H(\lambda)x}{\lambda-\alpha-i\eta} = \widehat{e^{\tau\cdot} * f}(\lambda). \tag{2.41}$$

From assumption (ii) and [3, Corollary 2.5.2], it follows that $(e^{\tau\cdot} * f)(t)$ is bounded, which implies

$$\lim_{\alpha\to 0+} \alpha\left(e^{\tau\cdot} * f\right)(t) = 0 \tag{2.42}$$

uniformly for $t \ge 0$. Finally, combining (2.40) with Theorem [7, Theorem 4.1], we complete the proof. □

3. Applications

In this section, we give some examples to illustrate our results.

First, we apply Corollary 2.3 to Example 2.1. As one will see, the previous result will be obtained by a different point of view.

Example 3.1. We reconsider Example 2.1. First, we note that

$$\alpha + \beta = 0. \tag{3.1}$$

This implies that condition $\mathrm{Re}(\alpha + \beta) \neq 0$ is not satisfied. Therefore, part (a) of Corollary 2.3 is not applicable, and this explains partially why the corresponding Volterra equation is not strongly stable. However, it is easy to check that conditions (i) and (ii) in Corollary 2.3 are satisfied. In particular, we have accordingly

$$H(\lambda) = \frac{\lambda + 1}{\lambda(\lambda + 3)}, \tag{3.2}$$

and hence the estimate

$$\|H(\lambda)\|^2 + \frac{|\alpha|^2}{\beta \cdot |\lambda + \alpha + \beta|^2} \cdot \|I - \lambda H(\lambda)\|^2 = \frac{|\lambda + 1|^2 + 4}{|\lambda(\lambda + 3)|^2} \leq \frac{13/9}{|\lambda|^2}, \quad \lambda \in \mathbb{C}_+. \tag{3.3}$$

Note $\sigma(A) = \{-2\}$ and

$$\left\{\lambda \in \mathbb{R} : i\lambda(i\lambda + \beta)(i\lambda + \alpha + \beta)^{-1} \in \sigma(A)\right\} = \{\lambda \in \mathbb{R} : i\lambda + 1 = -2\} = \emptyset. \tag{3.4}$$

Applying part (b) of Corollary 2.3, we know that the corresponding Volterra equation is asymptotically almost periodic.

Example 3.2. Consider the Volterra equation

$$\begin{gathered}
\frac{\partial u}{\partial t}(t,x) = \frac{\partial^2 u}{\partial x^2}(t,x) + \alpha \int_0^t e^{-\beta(t-s)} \frac{\partial^2 u}{\partial x^2}(s,x)\mathrm{d}s, \quad t > 0,\ 0 \leq x \leq \pi, \\
u(0,t) = u(\pi,t) = 0, \quad t \geq 0, \\
u(0,x) = u_0(x), \quad 0 \leq x \leq \pi,
\end{gathered} \tag{3.5}$$

where the constants satisfy

$$\beta > 0, \quad \alpha + \beta = 0. \tag{3.6}$$

Let $H = L^2[0,\pi]$, and define

$$A := \frac{\mathrm{d}^2}{\mathrm{d}x^2}, \quad D(A) = \left\{f \in H^2[0,\pi] : f(0) = f(\pi) = 0\right\}. \tag{3.7}$$

Then (3.5) can be formulated into the abstract form (1.1). It is well known that A is self-adjoint (see, e.g., [12, page 280, (b) of Example 3]) and that A generates an analytic C_0-semigroup. The self-adjointness of A implies

$$\left\|(\lambda - A)^{-1}\right\| = \frac{1}{\mathrm{dist}(\lambda, \sigma(A))}, \quad \lambda \in \rho(A). \tag{3.8}$$

On the other hand, we can compute

$$\sigma(A) = \sigma_p(A) = \left\{-n^2 : n = 1, 2, \ldots\right\}. \tag{3.9}$$

It follows immediately that condition (i) in Corollary 2.3 holds. Moreover, corresponding to (2.13), we have

$$H(\lambda) = \frac{\lambda + \beta}{\lambda}(\lambda + \beta - A)^{-1}. \tag{3.10}$$

Combining this with (3.6), (3.8), and (3.9), we estimate

$$\begin{aligned}
&\|H(\lambda)\|^2 + \frac{|\alpha|^2}{\beta \cdot |\lambda + \alpha + \beta|^2} \cdot \|I - \lambda H(\lambda)\|^2 \\
&\quad = \frac{1}{|\lambda|^2}\left[|\lambda + \beta|^2 \cdot \left\|(\lambda + \beta - A)^{-1}\right\|^2 + \beta\left\|I - (\lambda + \beta)(\lambda + \beta - A)^{-1}\right\|^2\right] \\
&\quad \le \frac{1}{|\lambda|^2}\left[\frac{|\lambda + \beta|^2}{|\lambda + \beta + 1|^2} + \beta\left(1 + \frac{|\lambda + \beta|^2}{|\lambda + \beta + 1|^2}\right)\right] \\
&\quad \le \frac{1 + 2\beta}{|\lambda|^2}, \quad \lambda \in \mathbb{C}_+.
\end{aligned} \tag{3.11}$$

Note that (2.15) becomes

$$\left\{\lambda \in \mathbb{R} : i\lambda + \beta = -n^2 \text{ for some } n \in \mathbb{N}\right\} = \emptyset. \tag{3.12}$$

Applying part (b) of Corollary 2.3, by (3.11), we conclude that (3.5) is asymptotically almost periodic (cf. [9, Remark 3.6]).

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