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Research Article

Integral Equations and Exponential Trichotomy of Skew-Product Flows

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We are interested in an open problem concerning the integral characterizations of the uniform exponential trichotomy of skew-product flows. We introduce a new admissibility concept which relies on a double solvability of an associated integral equation and prove that this provides several interesting asymptotic properties. The main results will establish the connections between this new admissibility concept and the existence of the most general case of exponential trichotomy. We obtain for the first time necessary and sufficient characterizations for the uniform exponential trichotomy of skew-product flows in infinite-dimensional spaces, using integral equations. Our techniques also provide a nice link between the asymptotic methods in the theory of difference equations, the qualitative theory of dynamical systems in continuous time, and certain related control problems.

1. Introduction

Exponential trichotomy is the most complex asymptotic property of evolution equations, being firmly rooted in bifurcation theory of dynamical systems. The concept proceeds from the central manifold theorem and mainly relies on the decomposition of the state space into a direct sum of three invariant closed subspaces: the stable subspace, the unstable subspace, and the neutral subspace such that the behavior of the solution on the stable and unstable subspaces is described by exponential decay backward and forward in time and, respectively, the solution is bounded on the neutral subspace. The concept of exponential trichotomy for differential equations has the origin in the remarkable works of Elaydi and Hájek (see [1, 2]). Elaydi and Hájek introduced the concept of exponential trichotomy for linear and nonlinear differential systems and proved a number of notable properties in these cases (see [1, 2]). These works were the starting points for the development of this subject in various directions (see [3–8], and the references therein). In [5] the author gave necessary and sufficient conditions for exponential trichotomy of difference equations by examining the existence of a bounded solution of the corresponding inhomogeneous system. Paper [4] brings a valuable contribution to the study of the exponential trichotomy. In this paper Elaydi and Janglajew obtained the first input-output characterization for exponential trichotomy (see Theorem 4, page 423). More precisely, the authors proved that a system $x(n+1) = A(n)x(n)$ of difference equations with $A(n)$ a $k \times k$ invertible matrix on $\mathbb{Z}$, has an (E-H)-trichotomy if and only if the associated inhomogeneous system $y(n+1) = A(n)y(n) + b(n)$ has at least one bounded solution on $\mathbb{Z}$ for every bounded input b. In [4] the applicability area of exponential trichotomy was extended, by introducing new concepts of exponential dichotomy and exponential trichotomy. The authors proposed two different methods: in the first approach the authors used the tracking method and in the second approach they introduced a discrete analogue of dichotomy and trichotomy in variation.

A new step in the study of the exponential trichotomy of difference equations was made in [3], where Cuevas and Vidal obtained the structure of the range of each trichotomy projection associated with a system of difference equations which has weighted exponential trichotomy. This approach allows them to deduce the connections between weighted exponential trichotomy and the (h, k) trichotomy on $\mathbb{Z}_+$ and $\mathbb{Z}_-$ as well as to present some applications to the case of nonhomogeneous linear systems. In [8] the authors deduce the explicit formula in terms of the trichotomy projections for the solution of the nonlinear system associated with a system of difference equations which has weighted exponential trichotomy. The first study for exponential trichotomy of variational difference equations was presented in [6], the methods being provided directly for the infinite-dimensional case. There we obtained necessary and sufficient conditions for uniform exponential trichotomy of variational difference equations in terms of the solvability of an associated discrete-time control system.

Starting with the ideas delineated by the pioneering work of Perron (see [9]) and developed later in remarkable works by Coppel (see [10]), Daleckii and Krein (see [11]), Massera and Schäffer (see [12]) one of the most operational tool in the study of the asymptotic behavior of an evolution equation is represented by the input-output conditions. These methods arise from control theory and often provide characterizations of the asymptotic properties of dynamical systems in terms of the solvability of some associated control systems (see [4, 6, 13–21]). According to our knowledge, in the existent literature, there are no input-output integral characterizations for uniform exponential trichotomy of skew-product flows. Moreover, the territory of integral admissibility for exponential trichotomy of skew-product flows was not explored yet. These facts led to a collection of open questions concerning this topic and, respectively, concerning the operational connotations and consequences in the framework of general variational systems.

The aim of the present paper is to present for the first time a study of exponential trichotomy of skew-product flows from the new perspective of the integral admissibility. We treat the most general case of exponential trichotomy of skew-product flows (see Definition 2.4) which is a direct generalization of the exponential dichotomy (see [13, 14, 19–22]) and is tightly related to the behavior described by the central manifold theorem. Our methods will be based on the connections between the asymptotic properties of variational difference equations, the qualitative behavior of skew-product flows, and control type techniques, providing an interesting interference between the discrete-time and the continuous-time behavior of variational systems. We also emphasize that our central purpose is to deduce a characterization for uniform exponential trichotomy without assuming a priori the existence of the projection families, without supposing the invariance with respect to the projection families or the invertibility on the unstable subspace or on the bounded subspace.

We will introduce a new concept of admissibility which relies on a double solvability of an associated integral equation and on the uniform boundedness of the norm of solution relative to the norm of the input function. Using detailed and constructive methods we will prove that this assures the existence of the uniform exponential trichotomy (with all its properties), without any additional hypothesis on the skew-product flow. Moreover, we will show that the admissibility is also a necessary condition for uniform exponential trichotomy. Thus, we deduce the premiere characterization of the uniform exponential trichotomy of skew-product flows in terms of the solvability of an associated integral equation. The results are obtained in the most general case, being applicable to any class of variational equations described by skew-product flows.

2. Basic Definitions and Preliminaries

In this section, for the sake of clarity, we will give some basic definitions and notations and we will present some auxiliary results.

Let X be a real or a complex Banach space. The norm on X and on $\mathcal{L}(X)$, the Banach algebra of all bounded linear operators on X, will be denoted by $\|\cdot\|$. The identity operator on X will be denoted by I.

Throughout the paper $\mathbb{R}$ denotes the set of real numbers and $\mathbb{Z}$ denotes the set of real integers. If $J \in \{\mathbb{R}, \mathbb{Z}\}$ then we denote $J_+ = \{x \in J : x \geq 0\}$ and $J_- = \{x \in J : x \leq 0\}$.

Notations

(i) We consider the spaces $\ell^\infty(\mathbb{Z}, X) := \{s : \mathbb{Z} \to X : \sup_{k\in\mathbb{Z}}\|s(k)\| < \infty\}$, $\Gamma(\mathbb{Z}, X) := \{s \in \ell^\infty(\mathbb{Z}, X) : \lim_{k\to\infty} s(k) = 0\}$, $\Delta(\mathbb{Z}, X) := \{s \in \ell^\infty(\mathbb{Z}, X) : \lim_{k\to-\infty} s(k) = 0\}$ and $c_0(\mathbb{Z}, X) := \Gamma(\mathbb{Z}, X) \cap \Delta(\mathbb{Z}, X)$, which are Banach spaces with respect to the norm $\|s\|_\infty := \sup_{k\in\mathbb{Z}}\|s(k)\|$.

(ii) If $p \in [1, \infty)$ then $\ell^p(\mathbb{Z}, X) = \{s : \mathbb{Z} \to X : \sum_{k=-\infty}^{\infty} \|s(k)\|^p < \infty\}$ is a Banach space with respect to the norm $\|s\|_p := (\sum_{k=-\infty}^{\infty} \|s(k)\|^p)^{1/p}$.

(iii) Let $\mathcal{F}(\mathbb{Z}, X)$ be the linear space of all $s : \mathbb{Z} \to X$ with the property that $s(k) = 0$, for all $k \in \mathbb{Z} \setminus \mathbb{Z}_+$ and the set $\{k \in \mathbb{Z} : s(k) \neq 0\}$ is finite.

Let (Θ, d) be a metric space and let $\mathcal{E} = X \times \Theta$.

Definition 2.1. A continuous mapping $\sigma : \Theta \times \mathbb{R} \to \Theta$ is called a *flow* on Θ if $\sigma(\theta, 0) = \theta$ and $\sigma(\theta, s + t) = \sigma(\sigma(\theta, s), t)$, for all $(\theta, s, t) \in \Theta \times \mathbb{R}^2$.

Definition 2.2. A pair $\pi = (\Phi, \sigma)$ is called *(linear) skew-product flow* on $\mathcal{E}$ if σ is a flow on Θ and the mapping $\Phi : \Theta \times \mathbb{R}_+ \to \mathcal{L}(X)$, called *cocycle*, satisfies the following conditions:

(i) $\Phi(\theta, 0) = I$, for all $\theta \in \Theta$;

(ii) $\Phi(\theta, s + t) = \Phi(\sigma(\theta, s), t)\Phi(\theta, s)$, for all $(\theta, t, s) \in \Theta \times \mathbb{R}_+^2$ (*the cocycle identity*);

(iii) there are $M \geq 1$ and $\omega > 0$ such that $\|\Phi(\theta, t)\| \leq Me^{\omega t}$, for all $(\theta, t) \in \Theta \times \mathbb{R}_+$;

(iv) for every $x \in X$ the mapping $(\theta, t) \to \Phi(\theta, t)x$ is continuous.

Example 2.3. Let $a : \mathbb{R} \to \mathbb{R}_+$ be a continuous increasing function with $\lim_{t\to\infty} a(t) < \infty$ and let $a_s(t) = a(t+s)$. We denote by Θ the closure of $\{a_s : s \in \mathbb{R}\}$ in $(C(\mathbb{R},\mathbb{R}), d)$, where $C(\mathbb{R},\mathbb{R})$ denotes the space of all continuous functions $u : \mathbb{R} \to \mathbb{R}$ and

$$d(f,g) := \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{d_n(f,g)}{1 + d_n(f,g)}, \tag{2.1}$$

where $d_n(f,g) = \sup_{t\in[-n,n]} |f(t) - g(t)|$.

Let X be a Banach space and let $\{T(t)\}_{t\geq 0}$ be a C_0-semigroup on X with the infinitesimal generator $A : D(A) \subset X \to X$. For every $\theta \in \Theta$ let $A(\theta) := \theta(0)A$. We define $\sigma : \Theta \times \mathbb{R} \to \Theta$, $\sigma(\theta,t)(s) := \theta(t+s)$ and we consider the system

$$\begin{aligned} \dot{x}(t) &= A(\sigma(\theta,t))x(t), \quad t \geq 0, \\ x(0) &= x_0. \end{aligned} \tag{A}$$

If $\Phi : \Theta \times \mathbb{R}_+ \to \mathcal{L}(X)$, $\Phi(\theta,t)x = T(\int_0^t \theta(s)ds)x$, then $\pi = (\Phi,\sigma)$ is a skew-product flow on $\mathcal{E} = X \times \Theta$. For every $x_0 \in D(A)$, we note that $x(t) := \Phi(\theta,t)x_0$, for all $t \geq 0$, is the strong solution of the system (A).

For other examples which illustrate the modeling of solutions of variational equations by means of skew-product flows as well as the existence of the perturbed skew-product flow we refer to [21] (see Examples 2.2 and 2.4). Interesting examples of skew-product flows which often proceed from the linearization of nonlinear equations can be found in [7, 13, 14, 22, 23], motivating the usual appellation of *linear* skew-product flows.

The most complex description of the asymptotic property of a dynamical system is given by the exponential trichotomy, which provides a complete chart of the qualitative behaviors of the solutions on each fundamental manifold: the stable manifold, the central manifold, and the unstable manifold. This means that the state space is decomposed at every point of the flow's domain—the base space—into a direct sum of three invariant closed subspaces such that the solution on the first and on the third subspace exponentially decays forward and backward in time, while on the central subspace the solution had a uniform upper and lower bound (see [1–6, 8]).

Definition 2.4. A skew-product flow $\pi = (\Phi,\sigma)$ is said to be *uniformly exponentially trichotomic* if there are three families of projections $\{P_k(\theta)\}_{\theta\in\Theta} \subset \mathcal{L}(X)$, $k \in \{1,2,3\}$ and two constants $K \geq 1$ and $\nu > 0$ such that

(i) $P_k(\theta)P_j(\theta) = 0$, for all $k \neq j$ and all $\theta \in \Theta$,

(ii) $P_1(\theta) + P_2(\theta) + P_3(\theta) = I$, for all $\theta \in \Theta$,

(iii) $\sup_{\theta\in\Theta} \|P_k(\theta)\| < \infty$, for all $k \in \{1,2,3\}$,

(iv) $\Phi(\theta,t)P_k(\theta) = P_k(\sigma(\theta,t))\Phi(\theta,t)$, for all $(\theta,t) \in \Theta \times \mathbb{R}_+$ and all $k \in \{1,2,3\}$,

(v) $\|\Phi(\theta,t)x\| \leq Ke^{-\nu t}\|x\|$, for all $t \geq 0$, $x \in \operatorname{Im} P_1(\theta)$ and all $\theta \in \Theta$,

(vi) $(1/K)\|x\| \leq \|\Phi(\theta,t)x\| \leq K\|x\|$, for all $t \geq 0$, $x \in \operatorname{Im} P_2(\theta)$ and all $\theta \in \Theta$,

(vii) $\|\Phi(\theta,t)x\| \geq (1/K)e^{\nu t}\|x\|$, for all $t \geq 0$, $x \in \operatorname{Im} P_3(\theta)$ and all $\theta \in \Theta$,

(viii) the restriction $\Phi(\theta,t)_| : \operatorname{Im} P_k(\theta) \to \operatorname{Im} P_k(\sigma(\theta,t))$ is an isomorphism, for all $(\theta,t) \in \Theta \times \mathbb{R}_+$ and all $k \in \{2,3\}$.

Remark 2.5. We note that this is a direct generalization of the classical concept of uniform exponential dichotomy (see [13, 14, 19–22, 24, 25]) and expresses the behavior described by the central manifold theorem. It is easily seen that for $P_2(\theta) = 0$, for all $\theta \in \Theta$, one obtains the uniform exponential dichotomy concept and the condition (iii) is redundant (see, e.g., [19, Lemma 2.8]).

Remark 2.6. If a skew-product flow is uniformly exponentially trichotomic with respect to the families of projections $\{P_k(\theta)\}_{\theta\in\Theta}$, $k \in \{1, 2, 3\}$, then

(i) $\Phi(\theta, t)\,\mathrm{Im}\; P_1(\theta) \subset \mathrm{Im}\; P_1(\sigma(\theta, t))$, for all $(\theta, t) \in \Theta \times \mathbb{R}_+$;

(ii) $\Phi(\theta, t)\,\mathrm{Im}\; P_k(\theta) = \mathrm{Im}\; P_k(\sigma(\theta, t))$, for all $(\theta, t) \in \Theta \times \mathbb{R}_+$ and all $k \in \{2, 3\}$.

Let $\pi = (\Phi, \sigma)$ be a skew-product flow on $\mathcal{E}$. At every point $\theta \in \Theta$ we associate with π three fundamental subspaces, which will have a crucial role in the study of the uniform exponential trichotomy.

Notation

For every $\theta \in \Theta$ we denote by $\mathcal{Q}(\theta)$ the linear space of all functions $\varphi : \mathbb{R}_- \to X$ with

$$\varphi(t) = \Phi(\sigma(\theta, s), t - s)\varphi(s), \quad \forall s \le t \le 0. \tag{2.2}$$

For every $\theta \in \Theta$ we consider the linear space:

$$\mathcal{S}(\theta) = \left\{ x \in X : \lim_{t\to\infty} \Phi(\theta, t)x = 0 \right\} \tag{2.3}$$

called *the stable subspace*. We also define

$$\mathcal{B}(\theta) = \left\{ x \in X : \sup_{t\ge 0} \|\Phi(\theta, t)x\| < \infty \text{ and there is } \varphi \in \mathcal{Q}(\theta) \text{ with } \varphi(0) = x \text{ and } \sup_{t\le 0} \|\varphi(t)\| < \infty \right\} \tag{2.4}$$

called *the bounded subspace* and, respectively,

$$\mathcal{U}(\theta) = \left\{ x \in X : \text{there is } \varphi \in \mathcal{Q}(\theta) \text{ with } \varphi(0) = x \text{ and } \lim_{t\to -\infty} \varphi(t) = 0 \right\} \tag{2.5}$$

called *the unstable subspace*.

Lemma 2.7. *(i) If for every $\theta \in \Theta$, $\mathcal{V}(\theta)$ denotes one of the subspaces $\mathcal{S}(\theta)$, $\mathcal{B}(\theta)$ or $\mathcal{U}(\theta)$, then $\Phi(\theta, t)\mathcal{V}(\theta) \subset \mathcal{V}(\sigma(\theta, t))$, for all $(\theta, t) \in \Theta \times \mathbb{R}_+$.*

(ii) If the skew-product flow $\pi = (\Phi, \sigma)$ is uniformly exponentially trichotomic with respect to three families of projections $\{P_k(\theta)\}_{\theta\in\Theta} \subset \mathcal{L}(X)$, $k \in \{1,2,3\}$, then these families are uniquely determined by the conditions in Definition 2.4. Moreover one has that

$$\operatorname{Im} P_1(\theta) = \mathcal{S}(\theta), \quad \operatorname{Im} P_2(\theta) = \mathcal{B}(\theta), \quad \operatorname{Im} P_3(\theta) = \mathcal{U}(\theta), \quad \forall\, \theta \in \Theta. \tag{2.6}$$

Proof. See [6, Lemma 5.4, Proposition 5.5, and Remark 5.3]. □

We will start our investigation by recalling a recent result obtained for the discrete-time case. Precisely, the discrete case was treated in [6], where we formulated a first resolution concerning the characterization of the uniform exponential trichotomy in terms of the solvability of a system of variational difference equations. Indeed, we associated with the skew-product flow $\pi = (\Phi, \sigma)$ the discrete input-output system $(S_\pi) = (S_\theta^\pi)_{\theta\in\Theta}$, where for every $\theta \in \Theta$

$$\gamma(n+1) = \Phi(\sigma(\theta, n), 1)\gamma(n) + s(n+1), \quad \forall\, n \in \mathbb{Z}, \tag{S_θ^π}$$

with $\gamma \in \ell^\infty(\mathbb{Z}, X)$ and $s \in \mathcal{F}(\mathbb{Z}, X)$.

Definition 2.8. The pair $(\ell^\infty(\mathbb{Z}, X), \mathcal{F}(\mathbb{Z}, X))$ is said to be *uniformly admissible* for the skew-product flow $\pi = (\Phi, \sigma)$ if there are $p \in (1, \infty)$ and $L > 0$ such that for every $\theta \in \Theta$ the following properties hold:

(i) for every $s \in \mathcal{F}(\mathbb{Z}, X)$ there are $\gamma \in \Gamma(\mathbb{Z}, X)$ and $\delta \in \Delta(\mathbb{Z}, X)$ such that the pairs (γ, s) and (δ, s) satisfy (S_θ^π);

(ii) if $s \in \mathcal{F}(\mathbb{Z}, X)$ and $\gamma \in \Gamma(\mathbb{Z}, X) \cup \Delta(\mathbb{Z}, X)$ is such that the pair (γ, s) satisfies (S_θ^π) then $\|\gamma\|_\infty \le L \ \ \|s\|_1$;

(iii) if $s \in \mathcal{F}(\mathbb{Z}, X)$ is such that $s(n) \in \mathcal{S}(\sigma(\theta, n)) \cup \mathcal{U}(\sigma(\theta, n))$, for all $n \in \mathbb{Z}_+$, and $\gamma \in c_0(\mathbb{Z}, X)$ is such that the pair (γ, s) satisfies (S_θ^π), then $\|\gamma\|_\infty \le L\|s\|_p$.

The first connection between an input-output discrete admissibility and the uniform exponential trichotomy of skew-product flows was obtained in [6] (see Theorem 5.8) and this is given by what follows.

Theorem 2.9. *Let $\pi = (\Phi, \sigma)$ be a skew-product flow on $\mathcal{E}$. π is uniformly exponentially trichotomic if and only if the pair $(\ell^\infty(\mathbb{Z}, X), \mathcal{F}(\mathbb{Z}, X))$ is uniformly admissible for π.*

The proof of this result relies completely on discrete-time arguments and essentially uses the properties of the associated system of variational difference equations. The natural question is whether we may study the uniform exponential trichotomy property of skew-product flows from a "continuous" point of view. On the other hand, in the spirit of the classical admissibility theory (see [10–12, 14, 15, 21]) it would be interesting to see if the uniform exponential trichotomy can be expressed in terms of the solvability of an integral equation. The aim of the next section will be to give a complete resolution to these questions. Thus, we are interested in solving for the first time the problem of characterizing the exponential trichotomy of skew-product flows in terms of the solvability of an integral equation and also in establishing the connections between the qualitative theory of difference equations and the continuous-time behavior of dynamical systems, pointing out how the discrete-time arguments provide interesting information in control problems related with the existence of the exponential trichotomy.

3. Main Results

Let X be a real or a complex Banach space. In this section, we will present a complete study concerning the characterization of uniform exponential trichotomy using a special solvability of an associated integral equation. We introduce a new and natural admissibility concept and we show that the trichotomic behavior of skew-product flows can be studied in the most general case, without any additional assumptions.

Notations

Let $C_b(\mathbb{R}, X) = \{f : \mathbb{R} \to X \mid f \textit{ continuous and bounded}\}$, which is a Banach space with respect to the norm $|||f||| := \sup_{t\in\mathbb{R}}\|f(t)\|$. We consider the spaces $\mathcal{L}(\mathbb{R}, X) := \{f \in C_b(\mathbb{R}, X) \mid \lim_{t\to\infty} f(t) = 0\}$, $\mathfrak{D}(\mathbb{R}, X) := \{f \in C_b(\mathbb{R}, X) \mid \lim_{t\to-\infty} f(t) = 0\}$ and let $C_0(\mathbb{R}, X) = \mathcal{L}(\mathbb{R}, X) \cap \mathfrak{D}(\mathbb{R}, X)$. Then $\mathcal{L}(\mathbb{R}, X), \mathfrak{D}(\mathbb{R}, X)$ and $C_0(\mathbb{R}, X)$ are closed linear subspaces of $C_b(\mathbb{R}, X)$. Let $\mathcal{C}(\mathbb{R}, X)$ be the space of all continuous functions $f : \mathbb{R} \to X$ with compact support and $\operatorname{supp} f \subset (0,\infty)$.

Let $p \in [1,\infty)$ and let $L^p(\mathbb{R}, X)$ be the linear space of all Bochner measurable functions $f : \mathbb{R} \to X$ with the property that $\int_{\mathbb{R}} \|f(s)\|^p \, ds < \infty$, which is a Banach space with respect to the norm

$$\|f\|_p := \left(\int_{\mathbb{R}} \|f(s)\|^p ds\right)^{1/p}. \tag{3.1}$$

Let (Θ, d) be a metric space and let $\pi = (\Phi, \sigma)$ be a skew-product flow. For every $\theta \in \Theta$ we consider the integral equation

$$f(t) = \Phi(\sigma(\theta, r), t - r)f(r) + \int_r^t \Phi(\sigma(\theta, \tau), t - \tau)u(\tau)d\tau, \quad \forall t \geq r, \tag{E_θ^π}$$

with $f : \mathbb{R} \to X$ and $u \in \mathcal{C}(\mathbb{R}, X)$.

Definition 3.1. The pair $(C_b(\mathbb{R}, X), \mathcal{C}(\mathbb{R}, X))$ is said to be *uniformly admissible* for the skew-product flow $\pi = (\Phi, \sigma)$ if there are $p \in (1,\infty)$ and $Q > 0$ such that for every $\theta \in \Theta$ the following properties hold:

(i) for every $u \in \mathcal{C}(\mathbb{R}, X)$ there are $f \in \mathcal{L}(\mathbb{R}, X)$ and $g \in \mathfrak{D}(\mathbb{R}, X)$ such that the pairs (f, u) and (g, u) satisfy (E_θ^π);

(ii) if $u \in \mathcal{C}(\mathbb{R}, X)$ and $f \in \mathcal{L}(\mathbb{R}, X) \cup \mathfrak{D}(\mathbb{R}, X)$ are such that the pair (f, u) satisfies (E_θ^π), then $|||f||| \leq Q \max\{\|u\|_1, \|u\|_p\}$;

(iii) if $u \in \mathcal{C}(\mathbb{R}, X)$ is such that $u(t) \in \mathcal{S}(\sigma(\theta, t)) \cup \mathcal{U}(\sigma(\theta, t))$, for all $t \geq 0$ and $f \in C_0(\mathbb{R}, X)$ has the property that the pair (f, u) satisfies (E_θ^π), then $|||f||| \leq Q\|u\|_p$.

Remark 3.2. In the above admissibility concept, the input space is a minimal one, because all the test functions u belong to the space $\mathcal{C}(\mathbb{R}, X)$.

In what follows we will establish the connections between the admissibility and the existence of uniform exponential trichotomy.

The first main result of this paper is as follows.

Theorem 3.3. *Let $\pi = (\Phi, \sigma)$ be a skew-product flow on $\mathcal{E}$. If the pair $(C_b(\mathbb{R}, X), \mathcal{C}(\mathbb{R}, X))$ is uniformly admissible for π, then π is uniformly exponentially trichotomic.*

Proof. We prove that the pair $(\ell^\infty(\mathbb{Z}, X), \mathcal{F}(\mathbb{Z}, X))$ is uniformly admissible for π.

Indeed, let $p \in (1, \infty)$ and $Q > 0$ be given by Definition 3.1. We consider a continuous function $\alpha : \mathbb{R} \to [0, 2]$ with the support contained in $(0, 1)$ and

$$\int_0^1 \alpha(\tau)\, d\tau = 1. \tag{3.2}$$

Let $M, \omega > 0$ be such that $\|\Phi(\theta, t)\| \leq Me^{\omega t}$, for all $(\theta, t) \in \Theta \times \mathbb{R}_+$. Since $\ell^1(\mathbb{Z}, X) \subset \ell^p(\mathbb{Z}, X) \subset \ell^\infty(\mathbb{Z}, X)$ there is $\lambda \geq 1$ such that

$$\|s\|_p \leq \lambda \|s\|_1, \quad \forall s \in \ell^1(\mathbb{Z}, X), \tag{3.3}$$

$$\|s\|_\infty \leq \lambda \|s\|_1, \quad \forall s \in \ell^1(\mathbb{Z}, X), \tag{3.4}$$

$$\|s\|_\infty \leq \lambda \|s\|_p, \quad \forall s \in \ell^p(\mathbb{Z}, X). \tag{3.5}$$

Let $\theta \in \Theta$.

Step 1. Let $s \in \mathcal{F}(\mathbb{Z}, X)$. We consider the function

$$u : \mathbb{R} \to X, \qquad u(t) = \alpha(t - [t])\Phi(\sigma(\theta, [t]), t - [t])s([t]). \tag{3.6}$$

Then u is continuous and

$$\|u(t)\| \leq \alpha(t - [t])Me^{\omega}\|s([t])\|, \quad \forall t \in \mathbb{R}. \tag{3.7}$$

Since $s \in \mathcal{F}(\mathbb{Z}, X)$ there is $n \in \mathbb{Z}_+$ such that $\{k \in \mathbb{Z} : s(k) \neq 0\} \subset \{0, \ldots, n\}$. Then, from (3.7) it follows that supp $u \subset (0, n+1)$, so $u \in \mathcal{C}(\mathbb{R}, X)$. According to our hypothesis it follows that there are $f \in \mathcal{L}(\mathbb{R}, X)$ and $g \in \mathfrak{D}(\mathbb{R}, X)$ such that the pairs (f, u) and (g, u) satisfy (E_θ^π).

Then, for every $n \in \mathbb{Z}$ we obtain that

$$\begin{aligned} f(n+1) &= \Phi(\sigma(\theta, n), 1)f(n) + \int_n^{n+1} \Phi(\sigma(\theta, \tau), n+1-\tau)u(\tau)d\tau \\ &= \Phi(\sigma(\theta, n), 1)f(n) + \Phi(\sigma(\theta, n), 1)s(n). \end{aligned} \tag{3.8}$$

Let

$$\gamma : \mathbb{Z} \longrightarrow X, \qquad \gamma(n) = f(n) + s(n). \tag{3.9}$$

From (3.8) we have that

$$\gamma(n+1) = \Phi(\sigma(\theta,n),1)\gamma(n) + s(n+1), \quad \forall n \in \mathbb{Z} \tag{3.10}$$

so the pair (γ, s) satisfies (S_θ^π). Moreover, since $f \in \mathcal{L}(\mathbb{R}, X)$ and $s \in \mathcal{F}(\mathbb{Z}, X)$, we deduce that $\gamma \in \Gamma(\mathbb{Z}, X)$. Since the pair (g, u) satisfies (E_θ^π) we obtain that

$$g(n+1) = \Phi(\sigma(\theta,n),1)g(n) + \Phi(\sigma(\theta,n),1)s(n), \quad \forall n \in \mathbb{Z}. \tag{3.11}$$

Taking

$$\delta : \mathbb{Z} \longrightarrow X, \qquad \delta(n) = g(n) + s(n) \tag{3.12}$$

we analogously obtain that $\delta \in \Delta(\mathbb{Z}, X)$ and the pair (δ, s) satisfies (S_θ^π).

Step 2. Let $s \in \mathcal{F}(\mathbb{R}, X)$ and let $\gamma \in \Gamma(\mathbb{Z}, X) \cup \Delta(\mathbb{Z}, X)$ be such that the pair (γ, s) satisfies (S_θ^π). We consider the functions $u, f : \mathbb{R} \to X$, given by

$$\begin{aligned} u(t) &= \alpha(t-[t])\Phi(\sigma(\theta,[t]),t-[t])s([t]), \\ f(t) &= \Phi(\sigma(\theta,[t]),t-[t])\gamma([t]) - \left(\int_t^{[t]+1} \alpha(\tau-[\tau])d\tau\right)\Phi(\sigma(\theta,[t]),t-[t])s([t]). \end{aligned} \tag{3.13}$$

Since $s \in \mathcal{F}(\mathbb{Z}, X)$ we have that $u \in \mathcal{C}(\mathbb{R}, X)$. Observing that for every $n \in \mathbb{Z}$

$$\begin{aligned} \lim_{t \nearrow n+1} f(t) &= \Phi(\sigma(\theta,n),1)\gamma(n), \\ f(n+1) &= \gamma(n+1) - s(n+1), \end{aligned} \tag{3.14}$$

we deduce that f is continuous. Moreover, since $s \in \mathcal{F}(\mathbb{Z}, X)$ and $\gamma \in \Gamma(\mathbb{Z}, X) \cup \Delta(\mathbb{Z}, X)$, from

$$\|f(t)\| \leq Me^{\omega}\left(\|\gamma([t])\| + \|s([t])\|\right), \quad \forall t \in \mathbb{R} \tag{3.15}$$

we obtain that $f \in \mathcal{L}(\mathbb{R}, X) \cup \mathfrak{D}(\mathbb{R}, X)$.

Let $r \in \mathbb{R}$. We prove that

$$f(t) = \Phi(\sigma(\theta,r),t-r)f(r) + \int_r^t \Phi(\sigma(\theta,\tau),t-\tau)u(\tau)d\tau, \quad \forall t \geq r. \tag{3.16}$$

We set $n = [r]$. If $[t] = n$, then, taking into account the way how f was defined, the relation (3.16) obviously holds. If $[t] \geq n+1$ then there is $k \in \mathbb{Z}$, $k \geq 1$ such that $[t] = n+k$. Then, we deduce that

$$\begin{aligned}\Phi(\sigma(\theta,r),t-r)f(r) &+ \int_r^t \Phi(\sigma(\theta,\tau),t-\tau)u(\tau)d\tau \\ &= \Phi(\sigma(\theta,n),t-n)\gamma(n) + \int_{n+1}^t \Phi(\sigma(\theta,\tau),t-\tau)u(\tau)d\tau.\end{aligned} \tag{3.17}$$

If $k = 1$ then $[t] = n+1$ and

$$\begin{aligned}\int_{n+1}^t \Phi(\sigma(\theta,\tau),t-\tau)u(\tau)d\tau &= \left(\int_{n+1}^t \alpha(\tau-[\tau])d\tau\right)\Phi(\sigma(\theta,[t]),t-[t])s([t]) \\ &= \Phi(\sigma(\theta,n+1),t-n-1)s(n+1) \\ &\quad - \left(\int_{\mathrm{t}}^{[\mathrm{t}]+1} \alpha(\tau-[\tau])d\tau\right)\Phi(\sigma(\theta,[t]),t-[t])s([t]).\end{aligned} \tag{3.18}$$

If $k \geq 2$ then

$$\begin{aligned}\int_{n+1}^t \Phi(\sigma(\theta,\tau),t-\tau)u(\tau)d\tau &= \sum_{j=1}^{k-1}\int_{n+j}^{n+j+1} \Phi(\sigma(\theta,\tau),t-\tau)u(\tau)d\tau \\ &\quad + \int_{n+k}^t \Phi(\sigma(\theta,\tau),t-\tau)u(\tau)d\tau \\ &= \sum_{j=1}^{k-1}\Phi(\sigma(\theta,n+j),t-n-j)s(n+j) \\ &\quad + \left(\int_{n+k}^t \alpha(\tau-[\tau])d\tau\right)\Phi(\sigma(\theta,n+k),t-n-k)s(n+k) \\ &= \sum_{j=1}^{k}\Phi(\sigma(\theta,n+j),t-n-j)s(n+j) \\ &\quad - \left(\int_t^{[t]+1} \alpha(\tau-[\tau])d\tau\right)\Phi(\sigma(\theta,[t]),t-[t])s([t]).\end{aligned} \tag{3.19}$$

From relations (3.18) and (3.19) we have that

$$\int_{n+1}^{t}\Phi(\sigma(\theta,\tau),t-\tau)u(\tau)d\tau=\sum_{j=1}^{k}\Phi(\sigma(\theta,n+j),t-n-j)s(n+j)$$
$$-\left(\int_{t}^{[t]+1}\alpha(\tau-[\tau])d\tau\right)\Phi(\sigma(\theta,[t]),t-[t])s([t]). \tag{3.20}$$

Then, from (3.17) and (3.20) it follows that

$$\begin{aligned}
&f(t)-\Phi(\sigma(\theta,r),t-r)f(r)-\int_{r}^{t}\Phi(\sigma(\theta,\tau),t-\tau)u(\tau)d\tau\\
&\quad=\Phi(\sigma(\theta,n+k),t-n-k)\gamma(n+k)-\Phi(\sigma(\theta,n),t-n)\gamma(n)\\
&\qquad-\sum_{j=1}^{k}\Phi(\sigma(\theta,n+j),t-n-j)s(n+j)\\
&\quad=\Phi(\sigma(\theta,n+k),t-n-k)\left[\gamma(n+k)-\Phi(\sigma(\theta,n),k)\gamma(n)-\sum_{j=1}^{k}\Phi(\sigma(\theta,n+j),k-j)s(n+j)\right].
\end{aligned} \tag{3.21}$$

Since the pair (γ,s) satisfies (S_{θ}^{π}) we have that

$$\gamma(n+k)=\Phi(\sigma(\theta,n),k)\gamma(n)+\sum_{j=1}^{k}\Phi(\sigma(\theta,n+j),k-j)s(n+j). \tag{3.22}$$

From (3.21) and (3.22) we obtain that the relation (3.16) holds for all $t\geq r$. Since $r\in\mathbb{R}$ was arbitrary we deduce that the pair (f,u) satisfies (E_{θ}^{π}). Then according to our hypothesis we have that

$$|||f|||\leq Q\max\left\{\|u\|_1,\|u\|_p\right\}. \tag{3.23}$$

In addition, we have that

$$\|u\|_1\leq\sum_{k\in\mathbb{Z}}\int_{k}^{k+1}\alpha(t-[t])\|\Phi(\sigma(\theta,[t]),t-[t])\|\|s([t])\|dt\leq Me^{\omega}\|s\|_1 \tag{3.24}$$

and that

$$\|u\|_p\leq 2Me^{\omega}\|s\|_p. \tag{3.25}$$

Taking $\tilde{Q} := 2\lambda Q M e^{\omega}$, from relations (3.23)–(3.25) and (3.3) it follows that

$$|||f||| \le \tilde{Q}\|s\|_1. \tag{3.26}$$

Observing that $f(n) = \gamma(n) - s(n)$, for all $n \in \mathbb{N}$, from (3.26) and (3.4) we successively deduce that

$$\|\gamma\|_\infty \le |||f||| + \|s\|_\infty \le \tilde{Q}\|s\|_1 + \lambda\|s\|_1. \tag{3.27}$$

Setting $L = \tilde{Q} + \lambda$, from relation (3.27) we deduce that

$$\|\gamma\|_\infty \le L\|s\|_1. \tag{3.28}$$

Step 3. Let $s \in \mathcal{F}(\mathbb{Z}, X)$ be such that $s(n) \in \mathcal{S}(\sigma(\theta, n)) \cup \mathcal{U}(\sigma(\theta, n))$, for all $n \in \mathbb{Z}_+$ and let $\gamma \in c_0(\mathbb{Z}, X)$ be such that the pair (γ, s) satisfies (S_θ^π). We consider the functions $u, f : \mathbb{R} \to X$, given by

$$\begin{gathered} u(t) = \alpha(t - [t])\Phi(\sigma(\theta, [t]), t - [t])s([t]), \\ f(t) = \Phi(\sigma(\theta, [t]), t - [t])\gamma([t]) - \left(\int_t^{[t]+1} \alpha(\tau - [\tau])d\tau\right)\Phi(\sigma(\theta, [t]), t - [t])s([t]). \end{gathered} \tag{3.29}$$

Using analogous arguments with those used in the Step 2 we obtain that $u \in C(\mathbb{R}, X)$, $f \in C_0(\mathbb{R}, X)$ and the pair (f, u) satisfies (E_θ^π). Moreover, using Lemma 2.7 we have that $u(t) \in \mathcal{S}(\sigma(\theta, t)) \cup \mathcal{U}(\sigma(\theta, t))$, for all $t \ge 0$. Then, according to our hypothesis we deduce that

$$|||f||| \le Q\|u\|_p. \tag{3.30}$$

Since $f(n) = \gamma(n) - s(n)$, for all $n \in \mathbb{N}$, using (3.30) and (3.5) we obtain that

$$\|\gamma\|_\infty \le |||f||| + \|s\|_\infty \le Q\|u\|_p + \lambda\|s\|_p. \tag{3.31}$$

Observing that

$$\|u\|_p \le 2Me^{\omega}\|s\|_p \tag{3.32}$$

from (3.31) and (3.32) we deduce that

$$\|\gamma\|_\infty \le (2QMe^{\omega} + \lambda)\|s\|_p \le L\|s\|_p, \tag{3.33}$$

where $L = 2\lambda QMe^{\omega} + \lambda$.

Finally, from Steps 1–3 and relations (3.28) and (3.33) we deduce that the pair $(\ell^\infty(\mathbb{Z}, X), \mathcal{F}(\mathbb{Z}, X))$ is uniformly admissible for the skew-product flow $\pi = (\Phi, \sigma)$. By applying Theorem 2.9 we conclude that π is uniformly exponentially trichotomic. □

The natural question arises whether the integral admissibility given by Definition 3.1 is also a necessary condition for the existence of the uniform exponential trichotomy. To answer this question, in what follows, our attention will focus on the converse implication of the result given by Theorem 3.3. Specifically, our study will motivate the admissibility concept introduced in this paper and will point out several qualitative aspects. First of all, we prove a technical result.

Proposition 3.4. *Let $\pi = (\Phi, \sigma)$ be a skew-product flow which is uniformly exponentially trichotomic with respect to the families of projections $\{P_k(\theta)\}_{\theta\in\Theta}, k \in \{1,2,3\}$. Let $\theta \in \Theta$, let $f \in C_b(\mathbb{R}, X)$, and let $u \in \mathcal{C}(\mathbb{R}, X)$ be such that the pair (f, u) satisfies (E_θ^π). We denote*

$$f_k(t) := P_k(\sigma(\theta,t))f(t), \quad u_k(t) := P_k(\sigma(\theta,t))u(t), \quad \forall t \in \mathbb{R},\ \forall k \in \{1,2,3\} \tag{3.34}$$

and let $\Phi_3(\sigma(\theta,s),t-s)_|^{-1}$ denote the inverse of the operator $\Phi(\sigma(\theta,s),t-s)_| : P_3(\sigma(\theta,s)) \to P_3(\sigma(\theta,t))$, for all $t \geq s$. The following assertions hold:

(i) *the functions f_1 and f_3 have the following representations*

$$f_1(t) = \int_{-\infty}^{t} \Phi(\sigma(\theta,\tau), t-\tau)u_1(\tau)d\tau, \quad \forall t \in \mathbb{R} \tag{3.35}$$

$$f_3(t) = -\int_{t}^{\infty} \Phi_3(\sigma(\theta,t), \tau-t)_|^{-1}u_3(\tau)d\tau, \quad \forall t \in \mathbb{R}; \tag{3.36}$$

(ii) *for every $p \in (1,\infty)$ there is $Q_p > 0$ which does not depend on θ, f or u such that*

$$|||f_k||| \leq Q_p\|u\|_p, \quad \forall k \in \{1,3\}. \tag{3.37}$$

Proof. Since $u \in \mathcal{C}(\mathbb{R}, X)$ there is $h > 0$ such that $\operatorname{supp} u \subset (0,h)$. Let $K, \nu \in (0,\infty)$ be given by Definition 2.4 and let $\lambda_k = \sup_{\theta\in\Theta}\|P_k(\theta)\|, k \in \{1,2,3\}$.

(i) Since the pair (f,u) satisfies (E_θ^π) we have that

$$f_k(t) = \Phi(\sigma(\theta,r),t-r)f_k(r) + \int_r^t \Phi(\sigma(\theta,\tau),t-\tau)u_k(\tau)d\tau, \quad \forall t \geq r,\ \forall k \in \{1,2,3\}. \tag{3.38}$$

Since $\operatorname{supp} u \subset (0,h)$ from (3.38) we have that $f_1(t) = \Phi(\sigma(\theta,r),t-r)f_1(r)$, for all $r \leq t \leq 0$. Let $t \leq 0$. Then we deduce that

$$\|f_1(t)\| \leq Ke^{-\nu(t-r)}\|f_1(r)\| \leq \lambda_1 K e^{-\nu(t-r)}|||f|||, \quad \forall r \leq t. \tag{3.39}$$

For $r \to -\infty$ in (3.39) we obtain that $f_1(t) = 0$, for all $t \leq 0$. This shows that relation (3.35) holds for all $t \leq 0$. For $t > 0$ from (3.38) we have that

$$f_1(t) = \int_0^t \Phi(\sigma(\theta,\tau),t-\tau)u_1(\tau)d\tau = \int_{-\infty}^t \Phi(\sigma(\theta,\tau),t-\tau)u_1(\tau)d\tau \tag{3.40}$$

so relation (3.35) holds for every $t \in \mathbb{R}$.

For every $t \geq r \geq h$ from (3.38) we have that $f_3(t) = \Phi(\sigma(\theta, r), t - r) f_3(r)$. This implies that

$$\lambda_3 |||f||| \geq \|f_3(t)\| \geq \frac{1}{K} e^{\nu(t-r)} \|f_3(r)\|, \quad \forall t \geq r \geq h \tag{3.41}$$

so we deduce that

$$\|f_3(r)\| \leq \lambda_3 K |||f||| e^{-\nu(t-r)}, \quad \forall t \geq r \geq h. \tag{3.42}$$

From relation (3.42) it follows that $f_3(r) = 0$, for all $r \geq h$. In particular, we have that relation (3.36) holds for $t \geq h$. For $t \leq h$ from (3.38) we obtain that

$$0 = f_3(h) = \Phi(\sigma(\theta, t), h - t) f_3(t) + \int_t^h \Phi(\sigma(\theta, \tau), h - \tau) u_3(\tau) d\tau \tag{3.43}$$

which implies that

$$f_3(t) = -\int_t^h \Phi_3(\sigma(\theta, t), \tau - t)_|^{-1} u_3(\tau) d\tau = -\int_t^\infty \Phi_3(\sigma(\theta, t), \tau - t)_|^{-1} u_3(\tau) d\tau \tag{3.44}$$

for all $t \leq h$. Thus, we conclude that relation (3.36) holds for every $t \in \mathbb{R}$.

(ii) Let $p \in (1, \infty)$ and let $q = p/(p-1)$. Setting $Q_p = K(1/\nu q)^{1/q} \max\{\lambda_1, \lambda_3\}$ and using Hölder's inequality we deduce that

$$\|f_k(t)\| \leq Q_p \|u\|_p, \quad \forall t \in \mathbb{R}, \ \forall k \in \{1, 3\}. \tag{3.45}$$

□

The second main result of the paper is as follows.

Theorem 3.5. *Let $\pi = (\Phi, \sigma)$ be a skew-product flow. If π is uniformly exponentially trichotomic, then the pair $(C_b(\mathbb{R}, X), \mathcal{C}(\mathbb{R}, X))$ is uniformly admissible for π.*

Proof. Let $K, \nu \in (0, \infty)$ be two constants and let $\{P_k(\theta)\}_{\theta \in \Theta}$, $k \in \{1, 2, 3\}$ be the families of projections given by Definition 2.4. We set $\lambda_k := \sup_{\theta \in \Theta} \|P_k(\theta)\|$. For every $k \in \{2, 3\}$ and every $(\theta, t) \in \Theta \times \mathbb{R}_+$ we denote by $\Phi_k(\theta, t)_|^{-1}$ the inverse of the operator $\Phi(\theta, t)_| : \operatorname{Im} P_k(\theta) \to \operatorname{Im} P_k(\sigma(\theta, t))$.

For every $\varphi : \mathbb{R} \to X$ and every $k \in \{1, 2, 3\}$ we denote by

$$\varphi_k(t) = P_k(\sigma(\theta, t)) \varphi(t), \quad \forall t \in \mathbb{R}. \tag{3.46}$$

Then $\varphi_k(t) \in \operatorname{Im} P_k(\sigma(\theta, t))$, for all $t \in \mathbb{R}$ and all $k \in \{1, 2, 3\}$ and $\varphi(t) = \sum_{k=1}^3 \varphi_k(t)$, for all $t \in \mathbb{R}$.

Let $p \in (1, \infty)$ and let $Q_p > 0$ be given by Proposition 3.4. We prove that all the properties from Definition 3.1 are fulfilled.

Let $\theta \in \Theta$.

Step 1. Let $u \in \mathcal{C}(\mathbb{R}, X)$, $u \neq 0$. We consider the functions $f, g : \mathbb{R} \to X$ defined by

$$\begin{aligned} f(t) &= \int_{-\infty}^{t} \Phi(\sigma(\theta,\tau), t-\tau) u_1(\tau) d\tau - \int_{t}^{\infty} \Phi_2(\sigma(\theta,t), \tau - t)_{|}^{-1} u_2(\tau) d\tau \\ &\quad - \int_{t}^{\infty} \Phi_3(\sigma(\theta,t), \tau - t)_{|}^{-1} u_3(\tau) d\tau. \\ g(t) &= \int_{-\infty}^{t} \Phi(\sigma(\theta,\tau), t-\tau) u_1(\tau) d\tau + \int_{-\infty}^{t} \Phi(\sigma(\theta,\tau), t-\tau) u_2(\tau) d\tau \\ &\quad - \int_{t}^{\infty} \Phi_3(\sigma(\theta,t), \tau - t)_{|}^{-1} u_3(\tau) d\tau. \end{aligned} \tag{3.47}$$

Since $u \in \mathcal{C}(\mathbb{R}, X)$ we have that f and g are correctly defined and continuous. Let $h > 0$ be such that supp $u \subset (0, h)$. Setting $x_1 = \int_0^h \Phi(\sigma(\theta,\tau), h-\tau) u_1(\tau) d\tau$, $x_2 = \int_0^h \Phi_2(\theta,\tau)_{|}^{-1} u_2(\tau) d\tau$ and $x_3 = \int_0^h \Phi_3(\theta,\tau)_{|}^{-1} u_3(\tau) d\tau$, we have that $x_1 \in \operatorname{Im} P_1(\sigma(\theta,h))$ and $x_k \in \operatorname{Im} P_k(\theta)$, for $k \in \{2,3\}$.

We observe that $f(t) = -\Phi_2(\sigma(\theta,t), -t)_{|}^{-1} x_2 - \Phi_3(\sigma(\theta,t), -t)_{|}^{-1} x_3$, for all $t \leq 0$, which implies that

$$\|f(t)\| \leq K(\|x_2\| + \|x_3\|), \quad \forall t \leq 0. \tag{3.48}$$

In addition, we have that $f(t) = \Phi(\sigma(\theta,h), t-h) x_1$, for all $t \geq h$. This implies that

$$\|f(t)\| \leq K e^{-\nu(t-h)} \|x_1\|, \quad \forall t \geq h. \tag{3.49}$$

Since f is continuous from relations (3.48) and (3.49) it follows that $f \in \mathcal{L}(\mathbb{R}, X)$. Using similar arguments we deduce that $g \in \mathfrak{D}(\mathbb{R}, X)$. An easy computation shows that the pairs (f, u) and (g, u) satisfy (E_θ^π).

If $u = 0$, then we take $f = g \equiv 0$.

Step 2. Let $u \in \mathcal{C}(\mathbb{R}, X)$ and let $f \in \mathcal{L}(\mathbb{R}, X) \cup \mathfrak{D}(\mathbb{R}, X)$ be such that the pair (f, u) satisfies (E_θ^π).

Suppose that $f \in \mathcal{L}(\mathbb{R}, X)$. From Proposition 3.4 we have that

$$|||f_k||| \leq Q_p \|u\|_p, \quad \forall k \in \{1,3\}. \tag{3.50}$$

Let $h > 0$ be such that supp $u \subset (0, h)$. Since

$$f_2(t) = \Phi(\sigma(\theta,r), t-r) f_2(r) + \int_r^t \Phi(\sigma(\theta,\tau), t-\tau) u_2(\tau) d\tau, \quad \forall t \geq r \tag{3.51}$$

for $t \geq h$ we deduce that $f_2(t) = \Phi(\sigma(\theta, h), t - h)f_2(h)$. Since $f \in \mathcal{L}(\mathbb{R}, X)$ we obtain that

$$\frac{1}{K}\|f_2(h)\| \leq \|f_2(t)\| \leq \lambda_2\|f(t)\| \longrightarrow 0, \quad \text{as } t \longrightarrow \infty \tag{3.52}$$

so $f_2(h) = 0$. This implies that $f_2(t) = 0$, for all $t \geq h$. Moreover, using (3.51), for $t < h$ we have that

$$0 = f_2(h) = \Phi(\sigma(\theta, t), h - t)f_2(t) + \int_t^h \Phi(\sigma(\theta, \tau), h - \tau)u_2(\tau)d\tau \tag{3.53}$$

which implies that

$$f_2(t) = -\int_t^h \Phi_2(\sigma(\theta, t), \tau - t)_|^{-1} u_2(\tau)d\tau, \quad \forall t \leq h. \tag{3.54}$$

Then, we obtain that

$$\|f_2(t)\| \leq K\lambda_2 \int_t^h \|u(\tau)\|d\tau \leq K\lambda_2\|u\|_1. \tag{3.55}$$

Setting $Q = 2Q_p + K\lambda_2$ from relations (3.50) and (3.55) it follows that

$$|||f||| \leq \sum_{k=1}^{3} |||f_k||| \leq Q \max\left\{\|u\|_p, \|u\|_1\right\}. \tag{3.56}$$

The case $f \in \mathfrak{D}(\mathbb{R}, X)$ can be treated using similar arguments with those used above.

Step 3. Let $u \in \mathcal{C}(\mathbb{R}, X)$ be such that $u(t) \in \mathcal{S}(\sigma(\theta, t)) \cup \mathcal{U}(\sigma(\theta, t))$, for all t ≥ 0 and let $f \in C_0(\mathbb{R}, X)$ be such that the pair (f, u) satisfies (E_θ^π).

Since $u \in \mathcal{C}(\mathbb{R}, X)$ and $u(t) \in \mathcal{S}(\sigma(\theta, t)) \cup \mathcal{U}(\sigma(\theta, t))$, for all $t \geq 0$, using Lemma 2.7 we deduce that $u_2(t) = 0$, for all $t \in \mathbb{R}$. Since the pair (f, u) satisfies (E_θ^π) we obtain that

$$f_2(t) = \Phi(\sigma(\theta, r), t - r)f_2(r), \quad \forall t \geq r. \tag{3.57}$$

Let $r \in \mathbb{R}$. Since $f \in C_0(\mathbb{R}, X)$, using relation (3.57) we have that

$$\frac{1}{K}\|f_2(r)\| \leq \|f_2(t)\| \leq \lambda_2\|f(t)\| \longrightarrow 0, \quad \text{as } t \longrightarrow \infty \tag{3.58}$$

so $f_2(r) = 0$, for all $r \in \mathbb{R}$. This shows that, in this case, $f = f_1 + f_3$.

If $Q > 0$ is given by Step 2, we deduce that

$$|||f||| \leq |||f_1||| + |||f_3||| \leq 2Q_p\|u\|_p \leq Q\|u\|_p \tag{3.59}$$

and the proof is complete. □

The central result of this paper is as follows.

Theorem 3.6. *A skew-product flow $\pi = (\Phi, \sigma)$ is uniformly exponentially trichotomic if and only if the pair $(C_b(\mathbb{R}, X), C(\mathbb{R}, X))$ is uniformly admissible for π.*

Proof. This follows from Theorems 3.3 and 3.5. □

Remark 3.7. The above result establishes for the first time in the literature a necessary and sufficient condition for the existence of the uniform exponential trichotomy of skew-product flows, based on an input-output admissibility with respect to the associated integral equation. The chart described by our method allows a direct analysis of the asymptotic behavior of skew-product flows, without assuming a priori the existence of a projection families, invariance properties or any reversibility properties. Moreover, the study is done in the most general case, without any additional assumptions concerning the flow or the cocycle.

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