Hindawi Publishing Corporation
Advances in Difference Equations
Volume 2011, Article ID 913403, 6 pages
doi:10.1155/2011/913403

Research Article

Notes on Interpolation Inequalities

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Received 3 October 2010; Accepted 16 November 2010

Academic Editor: Toka Diagana

An easy proof of the John-Nirenberg inequality is provided by merely using the Calderón-Zygmund decomposition. Moreover, an interpolation inequality is presented with the help of the John-Nirenberg inequality.

1. Introduction

It is well known that various interpolation inequalities play an important role in the study of operational equations, partial differential equations, and variation problems (see, e.g., [1–6]). So, it is an issue worthy of deep investigation.

Let Q_0 be either R^n or a fixed cube in R^n. For $f \in L^1_{\mathrm{loc}}(Q_0)$, write

$$\|f\|_{\mathrm{BMO}} := \sup_{Q \subset Q_0} \frac{1}{|Q|} \int_Q \left| f - f_Q \right| dx, \tag{1.1}$$

where the supremum is taken over all cubes $Q \subset Q_0$ and $f_Q := (1/|Q|) \int_Q f\,dx$.

Recall that $\mathrm{BMO}(Q_0)$ is the set consisting of all locally integrable functions on Q_0 such that $\|f\|_{\mathrm{BMO}} < \infty$, which is a Banach space endowed with the norm $\|\cdot\|_{\mathrm{BMO}}$. It is clear that any bounded function on Q_0 is in $\mathrm{BMO}(Q_0)$, but the converse is not true. On the other hand, the BMO space is regarded as a natural substitute for L^∞ in many studies. One of the important features of the space is the John-Nirenberg inequality. There are several versions of its proof; see, for example, [2, 7–9]. Stimulated by these works, we give, in this paper, an easy proof of the John-Nirenberg inequality by using the Calderón-Zygmund decomposition only. Moreover, with the help of this inequality, an interpolation inequality is showed for L^p and BMO norms.

2. Results and Proofs

Lemma 2.1 (John-Nirenberg inequality). *If $f \in BMO(Q_0)$, then there exist positive constants c_1, c_2 such that, for each cube $Q \subset Q_0$,*

$$\left|\left\{x \in Q : \left|f(x) - f_Q\right| > t\right\}\right| \leq c_1 \exp\left\{-\frac{c_2}{\|f\|_{BMO}} t\right\}|Q|, \quad t > 0. \tag{2.1}$$

Proof. Without loss of generality, we can and do assume that $\|f\|_{\mathrm{BMO}} = 1$.

For each $t > 0$, let $F(t)$ denote the least number for which we have

$$\left|\left\{x \in Q : \left|f(x) - f_Q\right| > t\right\}\right| \leq F(t)|Q|, \tag{2.2}$$

for any cube $Q \subset Q_0$. It is easy to see that $F(t) \leq 1(t > 0)$ and $F(t)$ is decreasing.

Fix a cube $Q \subset Q_0$. Applying the Calderón-Zygmund decomposition (cf., e.g., [2, 9]) to $|f(x) - f_Q|$ on Q, with 2^n as the separating number, we get a sequence of disjoint cubes $\{Q_j\}$ and E such that

$$Q = \left(\bigcup_j Q_j\right) \cup E, \tag{2.3}$$

$$\left|f(x) - f_Q\right| \leq 2^n, \quad \text{for a.e. } x \in E, \tag{2.4}$$

$$2^n < \frac{1}{\left|Q_j\right|} \int_{Q_j} \left|f - f_Q\right| dx \leq 4^n. \tag{2.5}$$

Using (2.5), we have

$$\sum_j \left|Q_j\right| < \frac{1}{2^n}|Q|. \tag{2.6}$$

From (2.3), (2.4), and (2.6), we deduce that for $t > 4^n$,

$$\begin{aligned}
\left|\left\{x \in Q : \left|f(x) - f_Q\right| > t\right\}\right| &= \left|\bigcup_j \left\{x \in Q_j : \left|f(x) - f_Q\right| > t\right\}\right| \\
&\leq \sum_j \left|\left\{x \in Q_j : \left|f(x) - f_Q\right| > t - 4^n\right\}\right| \\
&= \sum_j \left|Q_j\right| \frac{1}{\left|Q_j\right|} \left|\left\{x \in Q_j : \left|f(x) - f_Q\right| > t - 4^n\right\}\right| \\
&\leq \frac{1}{2^n} F\left(t - 4^n\right)|Q|.
\end{aligned} \tag{2.7}$$

This yields that

$$F(t) \leq \frac{1}{2^n} F(t - 4^n), \quad t > 4^n. \tag{2.8}$$

Let $\gamma = [(t-1)4^{-n}]$ $(t > 4^n)$, $\mu = 1 + \gamma 4^n$. Then $0 < \mu \leq t$. By iterating, we get

$$\begin{aligned} F(t) &\leq F(\mu) = F\left(1 + \gamma 4^n\right) \leq 2^{-n\gamma} \leq 2^{-n((t-1)4^{-n}-1)} \\ &= 2^{n(1+4^{-n})} \exp\left(-(\log 2) n 4^{-n} t\right), \quad t > 4^n. \end{aligned} \tag{2.9}$$

Thus, letting

$$c_1 = 2^{n(1+4^{-n})}, \qquad c_2 = (\log 2) n 4^{-n} \tag{2.10}$$

gives that

$$F(t) \leq c_1 e^{-c_2 t}, \quad t > 0, \tag{2.11}$$

since

$$F(t) \leq 1 \leq c_1 e^{-c_2 t}, \quad 0 < t \leq 4^n. \tag{2.12}$$

This completes the proof. □

Remark 2.2. (1) As we have seen, the recursive estimation (2.8) justifies the desired exponential decay of $F(t)$.

(2) There exists a gap in the proof of the John-Nirenberg inequality given in [2]. Actually, for a decreasing function $G(t) : (0, \infty) \to [0, 1]$, the following estimate:

$$G(2 \cdot 2^n \alpha) \leq \frac{1}{\alpha} G(2^n \alpha), \quad \alpha > 1 \tag{2.13}$$

does not generally imply such a property, that is, the existence of constants $c_1, c_2 > 0$ such that

$$G(t) \leq c_1 e^{-c_2 t}, \quad t > 0. \tag{2.14}$$

We present the following function as a counter example:

$$G(t) = \exp\left\{-\left(\log \frac{5}{3}\right)^{-1} \log^2(t+1)\right\}, \quad t > 0. \tag{2.15}$$

In fact, it is easy to see that there are no constants $c_1, c_2 > 0$ such that (2.14) holds. On the other hand, we have

$$\frac{G'(t)}{G(t)} = \left\{-\left(\log\frac{5}{3}\right)^{-1} 2\frac{\log(t+1)}{t+1}\right\}, \quad t > 0. \tag{2.16}$$

Integrating both sides of the above equation from $2^n\alpha$ to $2 \cdot 2^n\alpha$, we obtain

$$\begin{aligned}
G(2\cdot 2^n\alpha) &= \exp\left\{-2\left(\log\frac{5}{3}\right)^{-1}\int_{2^n\alpha}^{2\cdot 2^n\alpha}\frac{\log(t+1)}{t+1}dt\right\}G(2^n\alpha) \\
&= \exp\left\{-\left(\log\frac{5}{3}\right)^{-1}\left(\log^2(2\cdot 2^n\alpha+1)-\log^2(2^n\alpha+1)\right)\right\}G(2^n\alpha) \\
&= \exp\left\{-\left(\log\frac{5}{3}\right)^{-1}\log((2\cdot 2^n\alpha+1)(2^n\alpha+1))\cdot\log\left(\frac{2\cdot 2^n\alpha+1}{2^n\alpha+1}\right)\right\}G(2^n\alpha) \\
&\le \exp\{-\log((2\cdot 2^n\alpha+1)(2^n\alpha+1))\}G(2^n\alpha) \\
&= \frac{1}{(2\cdot 2^n\alpha+1)(2^n\alpha+1)}G(2^n\alpha) \\
&\le \frac{1}{\alpha}G(2^n\alpha),
\end{aligned} \tag{2.17}$$

where the fact that

$$\frac{2\cdot 2^n\alpha+1}{2^n\alpha+1} > \frac{5}{3} \quad (\alpha > 1) \tag{2.18}$$

is used to get the first inequality above. This means that

$$G(2\cdot 2^n\alpha) \le \frac{1}{\alpha}G(2^n\alpha), \quad \alpha > 1. \tag{2.19}$$

Next, we make use of the John-Nirenberg inequality to obtain an interpolation inequality for L^p and BMO norms.

Theorem 2.3. *Suppose that $1 \le p < r < \infty$ and $f \in L^p(Q_0) \cap BMO(Q_0)$. Then we have*

$$\|f\|_{L^r} \le (\text{const})\|f\|_{L^p}^{p/r}\|f\|_{BMO}^{1-p/r}. \tag{2.20}$$

Proof. If $\|f\|_{\mathrm{BMO}} = 0$, the proof is trivial; so we assume that $\|f\|_{\mathrm{BMO}} \neq 0$. In view of the Calderón-Zygmund decomposition theorem, there exists a sequence of disjoint cubes $\{Q_j\}$ and E such that

$$Q_0 = \left(\bigcup_j Q_j\right) \cup E, \tag{2.21}$$

$$|f(x)|^p \leq \|f\|_{\mathrm{BMO}}^p \quad \text{for a.e. } x \in E, \tag{2.22}$$

$$\|f\|_{\mathrm{BMO}}^p < \frac{1}{|Q_j|}\int_{Q_j} |f(x)|^p dx \leq 2^n \|f\|_{\mathrm{BMO}}^p. \tag{2.23}$$

From (2.23), we get

$$\begin{gathered}
\sum_j |Q_j| < \frac{1}{\|f\|_{\mathrm{BMO}}^p}\int_{Q_0} |f(x)|^p dx = \frac{\|f\|_{L_p}^p}{\|f\|_{\mathrm{BMO}}^p}, \\
|f|_{Q_j} = \frac{1}{|Q_j|}\int_{Q_j} |f(x)| dx \leq \left(\frac{1}{|Q_j|}\int_{Q_j} |f(x)|^p dx\right)^{1/p} \leq 2^{n/p}\|f\|_{\mathrm{BMO}}.
\end{gathered} \tag{2.24}$$

Using (2.21)–(2.24), together with Lemma 2.1, yields that, for $\lambda > 2^{n/p}\|f\|_{\mathrm{BMO}}$,

$$\begin{aligned}
|\{x \in Q_0 : |f(x)| > \lambda\}| &= \left|\bigcup_j \{x \in Q_j : |f(x)| > \lambda\}\right| \\
&\leq \sum_j \left|\left\{x \in Q_j : \left|f(x) - f_{Q_j}\right| > \lambda - \left|f_{Q_j}\right|\right\}\right| \\
&\leq \sum_j |Q_j| \frac{1}{|Q_j|}\left|\left\{x \in Q_j : \left|f(x) - f_{Q_j}\right| > \lambda - 2^{n/p}\|f\|_{\mathrm{BMO}}\right\}\right| \\
&\leq \sum_j c_1 \exp\left\{-\frac{c_2}{\|f\|_{\mathrm{BMO}}}\left(\lambda - 2^{n/p}\|f\|_{\mathrm{BMO}}\right)\right\}|Q_j| \\
&\leq c_1 \exp\left\{-\frac{c_2}{\|f\|_{\mathrm{BMO}}}\left(\lambda - 2^{n/p}\|f\|_{\mathrm{BMO}}\right)\right\}\frac{\|f\|_{L^p}^p}{\|f\|_{\mathrm{BMO}}^p}.
\end{aligned} \tag{2.25}$$

From (2.25), we obtain

$$
\begin{aligned}
\|f\|_{L^r}^r &= r\int_0^\infty \lambda^{r-1}\left|\left\{x\in Q_0 : |f(x)|>\lambda\right\}\right| d\lambda \\
&= r\int_0^{2^{n/p}\|f\|_{\mathrm{BMO}}} \lambda^{r-1}\left|\left\{x\in Q_0 : |f(x)|>\lambda\right\}\right| d\lambda \\
&\quad + r\int_{2^{n/p}\|f\|_{\mathrm{BMO}}}^\infty \lambda^{r-1}\left|\left\{x\in Q_0 : |f(x)|>\lambda\right\}\right| d\lambda \\
&\le r\int_0^{2^{n/p}\|f\|_{\mathrm{BMO}}} \lambda^{r-1}\frac{\|f\|_{L^p}^p}{\lambda^p} d\lambda \\
&\quad + r\int_{2^{n/p}\|f\|_{\mathrm{BMO}}}^\infty \lambda^{r-1} c_1 \exp\left\{-\frac{c_2}{\|f\|_{\mathrm{BMO}}}\left(\lambda - 2^{n/p}\|f\|_{\mathrm{BMO}}\right)\right\}\frac{f\|f\|_{L^p}^p}{\|f\|_{\mathrm{BMO}}^p} d\lambda \\
&= \frac{r}{r-p}2^{(n/p)(r-p)}\|f\|_{\mathrm{BMO}}^{r-p}\|f\|_{L^p}^p + \frac{rc_1}{c_2}2^{(n/p)(r-1)}\|f\|_{\mathrm{BMO}}^{r-p}\|f\|_{L^p}^p \\
&\le (\mathrm{const})\|f\|_{\mathrm{BMO}}^{r-p}\|f\|_{L^p}^p.
\end{aligned}
\tag{2.26}
$$

Hence, the proof is complete. □

Acknowledgments

The authors would like to thank the referees for helpful comments and suggestions. The work was supported partly by the NSF of China (11071042) and the Research Fund for Shanghai Key Laboratory for Contemporary Applied Mathematics (08DZ2271900).

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