Hindawi Publishing Corporation
Advances in Difference Equations
Volume 2011, Article ID 806458, 15 pages
doi:10.1155/2011/806458

Research Article

Multiple Periodic Solutions for Difference Equations with Double Resonance at Infinity

Xiaosheng Zhang[1] and Duo Wang[2, 3]

[1] *School of Mathematical Sciences, Capital Normal University, Beijing 100048, China*
[2] *School of Mathematical Sciences, Peking University, Beijing 100875, China*
[3] *School of Mathematical Sciences, South China Normal University, Guangzhou 510631, China*

Correspondence should be addressed to Xiaosheng Zhang, zhangxs007@gmail.com

Received 6 November 2010; Accepted 19 January 2011

Academic Editor: John Graef

By using variational methods and Morse theory, we study the multiplicity of the periodic solutions for a class of difference equations with double resonance at infinity. To the best of our knowledge, investigations on double-resonant difference systems have not been seen in the literature.

1. Introduction

Denote by $\mathbf{Z}$ the set of integers. For a given positive integer p, consider the following periodic problem on difference equation:

$$\begin{aligned} -\Delta^2 x(k-1) &= f(k, x(k)), \\ x(k+p) &= x(k), \end{aligned} \qquad k \in \mathbf{Z}, \tag{1.1}$$

where Δ is the forward difference operator defined by $\Delta x(k) = x(k+1) - x(k)$ and $\Delta^2 x(k) = \Delta(\Delta x(k))$ for $k \in \mathbf{Z}$. In this paper, we always assume that

(f_1) $f : \mathbf{Z} \times \mathbf{R} \to \mathbf{R}$ is C^1-differentiable with respect to the second variable and satisfies $f(k+p, t) = f(k, t)$ for $(k, t) \in \mathbf{Z} \times \mathbf{R}$ and $f(k, 0) \equiv 0$ for $k \in \mathbf{Z}$.

As a natural phenomenon, resonance may take place in the real world such as machinery, construction, electrical engineering, and communication. In a system described by a mathematical model, the feature of resonance lies in the interaction between the linear spectrum and the nonlinearity. It is known (see [1]) that the eigenvalue problem

$$-\Delta^2 x(k-1) = \lambda x(k), \quad x(k+p) = x(k), \quad k \in \mathbf{Z} \tag{1.2}$$

possess $p_1 + 1$ distinct eigenvalues $\lambda_l = 4\sin^2(l\pi/p), l = 0, 1, 2, \dots, p_1$, where $p_1 = [p/2]$, that is, the integer part of $p/2$.

For $a, b \in \mathbf{Z}$ with $a < b$, define $\mathbf{Z}(a, b) = \{a, a+1, \dots b\}$. Now, we suppose that

(f_2) $p > 3$, and there exists some $h \in \mathbf{Z}(0, p_1 - 1)$ such that

$$\lambda_h \leq \liminf_{|t|\to\infty} \frac{f(k,t)}{t} \leq \limsup_{|t|\to\infty} \frac{f(k,t)}{t} \leq \lambda_{h+1} \quad \text{for } k \in \mathbf{Z}(1, p). \tag{1.3}$$

Remark 1.1. The assumption (f_2) characterizes problem (1.1) as double resonant between two consecutive eigenvalues at infinity. Problem (1.1) is the discrete analogue of the differential equation with double resonance

$$\begin{gathered} -\ddot{z}(t) = g(t, z), \\ z(0) - z(2\pi) = \dot{z}(0) - \dot{z}(2\pi) = 0, \end{gathered} \tag{1.4}$$

whose solvability has been studied in [2], where $g : [0, 2\pi] \times \mathbf{R} \to \mathbf{R}$ is a differentiable function satisfying

$$h^2 \leq \liminf_{|z|\to\infty} \frac{g(t,z)}{z} \leq \limsup_{|z|\to\infty} \frac{g(t,z)}{z} \leq (h+1)^2, \tag{1.5}$$

for some $h \in \mathbf{N} \cup \{0\}$ and uniformly for a.e. $t \in [0, 2\pi]$.

Recently, many authors have studied the boundary value problems on nonlinear differential equations with double resonance(see [2–5]). It is well known that in different fields of research, such as computer science, mechanical engineering, control systems, artificial or biological neural networks, and economics, the mathematical modelling of important questions leads naturally to the consideration of nonlinear difference equations. For this reason, in recent years the solvability of nonlinear difference equations have been extensively investigated(see [1, 6–8] and the references cited therein). However, to the best of our knowledge, investigations on double resonant difference systems have not been seen in the literature.

In this paper, several theorems on the multiplicity of the periodic solutions to the double resonant system (1.1) are obtained via variational methods and Morse theory. The research here was mainly motivated by the works [2, 4].

We need the following assumptions (f_3) and (f_4):

(f_3) $p > 3$, and there exists some $h \in \mathbf{Z}(0, p_1 - 1)$ such that

$$\begin{aligned} &\text{(i)} \quad \liminf_{|t|\to\infty} |t|\left(\frac{f(k,t)}{t} - \lambda_h\right) > 0, \\ &\text{(ii)} \quad \limsup_{|t|\to\infty} |t|\left(\frac{f(k,t)}{t} - \lambda_{h+1}\right) < 0, \end{aligned} \quad \text{for } k \in \mathbf{Z}(1,p) \tag{1.6}$$

($f_4^{\pm}$) for some $m \in \mathbf{Z}(0, p_1)$,

$$\pm \int_0^t (f(k,s) - \lambda_m s)\,ds \geq 0 \quad \text{for } |t| > 0 \text{ small, } k \in \mathbf{Z}(1,p). \tag{1.7}$$

Remark 1.2. The assumption (f_3) implies (f_2) and will be employed to control the resonance at infinity. We will need (f_4) in the case that (1.1) is also resonant at the origin.

Now, the main results of this paper are stated as follows.

Theorem 1.3. *Assume that (f_1) and (f_3) hold. Then, problem* (1.1) *has at least two nontrivial p-periodic solutions in each of the following two cases:*

(i) $h \in \mathbf{Z}(1, p_1 - 1)$ *and* $f'(k,0) < \lambda_0$ *for* $k \in \mathbf{Z}(1,p)$,

(ii) $h \in \mathbf{Z}(0, p_1 - 2)$ *and* $f'(k,0) > \lambda_{p_1}$ *for* $k \in \mathbf{Z}(1,p)$.

Theorem 1.4. *Assume that (f_1) and (f_3) hold. If there exists $m \in \mathbf{Z}(0, p_1 - 1)$ with $m \neq h$ such that $\lambda_m < f'(k,0) < \lambda_{m+1}$ for $k \in \mathbf{Z}(1,p)$, then problem* (1.1) *has at least two nontrivial p-periodic solutions.*

Theorem 1.5. *Assume that (f_1) and (f_3) hold. If there exists $m \in \mathbf{Z}(0, p_1 - 1)$ such that $f'(k,0) \equiv \lambda_m$ for $k \in \mathbf{Z}(1,p)$. Then problem* (1.1) *has at least two nontrivial p-periodic solutions in each of the following two cases:*

(i) $h \in \mathbf{Z}(0, p_1 - 2)$ *and* (f_4^+) *with* $m \neq h$,

(ii) $h \in \mathbf{Z}(1, p_1 - 1)$ *and* (f_4^-) *with* $m \neq h + 1$.

In Section 3, we will prove the main results, before which some preliminary results on Morse theory will be collected in Section 2. Some fundamental facts relative to (1.1) revealed here will benefit the further investigations in this direction, which will be remarked in Section 4.

2. Preliminary Results on Critical Groups

In this section, we recall some basic facts in Morse theory which will be used in the proof of the main results. For the systematic discussion on Morse theory, we refer the reader to the monograph [9] and the references cited therein. Let H be a Hilbert space and $\Phi \in C^2(H, \mathbf{R})$

be a functional satisfying the compactness condition (PS), that is, every sequence $\{u_n\}$ such that $\{\Phi(u_n)\}$ is bounded and that $\Phi'(u_n) \to 0$ as $n \to \infty$ contains a convergent subsequence. Denote by $H_q(X,Y)$ the qth singular relative homology group of the topological pair (X,Y) with integer coefficients. Let u_0 be an isolated critical point of Φ with $\Phi(u_0) = c$, $c \in \mathbf{R}$, and U be a neighborhood of u_0. For $q \in \mathbf{N} \cup \{0\}$, the group

$$C_q(\Phi, u_0) : = H_q(\Phi^c \cap U, \Phi^c \cap U \setminus \{u_0\}) \tag{2.1}$$

is called the qth critical group of Φ at u_0, where $\Phi^c = \{u \in H : \ \Phi(u) \leq c\}$.

If the set of critical points of Φ, denoted by $\mathcal{K} := \{u \in H : \Phi'(u) = 0\}$, is finite and $a < \inf \Phi(\mathcal{K})$, the critical groups of Φ at infinity are defined by (see [10])

$$C_q(\Phi, \infty) := H_q(H, \Phi^a), \quad q \in \mathbf{N} \cup \{0\}. \tag{2.2}$$

For $q \in \mathbf{N} \cup \{0\}$, we call $\beta_q := \dim C_q(\Phi, \infty)$ the Betti numbers of Φ and define the Morse-type numbers of the pair (H, Φ^a) by

$$M_q := M_q(H, \Phi^a) = \sum_{u \in \mathcal{K}} \dim C_q(\Phi, u). \tag{2.3}$$

The following facts (2.a)–(2.g) are derived from [6, Chapter 8].

(2.a) If $C_\mu(\Phi, \infty) \ncong 0$ for some $\mu \in \mathbf{N} \cup \{0\}$, then there exists $x_0 \in \mathcal{K}$ such that $C_\mu(\Phi, x_0) \ncong 0$,

(2.b) If $\mathcal{K} = \{x_0\}$, then $C_q(\Phi, \infty) \cong C_q(\Phi, x_0)$,

(2.c) $\sum_{j=0}^{q} (-1)^{q-j} M_j \geq \sum_{j=0}^{q} (-1)^{q-j} \beta_j$ for $q \in \mathbf{N} \cup \{0\}$,

(2.d) $\sum_{j=0}^{\infty} (-1)^j M_j = \sum_{j=0}^{\infty} (-1)^j \beta_j$.

If $x_0 \in \mathcal{K}$ and $\Phi''(x_0)$ is a Fredholm operator and the Morse index μ_0 and nullity v_0 of x_0 are finite, then we have

(2.e) $\dim C_q(\Phi, x_0) \cong 0$ for $q \notin \mathbf{Z}(\mu_0, \mu_0 + \nu_0)$,

(2.f) If $C_{\mu_0}(\Phi, x_0) \ncong 0$ then $C_q(\Phi, x_0) \cong \delta_{q,\mu_0} \mathbf{Z}$ and if $C_{\mu_0+\nu_0}(\Phi, x_0) \ncong 0$ then $C_q(\Phi, x_0) \cong \delta_{q,\mu_0+\nu_0} \mathbf{Z}$,

(2.g) If $m := \dim H < +\infty$, then $C_q(\Phi, x_0) \cong \delta_{q,0} \mathbf{Z}$ when x_0 is local minimum of Φ, while $C_q(\Phi, x_0) \cong \delta_{q,m} \mathbf{Z}$ when x_0 is the local maximum of Φ.

We say that Φ has a local linking at $x_0 \in \mathcal{K}$ if there exist the direct sum decompositions $H = H^+ \oplus H^-$ and $\epsilon > 0$ such that

$$\begin{aligned} \Phi(x) &> \Phi(x_0) \quad \text{if } x - x_0 \in H^+, \quad 0 < \|x - x_0\| \leq \epsilon, \\ \Phi(x) &\leq \Phi(x_0) \quad \text{if } x - x_0 \in H^-, \quad \|x - x_0\| \leq \epsilon. \end{aligned} \tag{2.4}$$

The following results were due to Su [5].

(2.h) Assume that Φ has a local linking at $x_0 \in \mathcal{K}$ with respect to $H = H^+ \oplus H^-$ and $k = \dim H^- < +\infty$. Then,

$$C_q(\Phi, x_0) \cong \delta_{q,\mu_0}\mathbf{Z}, \quad \text{if } k = \mu_0,$$
$$C_q(\Phi, x_0) \cong \delta_{q,\mu_0+v_0}\mathbf{Z}, \quad \text{if } k = \mu_0 + v_0. \tag{2.5}$$

3. Proofs of Main Results

In this section, we will establish the variational structure relative to problem (1.1) and prove the main results via Morse theory.

Denote $X := \{x = \{x(k)\}_{k\in\mathbf{Z}} : x(k) \in \mathbf{R} \text{ for } k \in \mathbf{Z}\}$ and

$$E := \{x \in X : x(k+p) = x(k) \text{ for } k \in \mathbf{Z}\}. \tag{3.1}$$

Equipped with the inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$ as follows:

$$\langle x, y\rangle = \sum_{k=1}^{p} x(k)y(k), \quad \|x\| = \left(\sum_{k=1}^{p}|x(k)|^2\right)^{1/2}, \quad x, y \in E, \tag{3.2}$$

$(E, \langle\cdot,\cdot\rangle)$ is linearly homeomorphic to $\mathbf{R}^p$. Throughout this paper, we always identify $x \in E$ with $x = (x(1), x(2), \ldots, x(p))^T \in \mathbf{R}^p$.

Define the operator $-\widetilde{\Delta}^2 : E \mapsto E$ by $-\widetilde{\Delta}^2 x = \{-\Delta^2 x(k-1)\}, x \in E$ and denote $E^{\,l} = \ker(-\widetilde{\Delta}^2 - \lambda_l I)$, $l = 0, 1, \ldots, p_1$, where I is the identity operator. Set

$$E^- = \oplus_{l=0}^{h-1} E^l, \quad E^+ = \left(\oplus_{l=0}^{h+1} E^l\right)^{\perp}, \quad E^v = E^- \oplus E^+, \tag{3.3}$$

then E has the decomposition $E = E^h \oplus E^{h+1} \oplus E^v$. In the rest of this paper, the expression $x = x^h + x^{h+1} + x^v$ for $x \in E$ always means $x^\dagger \in E^\dagger$, $\dagger = h, h+1, v$.

Remark 3.1. From the discussion in [1, Section 2], we see that $\dim E^0 = 1$, $\dim E^l = 2$, for $l = 1, 2, \ldots, p_1 - 1$ and $\dim E^{p_1} = 1$ if p is even or $\dim E^{p_1} = 2$ if p is odd.

Define a family of functionals $J_s : \ E \to \mathbf{R}$, $s \in [0, 1]$ by

$$J_s(x) = -\frac{1}{2}\left\langle \widetilde{\Delta}^2 x, x\right\rangle - \frac{1-s}{4}(\lambda_h + \lambda_{h+1})\|x\|^2 - s\sum_{k=1}^{p} F(k, x(k)) \quad \text{for } x \in E, \tag{3.4}$$

where $F(k,t) = \int_0^t f(k,\xi)d\xi$, $(k,t) \in \mathbf{Z}(1,p) \times \mathbf{R}$. Then, the Fréchet derivative of J_s at $x \in E$, denoted by $J_s'(x)$, can be described as (see [1])

$$\left\langle J_s'(x), y\right\rangle = -\left\langle \widetilde{\Delta}^2 x, y\right\rangle - \sum_{k=1}^{p} g_s(k, x(k))y(\mathrm{k}) \quad \text{for } y \in E, \tag{3.5}$$

where $s \in [0,1]$ and

$$g_s(k,t) = sf(k,t) + \frac{1-s}{2}(\lambda_h + \lambda_{h+1})t \quad \text{for } (k,t) \in \mathbf{Z} \times \mathbf{R}. \tag{3.6}$$

Remark 3.2. From (3.5) with $s = 1$, we know by computation(or see [1]) that $x \in E$ is a critical point of J_1 if and only if $\{x(k)\}_{k\in\mathbf{Z}}$ is a p-periodic solution of problem (1.1). Moreover, J_1 is C^2- differentiable and

$$\langle J_1''(x)y, z\rangle = -\left\langle \widetilde{\Delta}^2 y, z\right\rangle - \sum_{k=1}^{p} f_t(k, x(k))y(k)z(k), \qquad \forall y, z \in E, \tag{3.7}$$

where $f_t(k,t)$ is the derivative of $f(k,t)$ with respect to t.

Let $\alpha_k \in \mathbf{R}$, $k \in \mathbf{Z}(1,p)$ and $\mathcal{E}_0$ consist of $w \in E$ satisfying

$$\left\langle -\widetilde{\Delta}^2 w, z\right\rangle = \sum_{k=1}^{p} \alpha_k w(k)z(k) \quad \text{for } z \in E. \tag{3.8}$$

Remark 3.3. $\mathcal{E}_0$ is the solution space of the system $Bx = 0$, $x \in E$, where

$$B = \begin{pmatrix} 2-\alpha_1 & -1 & 0 & \dots & 0 & 0 & -1 \\ -1 & 2-\alpha_2 & -1 & \dots & 0 & 0 & 0 \\ & \dots & \dots & \dots & & \dots & \\ 0 & 0 & 0 & \dots & -1 & 2-\alpha_{p-1} & -1 \\ -1 & 0 & 0 & \dots & 0 & -1 & 2-\alpha_p \end{pmatrix}_{p\times p}. \tag{3.9}$$

Thus, $\dim \mathcal{E}_0 \leq 2$ since B possesses of non-degenerate $(p-2)$ order submatrixes.

Lemma 3.4. *If $\lambda_h \leq \alpha_k \leq \lambda_{h+1}$, $k \in \mathbf{Z}(1,p)$ and $w = w^h + w^{h+1}$ satisfies* (3.8)*, where $w^h \in E^h$ and $w^{h+1} \in E^{h+1}$, then either $w^h = 0$ or $w^{h+1} = 0$.*

Proof. Setting $z = w^h$ and $z = w^{h+1}$, respectively, in (3.8), we have

$$\begin{aligned} \lambda_h \sum_{k=1}^{p} \left(w^h(k)\right)^2 &= \sum_{k=1}^{p} \alpha_k \left(w^h(k)\right)^2 + \sum_{k=1}^{p} \alpha_k w^{h+1}(k) w^h(k), \\ \lambda_{h+1} \sum_{k=1}^{p} \left(w^{h+1}(k)\right)^2 &= \sum_{k=1}^{p} \alpha_k \left(w^{h+1}(k)\right)^2 + \sum_{k=1}^{p} \alpha_k w^h(k) w^{h+1}(k). \end{aligned} \tag{3.10}$$

Comparing the above two equalities, we get

$$\sum_{k=1}^{p} (\lambda_{h+1} - \alpha_k)\left(w^{h+1}(k)\right)^2 = \sum_{k=1}^{p} (\lambda_h - \alpha_k)\left(w^h(k)\right)^2, \tag{3.11}$$

which, by $\alpha_k \in [\lambda_h, \lambda_{h+1}]$, $k \in \mathbf{Z}(1,p)$, implies that

$$(\alpha_k - \lambda_h) w^h(k) \equiv 0, \quad (\alpha_k - \lambda_{h+1}) w^{h+1}(k) \equiv 0 \quad \text{for } k \in \mathbf{Z}(1,p). \tag{3.12}$$

On the other hand, by the definition of w^h and w^{h+1}, we have

$$\begin{gathered} \lambda_\dagger w^\dagger(k) = 2w^\dagger(k) - w^\dagger(k-1) - w^\dagger(k+1), \\ w^\dagger(k+p) = w^\dagger(k) \quad \text{for } k \in \mathbf{Z}(1,p), \end{gathered} \tag{3.13}$$

where $\dagger = h, h+1$. There are two cases to be considered.

Case 1. $w^h(k) \neq 0$ for $k \in \mathbf{Z}(1,p)$. Then by (3.12), $\alpha_k = \lambda_h$ and $w^{h+1}(k) = 0$ for $k \in \mathbf{Z}(1,p)$, that is, $w^{h+1} = 0$.

Case 2. There exists $k^* \in \mathbf{Z}(1,p)$ such that $w^h(k^*) = 0$. By (3.13), we have

$$w^h(k^*+1) = -w^h(k^*-1). \tag{3.14}$$

If $w^h(k^*+1) = 0$, then $w^h(k-1) = 0$ which, by (3.13), implies that $w^h(k) = 0$ for $k \in \mathbf{Z}(1,p)$, that is, $w^h = 0$. If $w^h(k^*+1) \neq 0$, then $w^h(k^*-1) \neq 0$. This, by (3.12), implies $\alpha_{k^*-1} = \alpha_{k^*+1} = \lambda_h$ and $w^{h+1}(k^*-1) = w^{h+1}(k^*+1) = 0$. Thus, by (3.13), $w^{h+1}(k) = 0$ for $k \in \mathbf{Z}(1,p)$, that is $w^{h+1} = 0$. The proof is complete. □

Set $\gamma_1 = (\lambda_{h+1} - \lambda_h)/2$, $\gamma_2 = (\lambda_{h+1} + \lambda_h\ )/2$ and $A = -\widetilde{\Delta}^2 - \gamma_2 I$. The following Lemmas 3.5–3.7 benefit from [4].

Lemma 3.5. *Assume that* (f_1) *and* (f_2) *hold. Let* $\{s_n\} \subset [0,1]$ *and* $\{x_n\} \subset E$ *satisfy* $\|x_n\| \to \infty$ *and* $J'_{s_n}(x_n) \to 0$ *as* $n \to \infty$. *Then,*

$$\limsup_{n\to\infty} \|Ax_n\| \|x_n\|^{-1} \leq \gamma_1. \tag{3.15}$$

Proof. From (f_2), we have

$$\lambda_h \leq \liminf_{|t|\gamma\to\infty} \frac{g_s(k,t)}{t} \leq \limsup_{|t|\to\infty} \frac{g_s(k,t)}{t} \leq \lambda_{h+1} \quad \text{for } (k,t) \in \mathbf{Z}\times\mathbf{R}, \tag{3.16}$$

where the limitation is uniformly in $s \in [0,1]$. It follows that for any $\epsilon > 0$, there exists $R > 0$ such that

$$|g_s(k,t) - \gamma_2 t| \leq (\gamma_1 + \epsilon)|t| \quad \text{for } |t| > R,\ k \in \mathbf{Z}(1,p),\ s \in [0,1]. \tag{3.17}$$

Thus, there exists $\eta > 0$ such that

$$|g_s(k,t) - \gamma_2 t| \leq (\gamma_1 + \epsilon)(|t| + \eta) \quad \text{for } k \in \mathbf{Z}(1,p),\ s \in [0,1]. \tag{3.18}$$

By the assumption on $\{x_n\}$, we have $\langle J'_{s_n}(x_n), Ax_n/\|Ax_n\|\rangle \to 0$ as $n \to \infty$. It follows from (3.5) that

$$\|Ax_n\| - \|Ax_n\|^{-1}\sum_{k=1}^{p}\left(g_{s_n}(k, x_n(k)) - \gamma_2 x_n(k)\right)(Ax_n)(k) \longrightarrow 0 \quad \text{as } n \longrightarrow \infty, \tag{3.19}$$

which, combining with (3.18), implies that

$$\limsup_{n\to\infty}\left\{\|Ax_n\| - \frac{\gamma_1+\epsilon}{\|Ax_n\|}\left[\sum_{k=1}^{p}|x_n(k)(Ax_n)(k)| + \eta\sum_{k=1}^{p}|(Ax_n)(k)|\right]\right\} \le 0. \tag{3.20}$$

By using, Holder inequality on the above two summations, we get

$$\limsup_{n\to\infty}\left\{\|Ax_n\| - (\gamma_1+\epsilon)\|x_n\| - \eta\sqrt{p}\right\} \le 0, \tag{3.21}$$

which leads to

$$\limsup_{n\to\infty}\frac{\|Ax_n\|}{\|x_n\|} \le \gamma_1 + \epsilon. \tag{3.22}$$

Note that $\epsilon > 0$ is arbitrarily small, we get (3.15), and the proof is complete. □

Lemma 3.6. *Under the conditions of Lemma 3.5, one further has*

$$\frac{\|x_n^v\|}{\|x_n\|} \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty. \tag{3.23}$$

Proof. Since E^h, E^{h+1}, and E^v are invariant with respect to A, we have

$$\begin{aligned}\|Ax_n\|^2 &= \left\|Ax_n^h\right\|^2 + \left\|Ax_n^{h+1}\right\|^2 + \|Ax_n^v\|^2\\ &= \gamma_1^2\left\|x_n^h + x_n^{h+1}\right\|^2 + \|Ax_n^v\|^2.\end{aligned} \tag{3.24}$$

If, for the contradiction, (3.23) is false, then there is a subsequence of $\{x_n\}$, called $\{x_n\}$ again, and a number $\delta > 0$, such that $\|x_n^v\|/\|x_n\| \ge \delta$, $n = 1, 2, \ldots$. Then,

$$\begin{aligned}\gamma_1^{-2}\frac{\|Ax_n\|^1}{\|x_n\|^2} &= \frac{\left(\|x_n^h + x_n^{h+1}\|/\|x_n^v\|\right)^2 + \gamma_1^{-2}(\|Ax_n^v\|/\|x_n^v\|)^2}{\left(\left\|x_n^h + x_n^{h+1}\right\|/\|x_n^v\|\right)^2 + 1}\\ &\ge \frac{(\|x_n^h + x_n^{h+1}\|/\|x_n^v\|)^2 + (\theta/\gamma_1)^2}{\left(\left\|x_n^h + x_n^{h+1}\right\|/\|x_n^v\|\right)^2 + 1},\end{aligned} \tag{3.25}$$

where $\theta = \inf\{\|Ax^v\|/\|x^v\| : x^v \in E^v \setminus \{0\}\}$.

By the fact that $-\gamma_1$ and γ_1 are two consecutive eigenvalues of A with corresponding eigenspace E^h and E^{h+1}, we have $\theta/\gamma_1 > 1$ and then, the function $\phi(t) = (t + (\theta/\gamma_1)^2)/(t+1)$ is strictly decreasing on $(0,\infty)$ with $\phi(t) \to 1$ as $t \to +\infty$. Besides, $\|x_n^h + x_n^{h+1}\|/\|x_n^v\| \le \|x_n\|/\|x_n^v\| \le 1/\delta$. So, by (3.25),

$$\gamma_1^{-2}\frac{\|Ax_n\|^2}{\|x_n\|^2} \ge \phi\left(\frac{1}{\delta^2}\right) > 1. \tag{3.26}$$

This contradict to (3.15) and the proof is complete. □

Lemma 3.7. *Under the assumption of Lemma 3.5, there exists a subsequence of $\{x_n\}$, still called $\{x_n\}$, such that*

$$\text{either } \frac{\|x_n^h\|}{\|x_n\|} \longrightarrow 1 \text{ or } \frac{\|x_n^{h+1}\|}{\|x_n\|} \longrightarrow 1 \quad \text{as} \quad n \longrightarrow \infty. \tag{3.27}$$

Proof. Since $\|x_n\| \to \infty$ as $n \to \infty$, we can assume (by passing to a subsequence if necessary) that

$$\begin{gathered} \text{for some } K \subset \mathbf{Z}(1,p) \quad \text{with } K \neq \emptyset,\ \lim_{n\to\infty} x_n(k) = \infty \quad \text{for } k \in K, \\ \text{if } K^c \equiv \mathbf{Z}(1,p) \setminus K \neq \emptyset,\ \{x_n(k)\} \text{ is bounded for } k \in K^c. \end{gathered} \tag{3.28}$$

Thus, (3.16) implies

$$\lambda_h \le \liminf_{n\to\infty} \frac{g_{s_n}(k, x_n(k))}{x_n(k)} \le \limsup_{n\to\infty} \frac{g_{s_n}(k, x_n(k))}{x_n(k)} \le \lambda_{h+1} \quad \text{for } k \in K, \tag{3.29}$$

which implies that there exists a subsequence of $\{x_n\}$, still called $\{x_n\}$, and $\alpha_k \in [\lambda_h, \lambda_{h+1}]$, $k \in K$, such that

$$\lim_{n\to\infty} \frac{g_{s_n}(k, x_n(k))}{x_n(k)} = \alpha_k \quad \text{for } k \in K. \tag{3.30}$$

Let $w_n = x_n/\|x_n\|$, then $\|w_n\| = 1$, and, by Lemma 3.6, there is a convergent subsequence of $\{x_n\}$, call it $\{x_n\}$ again, such that

$$w_n \longrightarrow w \in E^h \oplus E^{h+1} \quad \text{as} \quad n \longrightarrow \infty. \tag{3.31}$$

To prove (3.27), we only need to show that $w^h = 0$ or $w^{h+1} = 0$. For every $y \in E$, we have $\langle J'_{s_n}(x_n), y\rangle/\|x_n\| \to 0$ as $n \to \infty$, that is,

$$\left\langle -\tilde{\Delta}^2 w_n, y \right\rangle - \sum_{k=1}^{p} \frac{g_{s_n}(k, x_n(k))}{\|x_n\|} y(k) \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty. \tag{3.32}$$

If $K^c \neq \emptyset$, $g_{s_n}(k, x_n(k))/\|x_n\| \to 0$ as $n \to \infty$ for $k \in K^c$, then we can rewrite (3.32) as

$$\left\langle -\widetilde{\Delta}^2 w_n, y \right\rangle - \sum_{k\in K} \frac{g_{s_n}(k, x_n(k))}{x_n(k)} \frac{x_n(k)}{\|x_n\|} y(k) \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{3.33}$$

Letting $n \to \infty$ in (3.33) and using (3.30) and (3.31), we get

$$\left\langle -\widetilde{\Delta}^2 w, y \right\rangle = \sum_{k\in K} \alpha_k w(k) y(k) \quad \text{for } y \in E. \tag{3.34}$$

Since $w(k) = 0$ for $k \in K^c$, by setting $\alpha_k = \lambda_h$ for $k \in K^c$, we rewrite (3.34) as

$$\left\langle -\widetilde{\Delta}^2 w, y \right\rangle = \sum_{k=1}^{p} \alpha_k w(k) y(k) \quad \text{for } y \in E. \tag{3.35}$$

Obviously, if $K^c = \emptyset$, (3.35) still holds. By Lemma 3.4, $w^h = 0$ or $w^{h+1} = 0$ and the proof is complete. □

Lemma 3.8. *Assume that (f_1) and (f_3) hold. Let $\{s_n\} \subset [0,1]$ and $\{x_n\} \subset E$ satisfy $\|x_n\| \to \infty$ and $J'_{s_n}(x_n) \to 0$ as $n \to \infty$. Then, there exists a subsequence of $\{x_n\}$, still called $\{x_n\}$, such that*

$$\begin{aligned} &\text{either} \quad \Gamma_1 := \limsup_{n\to\infty} \sum_{k=1}^{p} \{ g_{s_n}(k, x_n(k)) - \lambda_h x_n(k) \} \frac{x_n^h(k)}{\left\| x_n^h \right\|} > 0 \\ &\text{or} \quad \Gamma_2 := \liminf_{n\to\infty} \sum_{k=1}^{p} \{ g_{s_n}(k, x_n(k)) - \lambda_{h+1} x_n(k) \} \frac{x_n^{h+1}(k)}{\left\| x_n^{h+1} \right\|} < 0. \end{aligned} \tag{3.36}$$

Proof. As that in the above proof, we can assume that $\{x_n\}$ satisfies (3.28). Noticing that (f_3) implies (f_2) and by Lemma 3.7, we have two cases to be considered.

Case 1. $\|x_n^h\|/\|x_n\| \to 1$ as $n \to \infty$. We have $\|x_n^h\| \to \infty$ as $n \to \infty$ and

$$\lim_{n\to\infty} \frac{\left\| x_n^{h+1} \right\|}{\left\| x_n^h \right\|} = 0, \qquad \lim_{n\to\infty} \frac{\|x_n^v\|}{\left\| x_n^h \right\|} = 0. \tag{3.37}$$

If $K^c \neq \emptyset$, then $\{x_n(k)\}$ and $\{g_{s_n}(k, x_n(k)) - \lambda_h x_n(k)\}$ are bounded for $k \in K^c$ and $n \in \mathbf{N}$. It follows that $x_n(k)/\|x_n^h\| \to 0$ as $n \to \infty$ for $k \in K^c$ and

$$\lim_{n\to\infty} \{ g_{s_n}(k, x_n(k)) - \lambda_h x_n(k) \} \frac{x_n^h(k)}{\left\| x_n^h \right\|} = 0 \quad \text{for } k \in K^c. \tag{3.38}$$

By (f_3(i)), there exist $M > 0$ and $\xi > 0$ such that $|t|(f(k,t)/t - \lambda_h) > \xi$ and $|t|(\lambda_{h+1} - \lambda_h) > \xi$ for $|t| > M$ and $k \in \mathbf{Z}(1,p)$. Then, for $|t| > M$, $k \in \mathbf{Z}(1,p)$ and $s \in [0,1]$,

$$\begin{aligned}\left(\frac{g_s(k,t)}{t} - \lambda_h\right)|t| &= s\left(\frac{f(k,t)}{t} - \lambda_h\right)|t| + \frac{1-s}{2}(\lambda_{h+1} - \lambda_h)|t| \\ &\geq s\,\xi + \frac{1-s}{2}\xi \geq \frac{\xi}{2}.\end{aligned} \tag{3.39}$$

Choose $N > 0$ such that $|x_n(k)| > M$ for $k \in K$ and $n > N$. It follows that

$$\begin{aligned}&\left\{g_{s_n}(k, x_n(k)) - \lambda_h x_n(k)\right\} x_n^h(k) \\ &\quad= \left\{\frac{g_{s_n}(k, x_n(k))}{x_n(k)} - \lambda_h\right\} x_n(k)(x_n(k) - z_n(k)) \\ &\quad\geq \left\{\frac{g_{s_n}(k, x_n(k))}{x_n(k)} - \lambda_h\right\} |x_n(k)|(|x_n(k)| - |z_n(k)|) \\ &\quad\geq \frac{\xi}{2}(|x_n(k)| - |z_n(k)|) \quad \text{for } k \in K,\ n > N,\end{aligned} \tag{3.40}$$

where $z_n = x_n^{h+1} + x_n^v$. Since E is a finite dimensional vector space and possesses another norm defined by $\|x\|_1 \equiv \sum_{k=1}^p |x(k)|$, $x \in E$, which is equivalent to $\|\cdot\|$, there exists a positive constant $C > 0$ such that $\|x\|_1 \geqslant C\|x\|$, $x \in E$. Thus, by (3.37)–(3.40),

$$\begin{aligned}\Gamma_1 &\geq \limsup_{n\to\infty} \frac{\xi}{2\left\|x_n^h\right\|}\left\{\sum_{k\in K}|x_n(k)| - \sum_{k\in K}|z_n(k)|\right\} \\ &= \limsup_{n\to\infty} \frac{\xi}{2\left\|x_n^h\right\|}\left\{\sum_{k=1}^{p}|x_n(k)| - \sum_{k=1}^{p}|z_n(k)|\right\} \\ &\geq \limsup_{n\to\infty} \frac{\xi}{2\left\|x_n^h\right\|}\left(C\|x_n\| - \sqrt{p}\|z_n\|\right) = \frac{C\xi}{2}.\end{aligned} \tag{3.41}$$

Obviously, if $K^c = \emptyset$, the above inequality still holds.

Case 2. $\|x_n^{h+1}\|/\|x_n\| \to 1$ as $n \to \infty$. By using (f_3(ii)), we can show that $\Gamma_2 < 0$ in the same way. The proof is complete. □

In the rest of this section, we will use the facts (2.a)–(2.h) stated in Section 2 to complete the proofs.

Lemma 3.9. *Let f satisfy (f_1) and (f_3). Then, for every $\widehat{s} \in [0,1]$, $J_{\widehat{s}}$ satisfies the (PS) condition and*

$$C_q(J_1, \infty) \cong \delta_{q,\mu}\mathbf{Z}, \quad \mu = 2h + 1. \tag{3.42}$$

Proof. First we have the following claim:

Claim 1. For any sequences $\{x_n\} \subset E$ and $\{s_n\} \subset [0,1]$, if $J'_{s_n}(x_n) \to 0$ as $n \to \infty$, then $\{x_n\}$ is bounded.

In fact, if $\{x_n\}$ is unbounded, there exists a subsequence, still called $\{x_n\}$, such that $\|x_n\| \to \infty$ as $n \to \infty$. By Lemma 3.8, there exists a subsequence, still called $\{x_n\}$, such that $\Gamma_1 > 0$ or $\Gamma_2 < 0$.

On the other hand, $\langle J'_{s_n}(x_n), x_n^\dagger / \|x_n^\dagger\| \rangle \to 0$ as $n \to \infty$, $\dagger = h, h+1$, that is

$$\left\langle -\widetilde{\Delta}^2 x_n, \frac{x_n^\dagger}{\left\|x_n^\dagger\right\|} \right\rangle - \sum_{k=1}^{p} g_{s_n}(k, x_n(k)) \frac{x_n^\dagger(k)}{\left\|x_n^\dagger\right\|} \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty,\ \dagger = h, h+1. \tag{3.43}$$

Note that $\langle -\widetilde{\Delta}^2 x_n, x_n^\dagger \rangle = \langle \lambda_\dagger x_n, x_n^\dagger \rangle$, $\dagger = h,\ h+1$, it follows that $\Gamma_1 = \Gamma_2 = 0$. This contradiction proves Claim 1.

Setting $s_n \equiv \widehat{s}, n = 1, 2, \ldots$ in Claim 1, we see that $J_{\widehat{s}}$ satisfies (PS) condition. Now, we start to prove (3.42). Define a functional $I : E \mapsto \mathbf{R}$ as

$$I(x) = \frac{1}{4}(\lambda_{h+1} + \lambda_h)\|x\|^2 - \sum_{k=1}^{p} F(k, x(k)). \tag{3.44}$$

Claim 2. There exist $M > 0$ such that

$$\inf\{\left\|J'_s(x)\right\| : \|x\| > M, \quad s \in [0,1]\} > 0. \tag{3.45}$$

In fact, if Claim 2 is not true, there exists $\{x_n\} \subset E$ and $\{s_n\} \subset [0,1]$ such that $\|x_n\| \to \infty$ and $J_{s_n}(x_n) \to 0$ as $n \to \infty$, which contradict Claim 1.

Noticing that $a_0 := \inf\{J_s(x) : s \in [0,1], \|x\| \leq M\} > -\infty$, we set $a < a_0$. Then, $x \in J_0^a = \{x \in E : J_0 \leq a\}$ implies $\|x\| > M$. Consider the flow $\sigma : [0,1] \times E \to E$ generated by

$$\frac{d\sigma}{ds} = -\frac{I(\sigma)}{\|J'_s(\sigma)\|^2} J'_s(\sigma), \quad \sigma(0,x) = x,\ x \in J_0^a. \tag{3.46}$$

The chain rule for differentiation reads $dJ_s(\sigma)/ds = \langle J'_s(\sigma),\ d\sigma/ds \rangle + I(\sigma)$. Thus,

$$\frac{dJ_s(\sigma)}{ds} = -\left\langle \frac{I(\sigma)}{\|J'_s(\sigma)\|^2} J'_s(\sigma), J'_s(\sigma) \right\rangle + I(\sigma) = 0 \quad \text{for } s \in [0,1], \tag{3.47}$$

and $J_s(\sigma(s,x)) \equiv J_0(x) \leq a$, $s \in [0,1]$, which implies that $\|\sigma(s,x)\| > M$, $s \in [0,1]$. Then, the flow $\sigma(s,x)$ is well defined on J_0^a and $\sigma(1,\cdot)$ is a homeomorphism of J_0^a to J_1^a and (see [11])

$$H_q(E, J_0^a) \cong H_q(E, J_1^a). \tag{3.48}$$

On the other hand,

$$J_0(x) = \frac{1}{2}\left\langle -\widetilde{\Delta}^2 x, x \right\rangle - \frac{1}{4}(\lambda_h + \lambda_{h+1})\|x\|^2. \tag{3.49}$$

Note that $x = 0$ is the unique critical point of J_0 with Morse index $\mu := \dim(E^- \oplus E^h) = 2h+1$(see Remark 3.1) and nullity $\nu = 0$. Then, by (2.b), (2.f) and (3.48), we have

$$C_q(J_1, \infty) \cong C_q(J_0, \infty) \cong C_q(J_0, 0) \cong \delta_{q,\mu}\mathbf{Z}. \tag{3.50}$$

The proof is completed. □

Proof of Theorem 1.3. By lemma 3.9, we get (3.42) which, by (2.a), implies that there exists $x_1 \in \mathcal{K}$ with

$$C_\mu(J_1, x_1) \neq 0, \quad \mu = 2h + 1. \tag{3.51}$$

Since $0 \leq h < p_1$, we have $1 \leq \mu < p$. Denote by μ_1 and ν_1 the Morse index and nullity of x_1. By (2.e), we get $\mu_1 \leq \mu \leq \mu_1 + \nu_1$.

Denote $\alpha_k = f_t(k, x_1(k)), k \in \mathbf{Z}(1, p)$. Then, from (3.7) and Remark 3.3, we see that $\nu_1 = \dim \ker J_1''(x_1) \leq 2$.

In Case (i), $x = 0$ is a local minimum of J_1, hence, by (2.g),

$$C_q(J_1, 0) \cong \delta_{q,0}\mathbf{Z}, \tag{3.52}$$

which, by comparing with (3.51), implies that $x_1 \neq 0$. Besides, $3 \leq \mu < p$ since $h \in \mathbf{Z}(1, p_1 - 1)$. Assume, for the contradiction, that x_1 is the unique nontrivial critical point of J_1, then $\mathcal{K} = \{0, x_1\}$. If $\mu = \mu_1$ or $\mu = \mu_1 + \nu_1$, we have, by (2.f),

$$C_q(J_1, x_1) = \delta_{q,\mu_1}\mathbf{Z}, \text{ or } C_q(J_1, x_1) = \delta_{q,\mu_1+\nu_1}\mathbf{Z}, \tag{3.53}$$

from which, (2.d) reads $(-1)^0 + (-1)^\mu = (-1)^\mu$, a contradiction.

If $\mu_1 < \mu < \mu_1 + \nu_1$, then $\nu_1 = 2$ and $\mu = \mu_1 + 1$. Since $\mu \geq 3$, we have $\mu_1 \geq 2$. Thus, (2.c) with $q = 1$ reads $-1 \geq 0$, also a contradiction.

In Case (ii), $x = 0$ is a local maximum of J_1, hence, by (2.g),

$$C_q(J_1, 0) \cong \delta_{q,p}\mathbf{Z}, \tag{3.54}$$

which, by comparing with (3.51), implies that $x_1 \neq 0$. Besides, $1 \leq \mu < p-3$ since $h \in \mathbf{Z}(0, p_1-2)$. Assume, for the contradiction, that x_1 is the unique nontrivial critical point of J_1, then $\mathcal{K} = \{0, x_1\}$. If $\mu = \mu_1$ or $\mu = \mu_1 + \nu_1$, then (3.53) holds, from which, (2.d) reads

$$(-1)^p + (-1)^\mu = (-1)^\mu, \tag{3.55}$$

a contradiction. If $\mu_1 < \mu < \mu_1 + \nu_1$, then $\nu_1 = 2, \mu = \mu_1 + 1$ and, by (2.f),

$$C_{\mu_1}(J_1, x_1) \cong C_{\mu_1+\nu_1}(J_1, x_1) \cong 0. \tag{3.56}$$

Note that $\mu \leq p-3$, we have $\mu_1 + 1 < \mu_1 + 2 \leq p-2$. Thus, (2.c) with $q = (\mu_1 + 1)$ and with $q = \mu_1 + 2$ reads

$$\dim C_\mu(J_1, x_1) \geq 1, \quad \dim C_\mu(J_1, x_1) \leq 1, \tag{3.57}$$

respectively, which implies that $C_q(J_1, x_1) \cong \delta_{q,\mu}\mathbf{Z}$. Then, (2.d) reads (3.55), also a contradiction. The proof is complete. □

Proof of Theorem 1.4. As above, there exists $x_1 \in \mathcal{K}$ with the Morse index μ_1, and nullity v_1 satisfying $\mu_1 \leq \mu \leq \mu_1 + \nu_1, 0 \leq v_1 \leq 2$, and (3.51) holds.

On the other hand, $x = 0$ is a nondegenerate critical point of J_1 with Morse index, denoted by μ_0. Thus, $C_q(J_1, 0) \cong \delta_{q,\mu_0}\mathbf{Z}, \mu_0 = 2m + 1$ and $\mu_0 \neq \mu$ since $m \neq h$, which, by comparing with (3.51), implies that $x_1 \neq 0$.

Assume for the contradiction, that x_1 is unique nontrivial critical point of J_1, then $\mathcal{K} = \{0, x_1\}$. If $\mu = \mu_1$ or $\mu = \mu_1 + \nu_1$, then (3.53) holds and (2.d) reads the contradicition $(-1)^{\mu_0} + (-1)^\mu = (-1)^\mu$.

Now, we consider the case $\mu_1 < \mu < \mu_1 + \nu_1$ where we have $\mu = \mu_1 + 1$ and $\nu_1 = 2$ with (3.56). Since $m \neq h$, we know that either $\mu_0 < \mu - 1$ or $\mu_0 > \mu + 1$. If $\mu_0 < \mu - 1$, (2.c) with $q = \mu_0 + 1$ reads contradiction $-1 \geq 0$. If $\mu_0 > \mu + 1$, by similar argument, we can get (3.57). Thus $C_q(J_1, x_1) \cong \delta_{q,\mu}\mathbf{Z}$ and (2.d) reads the contradiction $(-1)^{\mu_0} + (-1)^\mu = (-1)^\mu$. The proof is complete. □

The proof of the following lemma is similar to that of ([12]) and is omitted.

Lemma 3.10. *Let f satisfy (f_4^+) (or (f_4^-)). Then J_1 has a local linking at $x = 0$ with respect to the decomposition $E = H^- \oplus E^+$, where $E^- := \oplus_{l \leq m} E^l$ (or $E^- := \oplus_{l<m} E^l$, respectively).*

Proof of Theorem 1.5. Now $f'(k, 0) = \lambda_m, k \in \mathbf{Z}(1, p)$. Thus, $x = 0$ is a degenerate critical point of J_1. Let μ_0 and ν_0 denote the Morse index and nullity of 0. By Lemma 3.10 and (2.h), we have

$$C_q(J_1, 0) \cong \delta_{q,r}\mathbf{Z}, \tag{3.58}$$

where $r = \mu_0$ or $\mu_0 + \nu_0$ corresponding to the case (f_4^-) or the case (f_4^+), respectively. The rest of the proof is similar and is omitted. The proof is complete. □

4. Conclusion and Future Directions

It is known that there have been many investigations on the solvability of elliptic equations with double-resonance via variational methods, where the so called unique continuation property of the Laplace operator, proved by Robinson [4], plays an important role in proving the compactness of the corresponding functional (see [2–5] and the references cited therein). In this paper, the solvability of the periodic problem on difference equations with double resonance is first studied and the "unique continuation property" of the second-order difference operator is derived by proving Lemma 3.4.

In addition, under the double resonance assumption (f_1) and (f_2), some fundamental facts relative to (1.1) are revealed in Lemmas 3.5–3.7, on which, further investigations, employing new restrictions different from (f_3) and (f_4), may be based.

On the observations as above, it is reasonable to believe that the research in this paper will benefit the future study in this direction.

Acknowledgments

The authors are grateful for the referee's careful reviewing and helpful comments. Also the authors would like to thank Professor Su Jiabao for his helpful suggestions. This work is supported by NSFC(10871005) and BJJW(KM200610028001).

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