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Research Article

Weighted S-Asymptotically ω-Periodic Solutions of a Class of Fractional Differential Equations

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We study the existence of weighted S-asymptotically ω-periodic mild solutions for a class of abstract fractional differential equations of the form $u' = \partial^{-\alpha+1}Au + f(t,u)$, $1 < \alpha < 2$, where A is a linear sectorial operator of negative type.

1. Introduction

S-asymptotically ω-periodic functions have applications to several problems, for example in the theory of functional differential equations, fractional differential equations, integral equations and partial differential equations. The concept of S-asymptotic ω-periodicity was introduced in the literature by Henríquez et al. [1, 2]. Since then, it attracted the attention of many researchers (see [1–10]). In Pierri [10] a new S-asymptotically ω-periodic space was introduced. It is called the space of weighted S-asymptotically ω-periodic (or Sv-asymptotically ω-periodic) functions. In particular, the author has established conditions under which a Sv-asymptotically ω-periodic function is asymptotically ω-periodic and also discusses the existence of Sv-asymptotically ω-periodic solutions for an integral abstract Cauchy problem. The author has applied the results to partial integrodifferential equations.

We study in this paper sufficient conditions for the existence and uniqueness of a weighted S-asymptotically ω-periodic (mild) solution to the following semi-linear integrodifferential equation of fractional order

$$v'(t) = \int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)} Av(s)ds + f(t, v(t)), \quad t \geq 0, \tag{1.1}$$

$$v(0) = u_0 \in X, \tag{1.2}$$

where $1 < \alpha < 2$, $A : D(A) \subseteq X \to X$ is a linear densely defined operator of sectorial type on a complex Banach space X and $f : [0,\infty) \times X \to X$ is an appropriate function. Note that the convolution integral in (1.1) is known as the Riemann-Liouville fractional integral [11]. We remark that there is much interest in developing theoretical analysis and numerical methods for fractional integrodifferential equations because they have recently proved to be valuable in various fields of sciences and engineering. For details, including some applications and recent results, see the monographs of Ahn and MacVinish [12], Gorenflo and Mainardi [13] and Trujillo et al. [14–16] and the papers of Agarwal et al. [17–23], Cuesta [11, 24], Cuevas et al. [5, 6], dos Santos and Cuevas [25], Eidelman and Kochubei [26], Lakshmikantham et al. [27–30], Mophou and N'Guérékata [31], Ahmed and Nieto [32], and N'Guérékata [33]. In particular equations of type (1.1) are attracting increasing interest (cf. [5, 11, 24, 34]).

The existence of weighted S-asymptotically ω-periodic (mild) solutions for integrodifferential equation of fractional order of type (1.1) remains an untreated topic in the literature. Anticipating a wide interest in the subject, this paper contributes in filling this important gap. In particular, to illustrate our main results, we examine sufficient conditions for the existence and uniqueness of a weighted S-asymptotically ω-periodic mild solution to a fractional oscillation equation.

2. Preliminaries and Basic Results

In this section, we introduce notations, definitions and preliminary facts which are used throughout this paper. Let $(Z, \|\cdot\|_Z)$ and $(Y, \|\cdot\|_Y)$ be Banach spaces. The notation $\mathcal{B}(Z,Y)$ stands for the space of bounded linear operators from Z into Y endowed with the uniform operator topology denoted $\|\cdot\|_{\mathcal{B}(Z,Y)}$, and we abbreviate to $\mathcal{B}(Z)$ and $\|\cdot\|_{\mathcal{B}(Z)}$ whenever $Z = Y$. In this paper $C_b([0,\infty), Z)$ denotes the Banach space consisting of all continuous and bounded functions from $[0,\infty)$ into Z with the norm of the uniform convergence. For a closed linear operator B we denote by $\rho(B)$ the resolvent set and by $\sigma(B)$ the spectrum of B (that is, the complement of $\rho(B)$ in the complex plane). Set $(\lambda I - B)^{-1}$ the resolvent of B for $\lambda \in \rho(B)$.

2.1. Sectorial Linear Operators and the Solution Operator for Fractional Equations

A closed and linear operator A is said sectorial of type μ if there are $0 < \theta < \pi/2$, $M > 0$ and $\mu \in \mathbb{R}$ such that the spectrum of A is contained in the sector $\mu + \Sigma_\theta := \{\mu + \lambda : \lambda \in \mathbb{C},\ |\arg(-\lambda)| < \theta\}$ and $\|(\lambda - A)^{-1}\| \le M/|\lambda - \mu|$, for all $\lambda \notin \mu + \Sigma_\theta$.

In order to give an operator theoretical approach for the study of the abstract system we recall the following definition.

Definition 2.1 (see [17]). Let A be a closed linear operator with domain $D(A)$ in a Banach space X. One calls A the generator of a solution operator for (1.1)-(1.2) if there are $\mu \in \mathbb{R}$ and a strongly continuous function $S_\alpha : \mathbb{R}^+ \to \mathcal{B}(X)$ such that $\{\lambda^\alpha : \operatorname{Re}\lambda > \mu\} \subseteq \rho(A)$ and $\lambda^{\alpha-1}(\lambda^\alpha - A)^{-1}x = \int_0^\infty e^{-\lambda t} S_\alpha(t)x\,dt$, for all $\operatorname{Re}\lambda > \mu$, $x \in X$. In this case, $S_\alpha(t)$ is called the solution operator generated by A. By [35, Proposition 2.6], $S_\alpha(0) = I$. We observe that the power function λ^α is uniquely defined as $\lambda^\alpha = |\lambda|^\alpha e^{i \arg\lambda}$, with $-\pi < \arg(\lambda) < \pi$.

We note that if A is a sectorial of type μ with $0 \leq \theta \leq \pi(1-\alpha/2)$, then A is the generator of a solution operator given by $S_\alpha(t) := (1/2\pi i)\int_\gamma e^{\lambda t}\lambda^{\alpha-1}(\lambda^\alpha - A)^{-1}d\lambda$, $t > 0$, where γ is a suitable path lying outside the sector $\mu + \Sigma_\theta$ (cf. [11]). Recently, Cuesta [11, Theorem 1] proved that if A is a sectorial operator of type $\mu < 0$ for some $M > 0$ and $0 \leq \theta \leq \pi(1-\alpha/2)$, then there exists $C > 0$ such that

$$\|S_\alpha(t)\|_{\mathcal{B}(X)} \leq \frac{CM}{1+|\mu|t^\alpha}, \quad t \geq 0. \tag{2.1}$$

Remark 2.2. In the remainder of this paper, we always assume that A is a a sectorial of type $\mu < 0$ and M, C, are the constants introduced above.

2.2. Weighted S-Asymptotically ω-Periodic Functions

We recall the following definitions.

Definition 2.3 (see [1]). A function $f \in C_b([0,\infty), Z)$ is called S-asymptotically ω-periodic if there exists $\omega > 0$ such that $\lim_{t\to\infty}(f(t+\omega) - f(t)) = 0$. In this case, we say that ω is an asymptotic period of $f(\cdot)$.

Throughout this paper, $\mathrm{SAP}_\omega(Z)$ represents the space formed for all the Z-valued S-asymptotically ω-periodic functions endowed with the uniform convergence norm denoted $\|\cdot\|_\infty$. It is clear that $\mathrm{SAP}_\omega(Z)$ is a Banach space (see [1, Proposition 3.5]).

Definition 2.4 (see [10]). Let $v \in C_b([0,\infty),(0,\infty))$. A function $f \in C_b([0,\infty), Z)$ is called weighted S-asymptotically ω-periodic (or Sv-asymptotically ω-periodic) if $\lim_{t\to\infty}(f(t+\omega) - f(t))/v(t) = 0$.

In this paper, $\mathrm{SAP}_\omega(Z, v)$ represents the space formed by all the Sv-asymptotically ω-periodic functions endowed with the norm

$$\|f\|_{\mathrm{SAP}_\omega(Z,v)} = \|f\|_\infty + \|f\|_v = \sup_{t\geq 0}\|f(t)\|_Z + \sup_{t\geq 0}\frac{\|f(t+\omega) - f(t)\|_Z}{v(t)}. \tag{2.2}$$

Proposition 2.5. *The space* $\mathrm{SAP}^v_\omega(X)$ *is a Banach space.*

Proof. Let $(f_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $\mathrm{SAP}^v_\omega(X)$. From the definition of $\|\cdot\|_{S^v_\omega(Z)}$, there exists $f \in C_b([0,\infty), X)$ such that $f_n \to f$ in $C_b([0,\infty), X)$. Next, we prove that $f_n \to f$ in $\mathrm{SAP}^v_\omega(X)$.

By noting that $(f_n)_n$ is a Cauchy sequence, for $\varepsilon > 0$ given there exists $N_\varepsilon \in \mathbb{N}$ such that $\|f_n - f_m\|_{S^v_\omega(Z)} < \varepsilon$, for all $n, m \geq N_\varepsilon$, which implies

$$\begin{gathered} \|(f_n - f_m)(t)\| < \varepsilon, \quad \forall t \geq 0,\ \forall n, m \geq N_\varepsilon, \\ \frac{\|(f_n - f_m)(t+\omega) - (f_n - f_m)(t)\|}{v(t)} < \varepsilon, \quad \forall t \geq 0,\ \forall n, m \geq N_\varepsilon. \end{gathered} \tag{2.3}$$

Under the above conditions, for $t \geq 0$ and $n \geq N_\varepsilon$ we see that

$$
\begin{aligned}
&\|f_n(t) - f(t)\| + \frac{\|(f_n - f)(t+\omega) - (f_n - f)(t)\|}{v(t)} \\
&\quad = \lim_{m \to \infty}\left(\|f_n(t) - f_m(t)\| + \frac{\|(f_n - f_m)(t+\omega) - (f_n - f_m)(t)\|}{v(t)} \right) \\
&\quad \leq 2\epsilon,
\end{aligned}
\tag{2.4}
$$

which implies that $\|f_n - f\|_{S^v_\omega(Z)} \leq 2\epsilon$ for $n \geq N_\varepsilon$ and $\|f_n - f\|_{S^v_\omega(Z)} \to 0$ as $n \to \infty$.

To conclude the proof we need to show that $f \in \mathrm{SAP}^v_\omega(X)$. Let N_ε as above. Since $f_{N_\varepsilon} \in \mathrm{SAP}^v_\omega(X)$, there exits $L_\varepsilon > 0$ such that $\|f_{N_\varepsilon}(t+\omega) - f_{N_\varepsilon}(t)\|/v(t) < \varepsilon$ for all $t \geq L_\varepsilon$. Now, by using that $\|f_{N_\varepsilon} - f\|_{S^v_\omega(Z)} \leq 2\epsilon$, for $t \geq L_\epsilon$ we get

$$
\begin{aligned}
\frac{\|f(t+\omega) - f(t)\|}{v(t)} &\leq \frac{\|(f(t+\omega) - f_{N_\varepsilon}(t+\omega)) - (f(t) - f_{N_\varepsilon}(t))\|}{v(t)} \\
&\quad + \frac{\|f_{N_\varepsilon}(t+\omega) - f_{N_\varepsilon}(t)\|}{v(t)} \\
&< 2\varepsilon + \varepsilon,
\end{aligned}
\tag{2.5}
$$

which implies that $\lim_{t\to\infty}[(f(t+\omega) - f(t))/v(t)] = 0$. This completes the proof. □

Definition 2.6. A function $f \in C([0,\infty) \times Z, Y)$ is called uniformly Sv-asymptotically ω-periodic on bounded sets if for every bounded subset $K \subseteq Z$, the set $\{f(t,x) : t \geq 0, x \in K\}$ is bounded and $\lim_{t\to\infty}\|f(t+\omega,x) - f(t,x)\|_Y/v(t) = 0$, uniformly for $x \in K$. If $v \equiv 1$ we say that $f(\cdot)$ is uniformly S-asymptotically ω-periodic on bounded sets (see [1]).

To prove some of our results, we need the following lemma.

Lemma 2.7. *Let $v \in C_b([0,\infty),(0,\infty))$. Assume $f \in C([0,\infty) \times Z, Y)$ is uniformly Sv-asymptotically ω-periodic on bounded sets and there is $L > 0$ such that*

$$
\|f(t,x) - f(t,y)\|_Y \leq L\|x - y\|_Z, \quad \forall t \geq 0,\ \forall x,y \in Z. \tag{2.6}
$$

If $u \in \mathrm{SAP}_\omega(Z,v)$, then the function $t \to f(t,u(t))$ belongs to $\mathrm{SAP}_\omega(Y,v)$.

Proof. Using the fact that $\mathcal{R}(u) = \{u(t) : t \geq 0\}$ is bounded, it follows that $f(\cdot, u(\cdot)) \in C_b([0,\infty), Y)$. For $\epsilon > 0$ be given, we select $T_\epsilon > 0$ such that

$$\frac{\left\| f(t+\omega, z) - f(t,z) \right\|_Y}{v(t)} \leq \frac{\epsilon}{2}, \qquad \frac{\|u(t+\omega) - u(t)\|_Z}{v(t)} \leq \frac{\epsilon}{2L}, \tag{2.7}$$

for all $t \geq T_\epsilon$ and $z \in \mathcal{R}(u)$. Then, for $t \geq T_\epsilon$ we see that

$$\begin{aligned}
\frac{\left\| f(t+\omega, u(t+\omega)) - f(t, u(t)) \right\|_Y}{v(t)} &\leq \frac{\left\| f(t+\omega, u(t+\omega)) - f(t, u(t+\omega)) \right\|_Y}{v(t)} \\
&\quad + \frac{\left\| f(t, u(t+\omega)) - f(t, u(t)) \right\|_Y}{v(t)} \\
&\leq \frac{\epsilon}{2} + L\frac{\|u(t+\omega) - u(t)\|_Z}{v(t)} \\
&\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,
\end{aligned} \tag{2.8}$$

which proves the assertion. □

Lemma 2.8. *Let $v \in C_b([0,\infty),(0,\infty))$. Let $u \in \mathrm{SAP}_\omega(X,v)$ and $l_\alpha : [0,\infty) \to X$ be the function defined by*

$$l_\alpha(t) = \int_0^t S_\alpha(t-s)u(s)ds. \tag{2.9}$$

If $v(t)t^{\alpha-1} \to \infty$ as $t \to \infty$ and $\Theta := \sup_{t\geq 0}(1/v(t))\int_0^t (v(t-s)/(1+|\mu|s^\alpha))ds < \infty$, then $l_\alpha \in \mathrm{SAP}_\omega(X,v)$.

Proof. From the estimate $\|l_\alpha\|_\infty \leq CM|\mu|^{-1/\alpha}\pi/\alpha\sin(\pi/\alpha)$, it follows that $l_\alpha \in C_b([0,\infty),X)$. For $\varepsilon > 0$ be given we select $T_\epsilon > 0$ such that

$$\frac{\|u(t+\omega) - u(t)\|}{v(t)} \leq \varepsilon, \qquad \frac{CM(1+2^\alpha)\|u\|_\infty}{(\alpha-1)\left|\mu\right|v(t)t^{\alpha-1}} \leq \varepsilon, \tag{2.10}$$

for all $t \geq T_\epsilon$. Under these conditions, for $t \geq 2T_\epsilon$ we have that

$$\begin{aligned}
\frac{\|l_\alpha(t+\omega) - l_\alpha(t)\|}{v(t)} &\leq \frac{1}{v(t)} \int_0^\omega \|S_\alpha(t+\omega-s)\|_{\mathcal{B}(X)} \|u(s)\|_X ds \\
&\quad + \frac{1}{v(t)} \int_0^{T_\epsilon} \|S_\alpha(t-s)\|_{\mathcal{B}(X)} \|u(s+\omega) - u(s)\|_X ds \\
&\quad + \frac{1}{v(t)} \int_{T_\epsilon}^t \|S_\alpha(t-s)\|_{\mathcal{B}(X)} \|u(s+\omega) - u(s)\|_X ds \\
&\leq \frac{CM\|u\|_\infty}{v(t)} \left(\int_t^{t+\omega} \frac{1}{1+|\mu| s^\alpha} ds + 2\int_{t-T_\epsilon}^t \frac{1}{1+|\mu|s^\alpha} ds \right) \\
&\quad + \frac{CM\epsilon}{v(t)} \int_0^{t-T_\epsilon} \frac{v(t-s)}{1+|\mu|s^\alpha} ds \\
&\leq \frac{CM(1+2^\alpha)\|u\|_\infty}{(\alpha-1)|\mu|} \frac{1}{v(t)t^{\alpha-1}} + CM\varepsilon\Theta \\
&\leq \varepsilon(1 + CM\Theta),
\end{aligned} \tag{2.11}$$

which completes the proof. □

3. Existence of Weighted S-Asymptotically ω-Periodic Solutions

In this section we discuss the existence of weighted S-asymptotically ω-periodic solutions for the abstract system (1.1)-(1.2). To begin, we recall the definition of mild solution for (1.1)-(1.2).

Definition 3.1 (see [5]). A function $u \in C_b([0,\infty), X)$ is called a mild solution of the abstract Cauchy problem (1.1)-(1.2) if

$$u(t) = S_\alpha(t)u_0 + \int_0^t S_\alpha(t-s) f(s, u(s)) ds, \quad \forall t \in \mathbb{R}^+. \tag{3.1}$$

Now, we can establish our first existence result.

Theorem 3.2. *Assume $f : [0,\infty) \times X \to X$ is a uniformly S-asymptotically ω-periodic on bounded sets function and there is a mesurable bounded function $L_f : [0,\infty) \to \mathbb{R}^+$ such that*

$$\|f(t,x) - f(t,y)\| \leq L_f(t)\|x-y\|, \quad \forall t \in \mathbb{R}, \ \forall x, y \in X. \tag{3.2}$$

If $\Lambda := CM(\sup_{t\geq 0} \int_0^t L_f(s)/(1+|\mu|(t-s)^\alpha) ds) < 1$, then there exits a unique S-asymptotically ω-periodic mild solution $u(\cdot)$ of (1.1)-(1.2). Suppose, there is a function $L_u : [0,\infty) \to \mathbb{R}^+$ such that $(1+|\mu|(\cdot)^\alpha)L_u(\cdot) \in L^1([0,\infty))$ and $\|f(t+\omega,x) - f(t,x)\| \leq L_u(t)$, for every $x \in \mathcal{R}(u) = \{u(s) : s \geq 0\}$ and all $t \geq 0$. If $v \in C_b([0,\infty),(0,\infty))$ is such that $(1/(v(t)(1+|\mu|t^\alpha)))e^{2^\alpha CM \int_0^t L_f(s)ds} \to 0$ as $t \to \infty$, then $u(\cdot)$ is weighted S-asymptotically ω-periodic.

Proof. Let $F_\alpha : \mathrm{SAP}_\omega(X) \to C_b([0,\infty), X)$ be the operator defined by

$$F_\alpha u(t) = S_\alpha(t)u_0 + \int_0^t S_\alpha(t-s) f(s,u(s))ds := S_\alpha(t)u_0 + F_\alpha^1 u(t). \tag{3.3}$$

We show initially that F_α is $\mathrm{SAP}_\omega(X)$-valued. Since $S_\alpha(t)u_0 \to 0$, as $t \to \infty$, it is sufficient to show that the function F_α^1 is $\mathrm{SAP}_\omega(X)$-valued. Let $u \in \mathrm{SAP}_\omega(X)$. Using the fact that $f(\cdot, u(\cdot))$ is a bounded function, it follows that $F_\alpha^1 u \in C_b([0,\infty), X)$. For $\varepsilon > 0$ be given, we select a constant $T_\epsilon > 0$ such that

$$\begin{gathered}\sup_{t\geq T_\epsilon, s\geq 0}\left(\|f(t+\omega,u(s)) - f(t,u(s))\| + \|u(t+\omega)-u(t)\|\right) < \frac{\varepsilon}{2}, \\ 2CM\|f(\cdot,u(\cdot))\|_\infty \int_{T_\epsilon}^\infty \frac{1}{1+|\mu|s^\alpha} ds < \frac{\varepsilon}{2}.\end{gathered} \tag{3.4}$$

Then, for $t \geq 2L_\epsilon$ we see that

$$\begin{aligned}\left\|F_\alpha^1 u(t+\omega) - F_\alpha^1 u(t)\right\| \leq{}& \int_0^\omega \|S_\alpha(t+\omega-s) f(s,u(\mathrm{s}))\| ds \\ &+ \int_0^{T_\epsilon} \|S_\alpha(t-s)[f(s+\omega,u(s+\omega)) - f(s,u(s+\omega))]\| ds \\ &+ \int_0^{T_\epsilon} \|S_\alpha(t-s)[f(s,u(s+\omega)) - f(s,u(s))]\| ds \\ &+ \int_{T_\epsilon}^t \|S_\alpha(t-s)[f(s+\omega,u(s+\omega)) - f(s,u(s+\omega))]\| ds \\ &+ \int_{T_\epsilon}^t \|S_\alpha(t-s)[f(s,u(s+\omega)) - f(s,u(s))]\| ds \\ \leq{}& CM\|f(\cdot,u(\cdot))\|_\infty \left(\int_t^\infty \frac{1}{1+|\mu|s^\alpha} ds + \int_{t/2}^\infty \frac{1}{1+|\mu|s^\alpha} ds\right) \\ &+ \frac{\varepsilon}{2} CM \sup_{\tau\geq 0} \int_0^\tau \frac{L_f(\tau-s)}{1+|\mu|s^\alpha} ds \\ <{}& \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,\end{aligned} \tag{3.5}$$

which implies that $F_\alpha^1 u(t+\omega) - F_\alpha^1 u(t) \to 0$ as $t \to \infty$, $F_\alpha^1 u \in \mathrm{SAP}_\omega(X)$ and hence $F_\alpha(\mathrm{SAP}_\omega(X)) \subset \mathrm{SAP}_\omega(X)$. Moreover, from the above estimate it is easy to infer that $\|F_\alpha u - F_\alpha v\| \leq \Lambda\|u-v\|$, for all $u, v \in \mathrm{SAP}_\omega(X)$, F_α is a contraction and there exists a unique S-asymptotically ω-periodic mild solution $u(\cdot)$ of (1.1)-(1.2).

Next, we prove that last assertion. Let $\xi : [0,\infty) \to \mathbb{R}^+$ be the function defined by $\xi(t) = \|u(t+\omega) - u(t)\|/v(t)$. For $t \geq 0$, we get

$$\begin{aligned}
\xi(t) &\leq \frac{\|S_\alpha(t+\omega)u_0 - S_\alpha(t)u_0\|}{v(t)} + \frac{\left\|F_\alpha^1 u(t+\omega) - F_\alpha^1 u(t)\right\|}{v(t)} \\
&\leq \frac{2CM\|u_0\|}{v(t)(1+|\mu|t^\alpha)} + \frac{1}{v(t)}\int_0^\omega \|S_\alpha(t+\omega-s)\|_{\mathcal{B}(X)}\|f(s,u(s))\|ds \\
&\quad + \frac{1}{v(t)}\int_0^t \|S_\alpha(t-s)\|_{\mathcal{B}(X)}\|f(s+\omega,u(s+\omega)) - f(s,u(s))\|ds \\
&= \frac{2CM\|u_0\|}{v(t)(1+|\mu|t^\alpha)} + I_1 + I_2.
\end{aligned} \tag{3.6}$$

Concerning the quantities I_1 and I_2, we note that

$$\begin{aligned}
I_1 &\leq \frac{CM\|f(\cdot,u(\cdot))\|_\infty}{v(t)}\left(\int_t^{t+\omega}\frac{1}{1+|\mu|s^\alpha}ds\right) \leq \frac{CM\omega\|f(\cdot,u(\cdot))\|_\infty}{v(t)(1+|\mu|t^\alpha)}, \\
I_2 &\leq \frac{1}{v(t)}\int_0^t \|S_\alpha(t-s)\|_{\mathcal{B}(X)}\|f(s+\omega,u(s+\omega)) - f(s,u(s+\omega))\|ds \\
&\quad + \frac{1}{v(t)}\int_0^t \|S_\alpha(t-s)\|_{\mathcal{B}(X)}\|f(s,u(s+\omega)) - f(s,u(s))\|ds \\
&\leq \frac{CM}{v(t)}\int_0^t \frac{L_u(s)}{1+|\mu|(t-s)^\alpha}ds + \frac{CM}{v(t)}\int_0^t \frac{L_f(s)v(s)\xi(s)}{1+|\mu|(t-s)^\alpha}ds.
\end{aligned} \tag{3.7}$$

Using the estimates (3.7) in (3.6), we see that

$$\begin{aligned}
v(t)(1+|\mu|t^\alpha)\xi(t) &\leq CM(2\|u_0\| + \|f(\cdot,u(\cdot))\|_\infty) + CM\int_0^t \frac{1+|\mu|t^\alpha}{1+|\mu|(t-s)^\alpha}L_u(s)ds \\
&\quad + CM\int_0^t \frac{1+|\mu|t^\alpha}{1+|\mu|(t-s)^\alpha}L(s)v(s)\xi(s)ds \\
&\leq CM\left(\|u_0\| + \|f(\cdot,u(\cdot))\|_\infty + 2^\alpha\int_0^t (1+|\mu|s^\alpha)L_u(s)ds\right) \\
&\quad + 2^{\alpha-1}CM\int_0^t L_f(s)v(s)(1+|\mu|s^\alpha)\xi(s)ds \\
&\leq P + 2^{\alpha-1}CM\int_0^t L_f(s)v(s)(1+|\mu|s^\alpha)\xi(s)ds,
\end{aligned} \tag{3.8}$$

where P is a positive constant independent of t. Finally, by using the Gronwall-Bellman inequality we infer that

$$\lim_{t\to\infty}\frac{\|u(t+\omega)-u(t)\|}{v(t)}=0, \tag{3.9}$$

which shows that $u\in \mathrm{SAP}_{\omega}(X,v)$. This completes the proof. □

Example 3.3. We set $X = L^2[0,\pi]$, $A = -\rho^{\alpha} I$ with $0<\rho<1$. Let $g:\mathbb{R}\to\mathbb{R}$ be a function such that $|g(x)-g(y)|\le L_g\|x-y\|$, for all $x,y\in\mathbb{R}$ and let $f:[0,\infty)\times X\to X$ be defined by $f(t,x)(\xi)=e^{-t^{\alpha}}g(x(\xi))$, $\xi\in[0,\pi]$. We observe that

$$\|f(t+\omega,x)-f(t,x)\|_{L^2}\le\sqrt{2}\left(e^{-(t+\omega)^{\alpha}}-e^{-t^{\alpha}}\right)\left(\sqrt{L_g}\|x\|_{L^2}+|g(0)|\sqrt{\pi}\right), \tag{3.10}$$

whence f is S-asymptotically ω-periodic on bounded sets. By Theorem 3.2 we conclude that if $L_g<\alpha\sin(\pi/\alpha)/\pi\rho^{-1}$, then there is a unique S-asymptotically ω-periodic mild solution $u(\cdot)$ of (1.1)-(1.2). Moreover $u\in\mathrm{SAP}_{\omega}(X,1/(1+\rho^{\alpha}t))$.

Theorem 3.4. *Let $v\in C_b([0,\infty),(0,\infty))$. Assume $G\in\mathrm{SAP}_{\omega}(\mathcal{B}(X),v)$, $1/v(t)t^{\alpha-1}\to 0$ as $t\to\infty$ and*

$$\Lambda := CM\|G\|_{\mathrm{SAP}_{\omega}(\mathcal{B}(X),v)}\left[\frac{|\mu|^{-1/\alpha}\pi}{\alpha\sin(\pi/\alpha)}+\omega\sup_{t\ge0}\left(\frac{1}{v(t)(1+|\mu|t^{\alpha})}\right)+2\Theta\right]<1, \tag{3.11}$$

where Θ is the constant introduced in Lemma 2.8.Then there is a unique weighted S-asymptotically ω-periodic mild solution of

$$\begin{gathered} u'(t)=\int_0^t\frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)}Au(s)ds+G(t)u(t),\quad t\ge0,\\ u(0)=u_0\in X. \end{gathered} \tag{3.12}$$

Proof. The proof is based in Lemmas 2.7 and 2.8. Let $\Gamma:\mathrm{SAP}_{\omega}(X,v)\to C_b([0,\infty),X)$ be the map defined by

$$\Gamma u(t)=S_{\alpha}(t)u_0+\int_0^t S_{\alpha}(t-s)G(s)u(s)ds=S_{\alpha}(t)u_0+\Gamma_1u(t),\quad t\ge0. \tag{3.13}$$

We show initially that Γ is $\mathrm{SAP}_{\omega}(X,v)$-valued. From the estimate

$$\frac{\|S_{\alpha}(t+\omega)u_0-S_{\alpha}(t)u_0\|}{v(t)}\le\frac{2CM\|u_0\|}{|\mu|}\frac{1}{v(t)t^{\alpha}} \tag{3.14}$$

we have that $S_{\alpha}(\cdot)u_0\in\mathrm{SAP}_{\omega}(X,v)$.

Let $u \in \mathrm{SAP}_\omega(X, v)$. From Lemma 2.7, we have that $s \to G(s)u(s)$ is a weighted S-asymptotically ω-periodic function and by Lemma 2.8 we obtain that $\Gamma u \in \mathrm{SAP}_\omega(X, v)$. Thus, the map Γ is $\mathrm{SAP}_\omega(X, v)$-valued. In order to prove that Γ is a contraction, we note that for $u \in \mathrm{SAP}_\omega(X, v)$ and $t \geq 0$,

$$\begin{aligned}
\|\Gamma_1 u(t)\| &\leq CM \int_0^t \frac{1}{1+|\mu|(t-s)^\alpha} \|G(s)\| \|u(s)\| ds \\
&\leq CM \left(\int_0^t \frac{1}{1+|\mu| s^\alpha} ds \right) \|G\|_\infty \|u\|_\infty \\
&\leq \frac{CM|\mu|^{-1/\alpha}\pi}{\alpha \sin(\pi/\alpha)} \|G\|_\infty \|u\|_\infty,
\end{aligned} \tag{3.15}$$

so that,

$$\|\Gamma_1 u\|_\infty \leq \frac{CM|\mu|^{-1/\alpha}\pi}{\alpha \sin(\pi/\alpha)} \|G\|_{\mathrm{SAP}_\omega(\mathcal{B}(X), v)} \|u\|_{\mathrm{SAP}_\omega(X, v)}. \tag{3.16}$$

On the another hand, for $t \geq 0$ we see that

$$\begin{aligned}
\frac{\|\Gamma_1 u(t+\omega) - \Gamma_1 u(t)\|}{v(t)} &\leq \frac{1}{v(t)} \left(\int_0^\omega \|S_\alpha(t+\omega-s)\|_{\mathcal{B}(X)} ds \right) \|G\|_\infty \|u\|_\infty \\
&\quad + \frac{1}{v(t)} \int_0^t \|S_\alpha(t-s)\|_{\mathcal{B}(X)} \|G(s+\omega)u(s+\omega) - G(s)u(s)\| ds \\
&\leq \frac{CM\omega}{v(t)(1+|\mu|t^\alpha)} \|G\|_\infty \|u\|_\infty \\
&\quad + \frac{CM}{v(t)} \int_0^t \frac{1}{1+|\mu|(t-s)^\alpha} \|G(s+\omega) - G(s)\|_{\mathcal{B}(X)} \|u(s+\omega)\| ds \\
&\quad + \frac{CM}{v(t)} \int_0^t \frac{1}{1+|\mu|(t-s)^\alpha} \|G(s)\|_{\mathcal{B}(X)} \|u(s+\omega) - u(s)\| ds \\
&\leq \frac{CM\omega}{v(t)(1+|\mu|t^\alpha)} \|G\|_\infty \|u\|_\infty \\
&\quad + CM \left(\frac{1}{v(t)} \int_0^t \frac{v(t-s)}{1+|\mu| s^\alpha} ds \right) \|G\|_v \|u\|_\infty \\
&\quad + CM \left(\frac{1}{v(t)} \int_0^t \frac{v(t-s)}{1+|\mu| s^\alpha} ds \right) \|G\|_\infty \|u\|_v,
\end{aligned} \tag{3.17}$$

from which we obtain that

$$\sup_{t\geq 0}\frac{\|\Gamma_1 u(t+\omega)-\Gamma_1 u(t)\|}{v(t)} \leq CM\|G\|_{\mathrm{SAP}_\omega(\mathcal{B}(X),v)}\left[\omega\sup_{t\geq 0}\left(\frac{1}{v(t)(1+|\mu|t^\alpha)}\right)+2\Theta\right]\|u\|_{\mathrm{SAP}_\omega(X,v)}. \tag{3.18}$$

By noting that $G(s)$ is a linear operator for all $t \geq 0$ and combining (3.16) and (3.18) we obtain that

$$\|\Gamma u_1 - \Gamma u_2\|_{\mathrm{SAP}_\omega(X,v)} \leq \Lambda\|u_1-u_2\|_{\mathrm{SAP}_\omega(X,v)}, \tag{3.19}$$

for all $u_1, u_2 \in \mathrm{SAP}_\omega(X,v)$, which shows that Γ is a contraction on $\mathrm{SAP}_\omega(X,v)$ and hence there is a unique Sv-asymptotically ω-periodic mild solution. The proof is complete. □

To complete this paper, we examine the existence and uniqueness of weighted S-asymptotically ω-periodic mild solutions for the following fractional differential equation

$$\partial_t^\alpha u(t,x) = \partial_x^2 u(t,x) - \nu u(t,x) + \partial_t^{\alpha-1}\left(\int_0^x \beta a(t)u(t,\xi)d\xi\right), \quad t\in\mathbb{R}^+,\ x\in[0,\pi], \tag{3.20}$$

with boundary conditions

$$u(t,0) = u(t,\pi) = 0, \qquad t\geq 0, \tag{3.21}$$

$$u(0,x) = u_0(x), \quad x\in[0,\pi], \tag{3.22}$$

where $u_0 \in L^2[0,\pi]$ and $a \in C_b([0,\infty),\mathbb{R})$. In what follows we consider the space $X = L^2[0,\pi]$ and let A be the operator given by $Au = u''-\nu u$, $(\nu > 0)$ with domain $D(A) = \{u \in X : u'' \in X, u(0) = u(\pi) = 0\}$. It is well known that A is sectorial of type negative.

Proposition 3.5. *Let $v \in C_b([0,\infty),(0,\infty))$ satisfying conditions of Lemma 2.8 and let $a \in \mathrm{SAP}_\omega(\mathbb{R},v)$. If $|\beta|$ is small enough, then the problems* (3.20)–(3.22) *has a unique Sv-asymptotically ω-periodic mild solution.*

Proof. Problem (3.20)–(3.22) can be expressed as an abstract fractional differential equation of the form (3.12), where $\mathrm{u}(t)(x) = u(t,x)$, for $t \geq 0$, $x \in [0,\pi]$. We define

$$(G(t)\phi)(\xi) = \beta a(t)\int_0^\xi \phi(\tau)d\tau, \quad \xi\in[0,\pi],\ t\geq 0. \tag{3.23}$$

We have the following estimates:

$$\|G(t)\phi\|_{L^2} \leq \pi|\beta||a(t)|\|\phi\|_{L^2}, \quad t\geq 0,\ \phi\in X, \tag{3.24}$$

$$\|G(t+\omega)\phi - G(t)\phi\|_{L^2} \leq \pi|\beta||a(t+\omega)-a(t)|\|\phi\|_{L^2}, \quad t\geq 0,\ \phi\in X. \tag{3.25}$$

estimate (3.25), we get

$$\frac{\|G(t+\omega)-G(t)\|_{\mathcal{B}(X)}}{v(t)} \leq \pi|\beta|\frac{|a(t+\omega)-a(t)|}{v(t)}, \quad t \geq 0. \tag{3.26}$$

Since $a \in \mathrm{SAP}_{\omega}(\mathbb{R}, v)$ we obtain that $G \in \mathrm{SAP}_{\omega}(\mathcal{B}(X), v)$. Moreover, we have the inequality

$$\|G\|_{\mathrm{SAP}_{\omega}(\mathcal{B}(X),v)} \leq \pi|\beta|\|a\|_{\mathrm{SAP}_{\omega}(\mathbb{R},v)}. \tag{3.27}$$

If we choose $|\beta|$ small enough, we have that condition (3.11) is fulfilled. By Theorem 3.4, the problems (3.20)–(3.22) has a unique Sv-asymptotically ω-periodic (mild) solution. This finishes the proof. □

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