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Research Article

Weighted Inequalities for Potential Operators with Lipschitz and BMO Norms

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Some Lipschitz norm and BMO norm inequalities for potential operator to the versions of differential forms are obtained, and some properties of a new kind of $A_r^{\lambda_3}(\lambda_1, \lambda_2, \Omega)$ weight are derived.

1. Introduction

In many situations, the process to study solutions of PDEs involves estimating the various norms of the operators. Hence, we are motivated to establish some Lipschitz norm inequalities and BMO norm inequalities for potential operator to the versions of differential forms.

We keep using the traditional notation.

Let Ω be a connected open subset of $\mathbf{R}^n$, let $e_1, e_2, \dots, e_n$ be the standard unit basis of $\mathbf{R}^n$, and let $\bigwedge^l = \bigwedge^l(\mathbf{R}^n)$ be the linear space of l-covectors, spanned by the exterior products $e_I = e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_l}$, corresponding to all ordered l-tuples $I = (i_1, i_2, \dots, i_l)$, $1 \le i_1 < i_2 < \cdots < i_l \le n$, $l = 0, 1, \dots, n$. We let $\mathbf{R} = \mathbf{R}^1$. The Grassman algebra $\wedge = \oplus \bigwedge^l$ is a graded algebra with respect to the exterior products. For $\alpha = \sum \alpha^I e_I \in \wedge$ and $\beta = \sum \beta^I e_I \in \wedge$, the inner product in $\wedge$ is given by $\langle \alpha, \beta \rangle = \sum \alpha^I \beta^I$ with summation over all l-tuples $I = (i_1, i_2, \dots, i_l)$ and all integers $l = 0, 1, \dots, n$. We define the Hodge star operator $\star : \wedge \to \wedge$ by the rule $\star 1 = e_1 \wedge e_2 \wedge \cdots \wedge e_n$ and $\alpha \wedge \star\beta = \beta \wedge \star\alpha = \langle \alpha, \beta \rangle(\star 1)$ for all $\alpha, \beta \in \wedge$. The norm of $\alpha \in \wedge$ is given by the formula $|\alpha|^2 = \langle \alpha, \alpha \rangle = \star(\alpha \wedge \star\alpha) \in \bigwedge^0 = \mathbf{R}$. The Hodge star is an isometric isomorphism on $\wedge$ with $\star : \bigwedge^l \to \bigwedge^{n-l}$ and $\star\star\, (-1)^{l(n-l)} : \bigwedge^l \to \bigwedge^l$. Balls are denoted by B, and ρB is the ball with the same center as B and with $\operatorname{diam}(\rho B) = \rho \operatorname{diam}(B)$. We do not distinguish balls from cubes throughout this paper. The n-dimensional Lebesgue measure of a set $E \subseteq \mathbf{R}^n$ is denoted by $|E|$.

We call $w(x)$ a weight if $w \in L^1_{\rm loc}(\mathbf{R}^n)$ and that is, $w > 0$. For $0 < p < \infty$ and a weight $w(x)$, we denote the weighted L^p-norm of a measurable function f over E by

$$\|f\|_{p,E,w^\alpha} = \left(\int_E |f(x)|^p w^\alpha dx\right)^{1/p}, \tag{1.1}$$

where α is a real number.

Differential forms are important generalizations of real functions and distributions; note that a 0-form is the usual function in $\mathbf{R}^n$. A differential l-form ω on Ω is a Schwartz distribution on Ω with values in $\bigwedge^l(\mathbf{R}^n)$. We use $D'(\Omega, \bigwedge^l)$ to denote the space of all differential l-forms $\omega(x) = \sum_I \omega_I(x) dx_I = \sum \omega_{i_1 i_2,\ldots,i_l}(x) dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_l}$. We write $L^p(\Omega, \bigwedge^l)$ for the l-forms with $\omega_I \in L^p(\Omega, \mathbf{R})$ for all ordered l-tuples I. Thus, $L^p(\Omega, \bigwedge^l)$ is a Banach space with norm

$$\|\omega\|_{p,\Omega} = \left(\int_\Omega |\omega(x)|^p dx\right)^{1/p} = \left(\int_\Omega \left(\sum |\omega_I(x)|^2\right)^{p/2} dx\right)^{1/p}. \tag{1.2}$$

For $\omega \in D'(\Omega, \bigwedge^l)$, the vector-valued differential form $\nabla\omega = (\partial\omega/\partial x_1, \ldots, \partial\omega/\partial x_n)$ consists of differential forms $\partial\omega/\partial x_i \in D'(\Omega, \bigwedge^l)$, where the partial differentiations are applied to the coefficients of ω. As usual, $W^{1,p}(\Omega, \bigwedge^l)$ is used to denote the Sobolev space of l-forms, which equals $L^p(\Omega, \bigwedge^l) \cap L^p_1(\Omega, \bigwedge^l)$ with norm

$$\|\omega\|_{W^{1,p}(\Omega,\bigwedge^l)} = \|\omega\|_{W^{1,p}(\Omega,\bigwedge^l)} = \operatorname{diam}(\Omega)^{-1}\|\omega\|_{p,\Omega} + \|\nabla\omega\|_{p,\Omega}. \tag{1.3}$$

The notations $W^{1,p}_{\rm loc}(\Omega, \mathbf{R})$ and $W^{1,p}_{\rm loc}(\Omega, \bigwedge^l)$ are self-explanatory. For $0 < p < \infty$ and a weight $w(x)$, the weighted norm of $\omega \in W^{1,p}(\Omega, \bigwedge^l)$ over Ω is denoted by

$$\|\omega\|_{W^{1,p}(\Omega,\bigwedge^l),w^\alpha} = \|\omega\|_{W^{1,p}(\Omega,\bigwedge^l),w^\alpha} = \operatorname{diam}(\Omega)^{-1}\|\omega\|_{p,\Omega,w^\alpha} + \|\nabla\omega\|_{p,\Omega,w^\alpha}, \tag{1.4}$$

where α is a real number. We denote the exterior derivative by $d : D'(\Omega, \bigwedge^l) \to D'(\Omega, \bigwedge^{l+1})$ for $l = 0, 1, \ldots, n$. Its formal adjoint operator $d^\star : D'(\Omega, \bigwedge^{l+1}) \to D'(\Omega, \bigwedge^l)$ is given by $d^\star = (-1)^{nl+1} \star d \star$ on $D'(\Omega, \bigwedge^{l+1})$, $l = 0, 1, \ldots, n$.

Let $u \in L^1_{\rm loc}(\Omega, \bigwedge^l)$, $l = 0, 1, \ldots, n$. We write $u \in \operatorname{loc}\operatorname{Lip}_k(\Omega, \bigwedge^l)$, $0 \le k \le 1$ if

$$\|u\|_{\operatorname{loc}\operatorname{Lip}_k,\Omega} = \sup_{\sigma Q \subset \Omega} |Q|^{-(n+k)/n} \|u - u_Q\|_{1,Q} < \infty, \tag{1.5}$$

for some $\sigma \ge 1$. Further, we write $\operatorname{Lip}_k(\Omega, \bigwedge^l)$ for those forms whose coefficients are in the usual Lipschitz space with exponent k and write $\|u\|_{\operatorname{Lip}_k,\Omega}$ for this norm. Similarly, for $u \in L^1_{\rm loc}(\Omega, \bigwedge^l)$, $l = 0, 1, \ldots, n$, we write $u \in \operatorname{BMO}(\Omega, \bigwedge^l)$ if

$$\|u\|_{\star,\Omega} = \sup_{\sigma Q \subset \Omega} |Q|^{-1} \|u - u_Q\|_{1,Q} < \infty, \tag{1.6}$$

for some $\sigma \ge 1$. When u is a 0-form, (1.6) reduces to the classical definition of $\operatorname{BMO}(\Omega)$.

Based on the above results, we discuss the weighted Lipschitz and BMO norms. For $u \in L^1_{\text{loc}}(\Omega, \bigwedge^l, w^\alpha)$, $l = 0, 1, \ldots, n$, we write $u \in \text{loc}\,\text{Lip}_k(\Omega, \bigwedge^l, w^\alpha)$, $0 \le k \le 1$ if

$$\|u\|_{\text{loc}\,\text{Lip}_k, \Omega, w^\alpha} = \sup_{\sigma Q \subset \Omega} (\mu(Q))^{-(n+k)/n} \|u - u_Q\|_{1, Q, w^\alpha} < \infty, \tag{1.7}$$

for some $\sigma > 1$, where Ω is a bounded domain, the Radon measure μ is defined by $d\mu = w(x)^\alpha dx$, w is a weight and α is a real number. For convenience, we will write the following simple notation $\text{loc}\,\text{Lip}_k(\Omega, \bigwedge^l)$ for $\text{loc}\,\text{Lip}_k(\Omega, \bigwedge^l, w^\alpha)$. Similarly, for $u \in L^1_{\text{loc}}(\Omega, \bigwedge^l, w^\alpha)$, $l = 0, 1, \ldots, n$, we write $u \in \text{BMO}(\Omega, \bigwedge^l, w^\alpha)$ if

$$\|u\|_{\star, \Omega, w^\alpha} = \sup_{\sigma Q \subset \Omega} (\mu(Q))^{-1} \|u - u_Q\|_{1, Q, w^\alpha} < \infty, \tag{1.8}$$

for some $\sigma > 1$, where the Radon measure μ is defined by $d\mu = w(x)^\alpha dx$, w is a weight, and α is a real number. Again, we use $\text{BMO}(\Omega, \bigwedge^l)$ to replace $\text{BMO}(\Omega, \bigwedge^l, w^\alpha)$ whenever it is clear that the integral is weighted.

From [1], if ω is a differential form defined in a bounded, convex domain M, then there is a decomposition

$$\omega = d(T\omega) + T(d\omega), \tag{1.9}$$

where T is called a homotopy operator. Furthermore, we can define the k-form $\omega_M \in D'(M, \bigwedge^k)$ by

$$\omega_M = |M|^{-1} \int_M \omega(y) dy, \quad k = 0, \quad \omega_M = d(T\omega), \quad k = 1, 2, \ldots, n, \tag{1.10}$$

for all $\omega \in L^p(M, \bigwedge^k)$, $1 \le p < \infty$.

For any differential k-form $\omega(x)$, we define the potential operator P by

$$P\omega(x) = \sum_I \int_E K(x, y) \omega_I(y) dy\, dx_I, \tag{1.11}$$

where the kernel $K(x, y)$ is a nonnegative measurable function defined for $x \ne y$, and the summation is over all ordered k-tuples I. It is easy to find that the case $k = 0$ reduces to the usual potential operator. That is,

$$Pf(x) = \int_E K(x, y) f(y) dy, \tag{1.12}$$

where $f(x)$ is a function defined on $E \subset R^n$. Associated with P, the functional φ is defined as

$$\varphi(B) = \sup_{x, y \in B, |x - y| \ge Cr} K(x, y), \tag{1.13}$$

where C is some sufficiently small constant and $B \subset E$ is a ball with radius r. Throughout this paper, we always suppose that φ satisfies the following conditions: there exists C_φ such that

$$\varphi(2B) \le C_\varphi \varphi(B) \quad \text{for all balls } B \subset E, \tag{1.14}$$

and there exists $\varepsilon > 0$ such that

$$\varphi(B_1)\mu(B_1) \le C_\varphi \left(\frac{r(B_1)}{r(B_2)}\right)^{\varepsilon} \varphi(B_2)\mu(B_2) \quad \text{for all balls } B_1 \subset B_2. \tag{1.15}$$

On the potential operator P and the functional φ, see [2] for details.

The nonlinear elliptic partial differential equation $d^\star A(x, du) = 0$ is called the homogeneous A-harmonic equation or the A-harmonic equation, and the differential equation

$$d^\star A(x, du) = B(x, du) \tag{1.16}$$

is called the nonhomogeneous A-harmonic equation for differential forms, where $A : \Omega \times \bigwedge^l(\mathbf{R}^n) \to \bigwedge^l(\mathbf{R}^n)$ and $B : \Omega \times \bigwedge^l(\mathbf{R}^n) \to \bigwedge^{l-1}(\mathbf{R}^n)$ satisfy the conditions

$$|A(x,\xi)| \le a|\xi|^{p-1}, \qquad \langle A(x,\xi), \xi\rangle \ge |\xi|^p, \qquad |B(x,\xi)| \le b|\xi|^{p-1}, \tag{1.17}$$

for almost every $x \in \Omega$ and all $\xi \in \bigwedge^l(\mathbf{R}^n)$. Here $a, b > 0$ are constants and $1 < p < \infty$ is a fixed exponent associated with (1.16). A solution to (1.16) is an element of the Sobolev space $W^{1,p}_{\text{loc}}(\Omega, \bigwedge^{l-1})$ such that

$$\int_\Omega A(x, du) \cdot d\varphi + B(x, du) \cdot \varphi = 0, \tag{1.18}$$

for all $\varphi \in W^{1,p}_{\text{loc}}(\Omega, \bigwedge^{l-1})$ with compact support. When u is a 0-form, that is, u is a function, (1.16) is equivalent to

$$\operatorname{div} A(x, \nabla u) = B(x, \nabla u). \tag{1.19}$$

Lots of results have been obtained in recent years about different versions of the A-harmonic equation, see [3–5].

2. The Estimate for Potential Operators with Lipschitz Norm and BMO Norm

In this section, we give the estimate for potential operators with Lipschitz norm and BMO norm applied to differential forms. The following strong type (p, p) inequality for potential operators appears in [6].

Lemma 2.1 (see [6]). *Let $u \in D'(E, \bigwedge^k)$, $k = 0, 1, \ldots, n-1$, be a differential form defined in a bounded, convex domain E, and let u_I be coefficient of u with $\operatorname{supp} u_I \subset E$ for all ordered k-tuples I. Assume that $1 < p < \infty$ and P is the potential operator with $k(x, y) = \varphi_\varepsilon(x - y)$ for any $\varepsilon > 0$, then there exists a constant C, independent of u, such that*

$$\|P(u) - (P(u))_E\|_{p,E} \le C|E| \operatorname{diam}(E) \|u\|_{p,E}. \tag{2.1}$$

We will establish the following estimate for potential operators.

Theorem 2.2. *Let $u \in D'(E, \bigwedge^k)$, $k = 0, 1, \ldots, n-1$, be a differential form defined in a bounded, convex domain E, and let u_I be coefficient of u with $\operatorname{supp} u_I \subset E$ for all ordered k-tuples I. Assume that $1 < p < \infty$ and P is the potential operator with $k(x, y) = \varphi_\varepsilon(x - y)$ for any $\varepsilon > 0$, then there exists a constant C, independent of ω, such that*

$$\|P(u)\|_{\star,E} \le \|P(u)\|_{\operatorname{loc}\operatorname{Lip}_k,E} \le C\|u\|_{p,E}. \tag{2.2}$$

Proof. By the definition of the Lipschitz norm, (2.1), and hölder's inequality with $1 = 1/p + (p-1)/p$, we have

$$\begin{aligned}
\|P(u)\|_{\operatorname{loc}\operatorname{Lip}_k,E} &= \sup_{\sigma B \subset E} (\mu(B))^{-(n+k)/n} \|P(u) - (P(u))_B\|_{1,B} \\
&\le \sup_{\sigma B \subset E} (\mu(B))^{-(n+k)/n} \left(\int_B |P(u) - (P(u))_B|^p dx \right)^{1/p} \left(\int_B 1^{p/(p-1)} dx \right)^{(p-1)/p} \\
&= \sup_{\sigma B \subset E} (\mu(B))^{-(n+k)/n+(p-1)/p} \|P(u) - (P(u))_B\|_{p,B} \\
&\le \sup_{\sigma B \subset E} (\mu(B))^{-(n+k)/n+(p-1)/p} C|B| \operatorname{diam}(B) \|u\|_{p,B} \\
&\le C|E|^{-(n+k)/n+(p-1)/p+1+1/n} \|u\|_{p,E} \\
&\le C\|u\|_{p,E},
\end{aligned} \tag{2.3}$$

since $-1/p - k/n + 1 + 1/n > 0$ and $|\Omega| < \infty$, where σ is a constant and $\sigma B \subset \Omega$.

By the definition of the BMO norm, we have

$$\begin{aligned}
\|P(u)\|_{\star,E} &= \sup_{\sigma B \subset E} (\mu(B))^{-1} \|P(u) - (P(u))_B\|_{1,B} \\
&= \sup_{\sigma B \subset E} (\mu(B))^{k/n} (\mu(B))^{-(n+k)/n} \|P(u) - (P(u))_B\|_{1,B} \\
&\le C \sup_{\sigma B \subset E} (\mu(B))^{-(n+k)/n} \|P(u) - (P(u))_B\|_{1,B} \\
&\le C\|P(u)\|_{\operatorname{loc}\operatorname{Lip}_k,E}.
\end{aligned} \tag{2.4}$$

We have completed the proof of Theorem 2.2. □

3. The $A_r^{\lambda_3}(\lambda_1, \lambda_2, \Omega)$ Weight

In this section, we introduce the $A_r^{\lambda_3}(\lambda_1, \lambda_2, \Omega)$ weight appeared in [7].

Definition 3.1. Let $w_1(x), w_2(x)$ be two locally integrable nonnegative functions in $E \subset \mathbf{R}^n$ and assume that $0 < w_1, \ w_2 < \infty$ almost everywhere. We say that $(w_1(x), w_2(x))$ belongs to the $A_r^{\lambda_3}(\lambda_1, \lambda_2, E)$ class, $1 < r < \infty$ and $0 < \lambda_1, \lambda_2, \lambda_3 < \infty$, or that $(w_1(x), w_2(x))$ is an $A_r^{\lambda_3}(\lambda_1, \lambda_2, E)$ weight, write $(w_1, w_2) \in A_r^{\lambda_3}(\lambda_1, \lambda_2, E)$ or $(w_1, w_2) \in A_r^{\lambda_3}(\lambda_1, \lambda_2)$ when it will not cause any confusion, if

$$\sup_B \left(\frac{1}{|B|}\int_B w_1^{\lambda_1} dx\right)\left(\frac{1}{|B|}\int_B \left(\frac{1}{w_2}\right)^{\lambda_2/(r-1)} dx\right)^{\lambda_3(r-1)} < \infty \tag{3.1}$$

for all balls $B \subset E \subset \mathbf{R}^n$.

The following results show that the $A_r^{\lambda_3}(\lambda_1, \lambda_2)$ weights have the properties similar to those of the A_r weights.

Theorem 3.2. *If* $1 < r < s < \infty$, *then* $A_r^{\lambda_3}(\lambda_1, \lambda_2) \subset A_s^{\lambda_3}(\lambda_1, \lambda_2)$.

Proof. Let $(w_1, w_2) \in A_r^{\lambda_3}(\lambda_1, \lambda_2)$. Since $1 < r < s < \infty$, by Hölder's inequality,

$$\begin{aligned}
\left(\int_B \left(\frac{1}{w_2}\right)^{\lambda_2/(s-1)} dx\right)^{\lambda_3(s-1)} &\le \left(\int_B \left(\frac{1}{w_2}\right)^{\lambda_2/(r-1)} dx\right)^{\lambda_3(r-1)} \left(\int_B 1^{\lambda_2/(s-r)} dx\right)^{\lambda_3(s-r)} \\
&= |B|^{\lambda_3(s-r)}\left(\int_B \left(\frac{1}{w_2}\right)^{\lambda_2/(r-1)} dx\right)^{\lambda_3(r-1)} \\
&= \frac{|B|^{\lambda_3(s-1)}}{|B|^{\lambda_3(r-1)}}\left(\int_B \left(\frac{1}{w_2}\right)^{\lambda_2/(r-1)} dx\right)^{\lambda_3(r-1)},
\end{aligned} \tag{3.2}$$

so that

$$\left(\frac{1}{|B|}\int_B \left(\frac{1}{w_2}\right)^{\lambda_2/(s-1)} dx\right)^{\lambda_3(s-1)} \le \left(\frac{1}{|B|}\int_B \left(\frac{1}{w_2}\right)^{\lambda_2/(r-1)} dx\right)^{\lambda_3(r-1)}. \tag{3.3}$$

Thus, we find that

$$\begin{aligned}
&\sup_B \left(\frac{1}{|B|}\int_B w_1^{\lambda_1} dx\right)\left(\frac{1}{|B|}\int_B \left(\frac{1}{w_2}\right)^{\lambda_2/(s-1)} dx\right)^{\lambda_3(s-1)} \\
&\quad\le \sup_B \left(\frac{1}{|B|}\int_B w_1^{\lambda_1} dx\right)\left(\frac{1}{|B|}\int_B \left(\frac{1}{w_2}\right)^{\lambda_2/(r-1)} dx\right)^{\lambda_3(r-1)},
\end{aligned} \tag{3.4}$$

for all balls $B \subset \mathbf{R}^n$ since $(w_1, w_2) \in A_r^{\lambda_3}(\lambda_1, \lambda_2)$. Therefore, $(w_1, w_2) \in A_s^{\lambda_3}(\lambda_1, \lambda_2)$, and hence $A_r^{\lambda_3}(\lambda_1, \lambda_2) \subset A_s^{\lambda_3}(\lambda_1, \lambda_2)$. □

Theorem 3.3. *If* $(w_1, w_2) \in A_r^{\lambda_3}(\lambda_1, \lambda_2),\ \lambda_1 \geq 1,\ \lambda_2, \lambda_3 > 0$ *and the measures* $\mu,\ \nu$ *are defined by* $d\mu = w_1(x)dx,\ d\nu = w_2(x)^{\lambda_2}dx$, *then*

$$\frac{|E|^{\lambda_3 r}}{|B|^{\lambda_1+\lambda_3(r-1)}} \leq C(r, \lambda_1, \lambda_2, \lambda_3, w_1, w_2)\frac{\mu(E)^{\lambda_3}}{\mu(B)^{\lambda_1}}, \tag{3.5}$$

where B *is a ball in* $\mathbf{R}^n$ *and* E *is a measurable subset of* B.

Proof. By Hölder's inequality, we have

$$\begin{aligned} |E| &= \int_E dx = \int_E w_2^{\lambda_2/r} w_2^{-\lambda_2/r}\,dx \\ &\leq \left(\int_E w_2^{\lambda_2}dx\right)^{1/r}\left(\int_E w_2^{\lambda_2/(1-r)}dx\right)^{(r-1)/r} \\ &= (\mu(E))^{1/r}\left(\int_E w_2^{\lambda_2/(1-r)}dx\right)^{(r-1)/r}. \end{aligned} \tag{3.6}$$

This implies

$$|E|^r \leq \mu(E)\left(\int_E w_2^{\lambda_2/(1-r)}dx\right)^{(r-1)}. \tag{3.7}$$

Note that $\lambda_1 \geq 1$, by Hölder's inequality again, we have

$$\frac{1}{|B|}\int_B w_1 dx \leq \left(\frac{1}{|B|}\int_B w_1^{\lambda_1}dx\right)^{1/\lambda_1}, \tag{3.8}$$

so that

$$1 = \frac{1}{\mu(B)}\int_B w_1 dx \leq \frac{|B|}{\mu(B)}\left(\frac{1}{|B|}\int_B w_1^{\lambda_1}dx\right)^{1/\lambda_1}. \tag{3.9}$$

Hence, we obtain

$$\mu(B)^{\lambda_1} \leq |B|^{\lambda_1-1}\int_B w_1^{\lambda_1}dx. \tag{3.10}$$

Since $(w_1, w_2) \in A_r^{\lambda_3}(\lambda_1, \lambda_2)$, there exists a constant $C(r, \lambda_1, \lambda_2, \lambda_3, w_1, w_2)$ such that

$$\left(\frac{1}{|B|}\int_B w_1^{\lambda_1}dx\right)\left(\frac{1}{|B|}\int_B \left(\frac{1}{w_2}\right)^{\lambda_2/(r-1)}dx\right)^{\lambda_3(r-1)} \leq C(r, \lambda_1, \lambda_2, \lambda_3, w_1, w_2). \tag{3.11}$$

Combining (3.7), (3.10), and (3.11), we deduce that

$$\begin{aligned} |E|^{\lambda_3 r}\mu(B)^{\lambda_1} &\le \mu(E)^{\lambda_3}|B|^{\lambda_1-1}\left(\int_E w_2^{\lambda_2/(1-r)}dx\right)^{\lambda_3(r-1)}\int_B w_1^{\lambda_1}dx \\ &= \mu(E)^{\lambda_3}|B|^{\lambda_1+\lambda_3(r-1)}\left(\frac{1}{|B|}\int_E w_2^{\lambda_2/(1-r)}dx\right)^{\lambda_3(r-1)}\left(\frac{1}{|B|}\int_B w_1^{\lambda_1}dx\right) \\ &\le C(r,\lambda_1,\lambda_2,\lambda_3,w_1,w_2)\mu(E)^{\lambda_3}|B|^{\lambda_1+\lambda_3(r-1)}. \end{aligned} \tag{3.12}$$

Hence,

$$\frac{|E|^{\lambda_3 r}}{|B|^{\lambda_1+\lambda_3(r-1)}} \le C(r,\lambda_1,\lambda_2,\lambda_3,w_1,w_2)\frac{\mu(E)^{\lambda_3}}{\mu(B)^{\lambda_1}}. \tag{3.13}$$

The desired result is obtained. □

If we choose $\lambda_1 = \lambda_2 = \lambda_3 = 1$ and $w_1 = w_2 = w$ in Theorem 3.3, we will obtain

$$\frac{|E|^r}{|B|^r} \le C(r,w)\frac{\mu(E)}{\mu(B)}, \tag{3.14}$$

which is called the strong doubling property of A_r weights; see [8].

4. The Weighted Inequality for Potential Operators

In this section, we are devoted to develop some two-weight norm inequalities for potential operator P to the versions of differential forms. We need the following lemmas.

Lemma 4.1 (see [9]). *If $w \in A_r(\Omega)$, then there exist constants $\beta > 1$ and C, independent of w, such that*

$$\|w\|_{\beta,B} \le C|B|^{(1-\beta)/\beta}\|w\|_{1,B}, \tag{4.1}$$

for all balls $B \subset \mathbf{R}^n$.

Lemma 4.2. *Let $0 < \alpha < \infty$, $0 < \beta < \infty$, and $s^{-1} = \alpha^{-1} + \beta^{-1}$. If f and g are measurable functions on $\mathbf{R}^n$, then*

$$\|fg\|_{s,E} \le \|f\|_{\alpha,E} \cdot \|g\|_{\beta,E}, \tag{4.2}$$

for any $E \subset \mathbf{R}^n$.

Lemma 4.3 (see [10]). *Let $\omega \in D'(E, \bigwedge^k)$, $k = 0, 1, \ldots, n$ be a solution of the nonhomogeneous A-harmonic equation in E, $\rho > 1$ and $0 < s, \ t < \infty$, then there exists a constant C, independent of ω, such that*

$$\|\omega\|_{s,B} \leq C|B|^{(t-s)/st}\|\omega\|_{t,\sigma Q}, \tag{4.3}$$

for all B with $\rho B \subset E$.

Theorem 4.4. *Let $u \in D'(E, \bigwedge^k, v)$, $k = 0, 1, 2, \ldots, n-1$, be a solution of the nonhomogeneous A-harmonic equation* (1.16) *in a bounded domain E and P is the potential operator with $k(x,y) = \varphi_\varepsilon(x-y)$ for any $\varepsilon > 0$, where the Radon measures μ and v are defined by $d\mu = w_1^{\alpha\lambda_1}(x)$, $dv = w_2^{\alpha\lambda_2\lambda_3/s}(x)$. Assume that $w_1^{\lambda_1}(x) \in A_r(\Omega)$ and $(w_1(x), w_2(x)) \in A_r^{\lambda_3}(\lambda_1, \lambda_2, \Omega)$ for some $r > 1$, $0 < \lambda_1, \lambda_2, \lambda_3 < \infty$ with $w_1(x) \geq \varepsilon > 0$ for any $x \in \Omega$, then there exists a constant C, independent of u, such that*

$$\|P(u)\|_{\star,E,w_1^{\alpha\lambda_1}} \leq C\|u\|_{1,\Omega,w_2^{\alpha\lambda_2\lambda_3/s}}, \tag{4.4}$$

where α is a constant with $0 < \alpha < 1$.

Proof. Since $w_1^{\lambda_1} \in A_r(\Omega)$, using Lemma 4.1, there exist constants $\beta > 1$ and $C_1 > 0$, such that

$$\left\|w_1^{\lambda_1}\right\|_{\beta,B} \leq C_1|B|^{(1-\beta)/\beta}\left\|w_1^{\lambda_1}\right\|_{1,B}, \tag{4.5}$$

for any ball $B \subset \mathbf{R}^n$.

Since $1 = 1/s + (s-1)/s$, by Lemma 4.2, we have

$$\begin{aligned}
\|P(u) - P(u)_B\|_{1,B,w_1^{\alpha\lambda_1}} &= \int_B |P(u) - P(u)_B| w_1^{\alpha\lambda_1} dx \\
&\leq \left(\int_B |P(u) - P(u)_B|^s w_1^{\alpha\lambda_1} dx\right)^{1/s}\left(\int_B w_1^{\alpha\lambda_1} dx\right)^{(s-1)/s} \\
&= \mu(B)^{(s-1)/s}\|P(u) - P(u)_B\|_{s,B,w_1^{\alpha\lambda_1}}.
\end{aligned} \tag{4.6}$$

Choose $t = s/(1 - \alpha/\beta)$ where $0 < \alpha < 1$, $\beta > 1$, then $1 < s < t$ and $\alpha t/(t-s) = \beta$. Since $1/s = 1/t + (t-s)/st$, by Lemma 4.2 and (4.5), we have

$$\begin{aligned}
\|P(u) - P(u)_B\|_{s,B,w_1^{\alpha\lambda_1}} &= \left(\int_B \left(|P(u) - P(u)_B| w_1^{\alpha\lambda_1/s}\right)^s dx\right)^{1/s} \\
&\leq \left(\int_B (|P(u) - P(u)_B|^t dx\right)^{1/t}\left(\int_B w_1^{\lambda_1\beta} dx\right)^{\alpha/(\beta s)} \\
&= \|P(u) - P(u)_B\|_{t,B} \cdot \left\|w_1^{\lambda_1}\right\|_{\beta,B}^{\alpha/s} \\
&\leq \|P(u) - P(u)_B\|_{t,B} \cdot C_2|B|^{(1-\beta)\alpha/(\beta s)}\left\|w_1^{\lambda_1}\right\|_{1,B}^{\alpha/s}.
\end{aligned} \tag{4.7}$$

From Lemma 2.1, we have

$$\|P(u)-(P(u))_B\|_{t,B} \le C_3|B|\operatorname{diam}(B)\|u\|_{t,B}. \tag{4.8}$$

Applying Lemma 4.3 (the weak reverse Hölder inequality for the solutions of the nonhomogeneous A-harmonic equation), we obtain

$$\|u\|_{t,B} \le C_4|B|^{(m-t)/mt}\|u\|_{m,\sigma_1 B}, \tag{4.9}$$

where σ_1 is a constant and $\sigma_1 B \subset \Omega$. Choosing $m = s/(\alpha\lambda_3(r-1)+s)$, then $m < 1 < s$. Using Hölder's inequality with $1/m = 1/1 + \alpha\lambda_3(r-1)/s$, we have

$$\begin{aligned}\|u\|_{m,\sigma_1 B} &= \left(\int_{\sigma_1 B}\left(|u|w_2^{\alpha\lambda_2\lambda_3/s}w_2^{-\alpha\lambda_2\lambda_3/s}\right)^m dx\right)^{1/m}\\ &\le \left(\int_{\sigma_1 B}|u|w_2^{\alpha\lambda_2\lambda_3/s}dx\right)\left(\int_{\sigma_1 B}\left(\frac{1}{w_2}\right)^{\lambda_2/(r-1)}dx\right)^{\alpha\lambda_3(r-1)/s}\\ &= \|u\|_{1,\sigma_1 B, w_2^{\alpha\lambda_2\lambda_3/s}}\left\|\left(\frac{1}{w_2}\right)^{\lambda_2}\right\|_{1/(r-1),\sigma_1 B}^{\alpha\lambda_3/s}.\end{aligned} \tag{4.10}$$

Since $(w_1, w_2) \in A_r^{\lambda_3}(\lambda_1, \lambda_2, \Omega)$, then

$$\begin{aligned}&\left\|w_1^{\lambda_1}\right\|_{1,B}^{\alpha/s}\cdot\left\|\left(\frac{1}{w_2}\right)^{\lambda_2}\right\|_{1/(r-1),\sigma_1 B}^{\alpha\lambda_3/s}\\ &\quad\le \left[\left(\int_{\sigma_1 B}w_1^{\lambda_1}dx\right)\left(\int_{\sigma_1 B}\left(\frac{1}{w_2}\right)^{\lambda_2/(r-1)}dx\right)^{\lambda_3(r-1)}\right]^{\alpha/s}\\ &\quad= \left[|\sigma_1 B|^{\lambda_3(r-1)+1}\left(\frac{1}{|\sigma_1 B|}\int_{\sigma_1 B}w_1^{\lambda_1}dx\right)\left(\frac{1}{|\sigma_1 B|}\int_{\sigma_1 B}\left(\frac{1}{w_2}\right)^{\lambda_2/(r-1)}dx\right)^{\lambda_3(r-1)}\right]^{\alpha/s}\\ &\quad\le C_5|\sigma_1 B|^{\alpha\lambda_3(r-1)/s+\alpha/s} \le C_6|B|^{\alpha\lambda_3(r-1)/s+\alpha/s}.\end{aligned} \tag{4.11}$$

Since $(m-t)/mt+\alpha\lambda_3(r-1)/s+\alpha/s+(s-1)/s+(1-\beta)\alpha/(\beta s) = 0$, combining with (4.6)–(4.11), we have

$$\begin{aligned}&\|P(u)-P(u)_B\|_{1,B,w_1^{\alpha\lambda_1}}\\ &\quad\le \mu(B)^{(s-1)/s}C_2|B|^{(1-\beta)\alpha/(\beta s)}C_3|B|\operatorname{diam}(B)C_4|B|^{(m-t)/mt}C_6|B|^{\alpha\lambda_3(r-1)/s+\alpha/s}\|u\|_{1,\sigma_1 B,w_2^{\alpha\lambda_2\lambda_3/s}}\\ &\quad\le C_7|B|\operatorname{diam}(B)\|u\|_{1,\sigma_1 B,w_2^{\alpha\lambda_2\lambda_3/s}}.\end{aligned} \tag{4.12}$$

From the definition of the BMO norm, we obtain

$$\begin{aligned}\|P(u)\|_{\star,E,w_1^{\alpha\lambda_1}} &= \sup_{\sigma_2 B\subset E}|B|^{-1}\|P(u)-(P(u))_B\|_{1,B,w_1^{\alpha\lambda_1}}\\ &\le \sup_{\sigma_2 B\subset E}|B|^{-1}C_7|B|\operatorname{diam}(B)\|u\|_{1,\sigma_1 B,w_2^{\alpha\lambda_2\lambda_3/s}}\\ &\le C_8\|u\|_{1,\sigma_1 B,w_2^{\alpha\lambda_2\lambda_3/s}},\end{aligned}\tag{4.13}$$

for all balls B with $\sigma_2 > \sigma_1$ and $\sigma_2 B \subset \Omega$. We have completed the proof of Theorem 4.4. □

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