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Research Article

The Solution of Embedding Problems in the Framework of GAPs with Applications on Nonlinear PDEs

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The paper presents a special class of embedding problems whoes solutions are important for the explicit solution of nonlinear partial differential equations. It is shown that these embedding problems are solvable and explicit solutions are given. Not only are the solutions new but also the mathematical framework of their construction which is defined by a nonstandard function theory built over nonstandard algebraical structures, denoted as "GAPs". These GAPs must not be neither associative nor division algebras, but the corresponding function theories built over them preserve the most important symmetries from the classical complex function theory in a generalized form: a generalization of the Cauchy-Riemannian differential equations exists as well as a generalization of the classical Cauchy Integral Theorem.

1. Introduction

Except some small areas, at time the nonlinear world is inaccessible for analytic methods, that is, for methods without using numerical algorithms, we only know some explicit solutions of a few nonlinear differential equations (see e.g., [1–13]), and we know precious few about the "embedding" of nonlinear structures without any symmetry in higher dimensional structures with symmetry ([14–16], etc.).

This fact legitimates the creation of a new mathematical theory, presented in [17], which allows an "analytic access" on wider regions of nonlinearity. It was shown that this theory succeeds in solving nonlinear partial differential equations (concretely the Einstein equations from General Relativity) if the problem defining parameters (concretely the stress energy tensor) have a certain symmetry which can be seen as a wide generalization of the classical symmetry of holomorphy. This new symmetry—in the following denoted as "$*$-symmetry" or "GAP-symmetry" allows the explicit solution of Einstein equations and also of a broad variety of other partial differential equations.

This leads to the following question as considered here: is it possible to embed an arbitrary n-dimensional nonlinear partial differential equation (which only symmetry is given by smoothness of all coefficient functions) in an N-dimensional nonlinear partial differential equation with $*$-symmetry, $N \geq n$. If this holds, we can solve the given n-dimensional system by embedding this system into the N-dimensional system, which is solvable on the base of $*$-symmetry. Since embedding of an equation means embedding of the coefficient functions (which usually are given by vector fields (tensor fields)), our question above boils down to the following question:

is it possible to embed an arbitrary n-dimensional smooth vector field (tensor field) into a "$*$-symmetrical" N-dimensional tensor field, $N \geq n$?

This question will be answered positively here.

For a better understanding of the practical meaning of this result, some analogies to the classical situation of holomorphy are given (see the following section and also more details in [17]).

(i) The simplest GAP is defined by the two-dimensional field $\mathbb{C}$ of the complex numbers. In this case, the $*$-symmetry is defined by the classical symmetry of holomorphy. It is a well-known fact, that an arbitrary smooth real-valued function $f : I \to \mathbb{R}$, $I \subseteq \mathbb{R}$, can be "embedded" in a complex valued function $F : G \to \mathbb{C}$, $G \subseteq \mathbb{C}$ in the sense that the real part of F for values in the region I is identically with f. Our Embedding Theorem is a generalization of this classical result, by generalizing the two-dimensional field $\mathbb{C}$ to an n-dimensional GAP (generally a nonassociative, noncommtative, nondivision algebra).

(ii) The classical symmetry of holomorphy leads immediately to related (partly equivalent) symmetries: the well-known Cauchy-Riemannian differential equations and the well-known Cauchy Integral Theorem. The GAP-formalism generalizes these symmetries: it is shown that in GAPs, analogies of the Cauchy-Riemannian differential equations exist (in the following denoted as Pseudo Cauchy-Riemannian differential equations or shorter: "PCRE") as well as analogies of the Cauchy Integral Theorem (in the following denoted as Pseudo Cauchy Integral Theorem or shorter: "PCIT").

(iii) The classical symmetries: Cauchy-Riemannian differential equations and Cauchy's Integral Theorem lead to a solution theory of the two-dimensional linear Laplace equation. In a similar way, their analogies—the PCRE and the PCIT—lead to a solution theory of n-dimensional nonlinear partial differential equations.

After these considerations, some remarks to the structure of the paper are presented as follows.

(1) In Section 2.1 the concept of "$*$-analyticity" is introduced: we will define the so-called $*$-analytic tensor fields by demanding that these fields allow a representation as generalized power series in very general algebras.

(2) In Section 2.2 we will specialize these algebras to the so-called "PAk-structures" by introducing symmetries which can be seen as wide generalizations/modifications of the classical associativity symmetry. In the function theories built up over special PAk-structures (so-called "pseudorings"), there exist generalizations of the well-known Cauchy-Riemannian differential equations as well as generalizations of the well-known Cauchy Integral Theorem.

(3) In Section 2.3 a further generalization is given by generalizing the PAk-structures to the so-called "GAPs." It is shown that also in this wide framework the main results of Section 2.2 hold.

(4) Section 2.4 solves the problem of calculating GAPs explicitly.

(5) Section 3 presents applications of the GAP-formalism on famous partial differential equations from Theoretical Physics: new explicit solutions of *Einstein equations* from General Relativity and *Navier-Stokes equations* are given. The request of a further generalization of these results (which would be important from a physical point of view) shows the necessity of "embedding low dimensional unsymmetrical structures into higher dimensional symmetrical structures," that is, the necessity of an Embedding formalism, which is developed in the following Section 4.

(6) Section 4 presents the Embedding Theorem: it is shown that an arbitrary n-dimensional smooth vector field always can be embedded into a special N-dimensional smooth vector field ($N \geq n$) restricted by special GAP-symmetries. In other words, The world of GAPs is wide enough to allow embedding for rather general (smooth) structures. It is remarkable that the Embedding Theorem not only shows the possibility of Embedding but also endows the tools for practical applications. A simple example is given.

Finally some remarks to the style of the presented paper, which is given by accentuation of constructiveness, are presented like it is demanded by the concrete problems: not only are statements to existence and uniqueness desired and given but furthermore the explicit construction of a wide variety of solutions. For this reasons, this paper is not written for pure mathematicians but for physicists and applied mathematicians.

2. Elements of GAP-Theory

In this chapter the most important mathematical concepts will be presented as necessary for our solution method. Some of these concepts have been presented in [17], where the interested reader can find the proofs as missing here. The larger part of concepts is new; the corresponding propositions of course all will be proved in the following.

2.1. The Concept of $*$-Analyticity in General Algebras

We start by remembering on same elementary terms of algebra theory: let V denote an arbitrary vector space built over the field $\mathbb{R}$ of real numbers, and $* : V \times V \to V$ an arbitrary binary distributive operation. The algebraic structure defined by V and $*$ will be denoted here as $(V;*)$. The $*$-operation on V-vectors can be described by means of an arbitrary V-base $\{b_i\}$, $i = 1, \ldots, \dim V$ according to $b_i * b_j = C^k_{(*)ij} b_k$, where $C^k_{(*)ij} \in \mathbb{R}$ denote the *structure constants* of the algebra $(V;*)$ with respect to the base $\{b_i\}$. Instead of $i = 1, \ldots, \dim V$, we will write in the future $i \in I_{1,\dim V}$. The associated base-independent object to the structure constants $C^k_{(*)ij}$, $i, j, k \in I_{1,\dim V}$ is the *structure constant tensor* $C_{(*)}$ of the algebra: $C_{(*)} = C^k_{(*)ij} b^i \otimes b^j \otimes b_k$, with $b^i \in V^*$, where V^* denotes the *dual space* of V and $\{b^i\}$ the dual base of $\{b_j\}$, defined by

Table 1

Symmetry	Definition	Corresponding $C_{(*)}$-symmetry
(anti)commutativity	$x * y = (-) y * x$	$C^k_{(*)ij} = (-) C^k_{(*)ji}$
associativity	$(x * y) * z = x * (y * z)$	$C^l_{(*)ij} C^m_{(*)lk} = C^l_{(*)jk} C^m_{(*)il}$
Jacobi symmetry	$(x * y) * z + (y * z) * x$ $+(z * x) * y = 0$	$C^l_{(*)ij} C^m_{(*)lk} + C^l_{(*)jk} C^m_{(*)li}$ $+C^l_{(*)ki} C^m_{(*)lj} = 0$
$e_{(*,L)}$-existence	$e_{(*,L)} * x = x$	$e^i_{(*,L)} C^k_{(*)ij} = \delta^k_j$
$e_{(*,R)}$-existence	$x * e_{(*,R)} = x$	$C^k_{(*)ij} e^j_{(*,R)} = \delta^k_j$

$b^i(b_j) = \delta^i_j$. By means of $C_{(*)}$ the product $x * y$, $x, y \in V$ can be developed as follows:

$$x * y = (x * y)^k b_k = \left(x^i b_i\right) * \left(y^j b_j\right) \overset{\text{distributivity}}{=} x^i y^j \left(b_i * b_j\right) \overset{\text{definition of } C_{(*)}}{=} x^i y^j C^k_{(*)ij} b_k$$
$$\Longrightarrow (x * y)^k = x^i C^k_{(*)ij} y^j, \quad k \in I_{1,\dim V}. \tag{2.1}$$

In the future we will denote the quadratic matrix $x^i C^k_{(*)ij}$ by $(x*)^k_j$ and the quadratic matrix $C^k_{(*)ij} y^j$ by $(*y)^k_i$ or shorter in formal denotation $x*$ and $*y$. If rank $x* < \dim V$ for a fixed element x, then equation $x * y = 0$ has a nonzero solution $y \in V$ and if rank $*y < \dim V$ for a fixed element y, then equation $x * y = 0$ has a nonzero solution $x \in V$. We remember that in the special case of a ring operation $*$ a nonzero element x with $x * y = 0, y$ another nonzero element, is called a *zero divisor*, and a ring without zero divisors is called a *division ring*. In this work we will use the denotation "zero divisor" also for more general algebras in the sense above.

If $(V; *)$ has a left unit element (shorter: "left unit"), we denote this element as $e_{(*,L)}$, defined by $e_{(*,L)} * x = x$, for all $x \in V$. If $(V; *)$ has a right unit element (shorter: "right unit"), we denote this element as $e_{(*,R)}$, defined by $x * e_{(*,R)} = x$, for all $x \in V$. If left unit and right unit are identical we will write $e_{(*)}$. The existence of a right unit or a left unit allows to define the following generalizations of the inverse element conception:

$$x * x^{-1_{(*,R,R)}} = e_{(*,R)}, \qquad x^{-1_{(*,L,R)}} * x = e_{(*,R)},$$
$$x * x^{-1_{(*,R,L)}} = e_{(*,L)}, \qquad x^{-1_{(*,L,L)}} * x = e_{(*,L)}. \tag{2.2}$$

Here the symbol $x^{-1_{(*,R,R)}}$ denotes the $\underline{\text{r}}$ight inverse of x in respect of the right unit $e_{(*,R)}$ (see the superscripts R, R), $x^{-1_{(*,L,R)}}$ denotes the $\underline{\text{l}}$eft inverse of x in respect of the $\underline{\text{r}}$ight unit $e_{(*,R)}$ (see the superscripts L, R), and so forth. The existence of a unit element $e_{(*)} = e_{(*,R)} = e_{(*,L)}$ allows the definition of an inverse element $x^{-1_{(*)}}$ for $x \in V$: $x^{-1_{(*)}} * x = x * x^{-1_{(*)}} = e_{(*)}$. The extension $(*)$ in $x^{-1_{(*)}}$ will help us to distinguish an inverse vector from the inverse A^{-1} of a quadratic matrix A.

The classical algebra symmetries (anti)commutativity, associativity, Jacobi-symmetry, existence of a left unit/right unit can be described by means of the structure constants as shown in Table 1.

The $C_{(*)}$ symmetries follow immediately from representing the vectors x, y, z in respect of a vector base $\{b_i\}$ and by calculating the base products $b_i * b_j, (b_i * b_j) * b_k$, and so forth, by the structure constants.

Let $V^p := V \otimes V \cdots \otimes V$ (p times) denote the p-repeated tensor product of the vector space V and $V_q := V^* \otimes V^* \cdots \otimes V^*$ (q times) the q-repeated tensor product of the dual space V^*. With $b_{i_1} \otimes b_{i_2} \cdots \otimes b_{i_p} \in V^p$, we denote an arbitrary V^p-base, and with $b^{i_1} \otimes b^{i_2} \cdots \otimes b^{i_q} \in V_q$ the corresponding dual base. Then a tensor $T \in V^p$ (also called as "tensor of type $(p, 0)$") has a representation $T = T^{i_1,\dots,i_p} b_{i_1} \otimes b_{i_2} \cdots \otimes b_{i_p}$ with $T^{i_1,\dots,i_p} = T(b^{i_1}, b^{i_2}, \dots, b^{i_p})$ and a tensor $S \in V_q$ (also called as "tensor of type $(0, q)$") has a representation $S = S_{i_1,\dots,i_q} b^{i_1} \otimes b^{i_2} \cdots \otimes b^{i_q}$ with $S_{i_1,\dots,i_q} = S(b_{i_1}, b_{i_2}, \dots, b_{i_q})$.

After these well-known assumptions, we will start with GAP-Theory by introducing two new product operations $V^p \times V \to V^p$ and $V_q \times V \to V_q$ as follows.

Definition 2.1. Given an arbitrary algebra $(V; *)$ with structure constant tensor $C_{(*)}$. Then the $*$-*associatedproduct of order p:* $\circledast_{(*)} : V^p \times V \to V^p$ is defined by

$$\left(b_{i_1} \otimes b_{i_2} \cdots \otimes b_{i_p}\right) \circledast_{(*)} b_j := C^k_{(*)i_1 j} b_k \otimes b_{i_2} \cdots \otimes b_{i_p} + b_{i_1} \otimes C^k_{(*)i_2 j} b_k \otimes b_{i_3} \cdots \otimes b_{i_p} + \cdots + b_{i_1} \otimes b_{i_2} \cdots \otimes b_{i_{p-1}} \otimes C^k_{(*)i_p j} b_k, \tag{2.3}$$

and the $*$-*dual associated product of order q*: $\circledast^{\text{dual}}_{(*)} : V_q \times V \to V_q$ by

$$\left(b^{i_1} \otimes b^{i_2} \cdots \otimes b^{i_q}\right) \circledast^{\text{dual}}_{(*)} b_j := C^{i_1}_{(*)kj} b^k \otimes b^{i_2} \cdots \otimes b^{i_q} + b^{i_1} \otimes C^{i_2}_{(*)kj} b^k \otimes b^{i_3} \cdots \otimes b^{i_q} + \cdots + b^{i_1} \otimes \cdots \otimes b^{i_{q-1}} \otimes C^{i_q}_{(*)kj} b^k. \tag{2.4}$$

It is necessary to write $\circledast_{(*)}$ instead of $\circledast$, because later we will deal with different product operations $*_k$, $k = 1, 2, \dots$ and with the corresponding $*_k$-associated operations $\circledast_{(*_k)}$. In the case $p = 1$, it holds by definition that $\circledast_{(*)} = *$; in the case $q = 1$, we write also $*^{\text{dual}}$ for $\circledast^{\text{dual}}_{(*)}$. Due to the linearity of the vector spaces V^p and V_q, the $\circledast$-products of the base vectors uniquely define products of the form $T \circledast_{(*)} z$ and $S \circledast^{\text{dual}}_{(*)} z$ for arbitrary tensors $T \in V^p$, $S \in V_q$, and $z \in V$. For this we look at the following results:

Proposition 2.2. *The components of products $T \circledast_{(*)} z$, $S \circledast^{dual} z$, $T \in V^p$, $S \in V_q$, and $z \in V$ are given by*

$$\begin{aligned}
\left(T \circledast_{(*)} z\right)^{i_1 i_2 \cdots i_p} &= \left(T^{k\, i_2 \cdots i_p} C^{i_1}_{(*)kj} + T^{i_1\, k i_3 \cdots i_p} C^{i_2}_{(*)kj} + \cdots + T^{i_1 i_2 \cdots i_{p-1}\, k} C^{i_p}_{(*)kj}\right) z^j, \\
\left(S \circledast^{dual}_{(*)} z\right)_{i_1 i_2 \cdots i_q} &= \left(S_{k\, i_2 \cdots i_q} C^k_{(*)i_1 j} + S_{i_1\, k i_3 \cdots i_q} C^k_{(*)i_2 j} + \cdots + S_{i_1 i_2 \cdots i_{q-1}\, k} C^k_{(*)i_q j}\right) z^j.
\end{aligned} \tag{2.5}$$

The following algebraical symmetries hold:

$$(T_1 + T_2) \circledast_{(*)} z = \left(T_1 \circledast_{(*)} z\right) + \left(T_2 \circledast_{(*)} z\right), \tag{2.6a}$$

$$(T_1 \otimes T_2) \circledast_{(*)} z = \left(T_1 \circledast_{(*)} z\right) \otimes T_2 + T_1 \otimes \left(T_2 \circledast_{(*)} z\right), \tag{2.6b}$$

$$(anti)\ symmetry\ of\ T \Longleftrightarrow (anti)\ symmetry\ of\ T \circledast_{(*)} z, \tag{2.6c}$$

$$T \circledast_{(*)} e_{(*,R)} = pT, \tag{2.6d}$$

$e_{(*,R)}$ the right unit of the operation $*, T_1, T_2, T \in V^p$, $z \in V$. Analog relations hold for the dual $*$-associated products.

Definition 2.3. We define the maps $P_{(*)}^{[\alpha]} : V^p \times V \to V^p$ and $P_{(*)}^{\text{dual}[\alpha]} : V_q \times V \to V_q$ as follows:

$$\begin{aligned} P_{(*)}^{[\alpha]}(A, z) &:= \left(\cdots\left(\left(A \circledast_{(*)} z\right) \circledast_{(*)} z\right) \cdots \circledast_{(*)} z\right) \circledast_{(*)} z, \quad \alpha\text{-times}, \\ P_{(*)}^{\text{dual}[\alpha]}(B, z) &:= \left(\cdots\left(\left(B \circledast_{(*)}^{\text{dual}} z\right) \circledast_{(*)}^{\text{dual}} z\right) \cdots \circledast_{(*)}^{\text{dual}} z\right) \circledast_{(*)}^{\text{dual}} z, \quad \alpha\text{-times}, \end{aligned} \tag{2.7}$$

$A \in V^p$ and $B \in V_q$ arbitrary constants, $z \in V$. The map $P_{(*)}^{[\alpha]}(A, z)$ is denoted as the $*$-*generalized power function* of the vector z, $P_{(*)}^{\text{dual}[\alpha]}(B, z)$ as the*dual $*$-generalized power function*.

From this definition, it follows in particular that $P_{(*)}^{[1]}(A, z) = A \circledast_{(*)} z$, $P_{(*)}^{[2]}(A, z) = (A \circledast_{(*)} z) \circledast_{(*)} z \neq A \circledast_{(*)} (z * z)$, $P_{(*)}^{[3]}(A, z) = ((A \circledast_{(*)} z) \circledast_{(*)} z) \circledast_{(*)} z$, which is different from $A \circledast_{(*)} ((z * z) * z)$ as well as from $(A \circledast_{(*)} (z * z)) \circledast_{(*)} z$, and so forth. Furthermore, it follows that

$$\begin{gathered} P_{(*)}^{[\alpha+1]}(A, z) = P_{(*)}^{[\alpha]}(A, z) \circledast_{(*)} z, \qquad P_{(*)}^{\text{dual}[\alpha+1]}(B, z) = P_{(*)}^{\text{dual}[\alpha]}(B, z) \circledast_{(*)}^{\text{dual}} z, \\ P_{(*)}^{[\alpha]}(z, z) = (\cdots((z * z) * z) \cdots * z) * z =: z^{[\alpha+1]}, \\ P_{(*)}^{[\alpha]}\left(e_{(*,L)}, z\right) = \left(\cdots\left(\left(e_{(*,L)} * z\right) * z\right) \cdots * z\right) * z = z^{[\alpha]}, \end{gathered} \tag{2.8}$$

where we have introduced the denotation $z^{[\alpha]}$ for reason of simplicity as well as to differ this expression from the components z^α, that is, the square bracket for the "order index" α refers to the generalized power operation.

By means of the generalized power functions, we will introduce now the concept of $*$-analytic tensor fields. For the following $\mathfrak{T}_q^p(W)$, $W \subseteq \mathbb{R}^n$ denotes the vector space of tensors of type (p, q) which are smooth in a region W.

Definition 2.4. Given an arbitrary algebra $(\mathbb{R}^n; *)$ and arbitrary tensors $T \in \mathfrak{T}^p(W)$, $S \in \mathfrak{T}_q(W)$, we denote T and S, respectively, as $*$-*analytic* in region W if there exists an element $z_0 \in W$, which allows the following representations of T, respectively, S:

$$\begin{aligned} T^{i_1 \cdots i_p}(z) &= \left\{ A_{[0]} + \sum_{\alpha \geq 1} P_{(*)}^{[\alpha]}\left(A_{[\alpha]}, z - z_0\right) \right\}^{i_1 \cdots i_p} \\ &= \left\{ A_{[0]} + A_{[1]} \circledast_{(*)} (z - z_0) + \left(A_{[2]} \circledast_{(*)} (z - z_0)\right) \circledast_{(*)} (z - z_0) \right. \\ &\qquad \left. + \left(\left(A_{[3]} \circledast_{(*)} (z - z_0)\right) \circledast_{(*)} (z - z_0)\right) \circledast_{(*)} (z - z_0) + \cdots \right\}^{i_1 \cdots i_p}, \end{aligned} \tag{2.9}$$

$$
\begin{aligned}
S_{i_1\cdots i_q}(z) &= \left\{ B_{[0]} + \sum_{\alpha\geq 1} P_{(*)}^{\mathrm{dual}[\alpha]}\left(B_{[\alpha]}, z - z_0\right) \right\}_{i_1\cdots i_q} \\
&= \Big\{ B_{[0]} + B_{[1]} \circledast_{(*)}^{\mathrm{dual}} (z - z_0) + \Big(B_{[2]} \circledast_{(*)}^{\mathrm{dual}} (z - z_0) \Big) \circledast_{(*)}^{\mathrm{dual}} (z - z_0) \\
&\quad + \Big(\Big(B_{[3]} \circledast_{(*)}^{\mathrm{dual}} (z - z_0) \Big) \circledast_{(*)}^{\mathrm{dual}} (z - z_0) \Big) \circledast_{(*)}^{\mathrm{dual}} (z - z_0) + \cdots \Big\}_{i_1\cdots i_q},
\end{aligned}
\tag{2.10}
$$

$z \in W$, $A_{[\alpha]}$ constants of type $(p, 0)$, $B_{[\alpha]}$ constants of type $(0, q)$. The set of all in region $W*$-analytic tensors T and S for a fixed z_0 is denoted as $\mathfrak{T}^p(W; *)_{(z_0)}$ and $\mathfrak{T}_q(W; *)_{(z_0)}$.

In other words, $*$-analytic tensor fields are Taylor series in respect of the (dual) $*$-generalized power function, that is, the Taylor coefficients must take a special place within the "chain structure" of the series. If the series above are infinite, we need a criterion for convergence. Of course convergence depends on the region W as well as on the concretely used algebra $(\mathbb{R}^n; *)$. For a simple convergence criterion, we introduce in the space $(\mathbb{R}^n)^p$ the following norm. (Since the vector space $\mathbb{R}^n$ is self-dual, that is, $\mathbb{R}^n = (\mathbb{R}^n)^*$, we can identify $(\mathbb{R}^n)^p$ with $(\mathbb{R}^n)_p = ((\mathbb{R}^n)^*)^p$, and the following norm also is a norm for elements $Y \in (\mathbb{R}^n)_p$ with $\|Y\| :=: \sum_{i_1\, i_2\cdots i_p} |Y_{i_1\, i_2\cdots i_p}|, p \geq 1$)

$$
\|X\| :=: \sum_{i_1\, i_2\cdots i_p} \left| X^{i_1\, i_2\cdots i_p} \right|, \quad X \in (\mathbb{R}^n)^p,\ p \geq 1. \tag{2.11}
$$

Proposition 2.5. *Given the norm* (2.11) *and the denotations of from above, then the series* (2.9) *is absolutely convergent in the region* $W \subset \mathbb{R}^n$ *if*

$$
\begin{gathered}
\left| C^l_{(*)kj} \right| \leq M^l, \quad \forall j, k \in I_{1,n}, \quad \textit{with } \sum_{l=1}^{n} M^l \leq \frac{1}{p}, \\
\sum_{\alpha\geq 0} \left\| A_{[\alpha]} \right\| \|z - z_0\|^{\alpha} < \infty, \quad z, z_0 \in W.
\end{gathered}
\tag{2.12}
$$

The series (2.10) *is absolutly convergent in the region* $W \subset \mathbb{R}^n$ *if*

$$
\begin{gathered}
\left| C^l_{(*)kj} \right| \leq N_k, \quad \forall j, l \in I_{1,n}, \quad \textit{with } \sum_{k=1}^{n} N_k \leq \frac{1}{q}, \\
\sum_{\alpha\geq 0} \left\| B_{[\alpha]} \right\| \|z - z_0\|^{\alpha} < \infty, \quad z, z_0 \in W.
\end{gathered}
\tag{2.13}
$$

We sum up in the following.

Until now two concepts have been introduced: the concept of $*$-associated Products $\circledast_{(*)}, \circledast_{(*)}^{\mathrm{dual}}$ and the concept of $*$-analytic tensor fields. We have seen that the operations $\circledast_{(*)}, \circledast_{(*)}^{\mathrm{dual}}$ always have some simple symmetries without restricting the underlying $*$-operation, and that specializing of $*$ cannot induce the classical symmetries of associativity, commutativity, and Jacobi-symmetry for the operations $\circledast_{(*)}, \circledast_{(*)}^{\mathrm{dual}}$. By means of the $*$-associated products, the so-called $*$-analytic tensor fields have been defined. These tensor fields always have symmetries without restricting the underlying $*$-operation, that is, $*$-analyticity is a stronger symmetry than smoothness.

2.2. $*$-Analyticity in PAk-Structures. Nonstandard Function Theory

Now we will define such symmetries of an algebra $(\mathbb{R}^n;*)$, so that the corresponding $*$-analytical tensor fields become interesting "nonstandard symmetries". In particular, we are interested in symmetries of the partial derivatives, the covariant derivatives, and the Lie-derivatives of $*$-analytical tensor fields.

Definition 2.6. An algebraic structure $(\mathbb{R}^n;*_1,*_2)$ is called *pseudoassociative of type k (PAk-structure)*, $k=1,2$, if the operations $*_1,*_2$ satisfy the following commutation relation:

$$k=1:(x*_1y)*_2z=(x*_1z)*_2y,\qquad k=2:x*_1(y*_2z)=z*_1(y*_2x),\tag{2.14}$$

for all $x,y,z\in\mathbb{R}^n$. In the special case $*_1=*_2=*$, we denote these structures as *pseudorings of type k*.

It is easy to see that a pseudoring of arbitrary type is associative if and only if it is commutative and a pseudoring with left unit is always commutative.

Proposition 2.7. *An algebraic structure $(\mathbb{R}^n;*_1,*_2)$ is a PAk-structure, $k=1,2$, if and only if the structure constant tensors $C_{(*_1)}$ and $C_{(*_2)}$ satisfy the following commutation relation (The following index symbol "[]" is used as an index commutator.):*

$$k=1:C^k_{(*_1)ij}C^m_{(*_2)kl}=C^k_{(*_1)il}C^m_{(*_2)kj},\quad \left(\textit{equivalent with } C^k_{(*_1)i[j}C^m_{(*_2)|k|l]}=0\right),\tag{2.15a}$$

$$k=2:C^k_{(*_2)ij}C^m_{(*_1)lk}=C^k_{(*_2)il}C^m_{(*_1)jk},\quad \left(\textit{equivalent with } C^k_{(*_2)i[j}C^m_{(*_1)l]k}=0.\right)\tag{2.15b}$$

This Proposition allows a characterization of PAk-structures by means of the corresponding structure constant tensors. The problem of the explicit construction of PAk-structure constant tensors will be solved later. Now we will analyze some relationships between the inverse of a matrix (written as $(\,)^{-1}$) and the (left/right) inverse elements of a "corresponding" vector. We will see that such relationships exist for matrices and vectors derived from PAk-structures.

Proposition 2.8. *Given a PA1-structure $\mathfrak{X}=(\mathbb{R}^n;*_1,*_2)$ with left unit $e_{(*_2,L)}$ and a PA2-structure $\mathfrak{Y}=(\mathbb{R}^n;*_1,*_2)$ with right unit $e_{(*_1,R)}$, then it holds for arbitrary vectors $x\in\mathbb{R}^n$:*

$$\begin{aligned}&\textit{in algebra } \mathfrak{X}:(x*_1)^{-1}=*_2y\quad \textit{with } x*_1y=e_{(*_2,L)},\\&\textit{in algebra } \mathfrak{Y}:(x*_2)^{-1}=y*_1\quad \textit{with } x*_2y=e_{(*_1,R)}.\end{aligned}\tag{2.16}$$

In pseudorings the relations above take the very simple form:

$$\begin{gathered}\text{Pseudoring of first type with left unit: }(x*)^{-1}=*x^{-1_{(*,R,L)}},\\ \text{Pseudoring of second type with right unit: }(x*)^{-1}=x^{-1_{(*,R,R)}}*,\\ \text{Commutative pseudoring with unit: }(x*)^{-1}=*x^{-1_{(*)}}.\end{gathered}\tag{2.17}$$

In the next step we will show that beside the algebraic characterization there also exists a geometrical characterization of PAk-structures. we will restrict here to a geometrical interpretation only of the pseudoringsymmetries of first type and look at a differentiable manifold $\mathfrak{M}$ with an affine connection. As well-known, the corresponding curvature tensor Ω can be written in local coordinates as

$$\Omega^m_{ilj} = \Gamma^m_{ij,l} - \Gamma^m_{il,j} + \Gamma^k_{ij}\Gamma^m_{kl} - \Gamma^k_{il}\Gamma^m_{kj} = \Gamma^m_{i[j,l]} + \Gamma^k_{i[j}\Gamma^m_{|k|l]}, \tag{2.18}$$

where Γ^j_{im} denote the affine connection coefficients in local coordinates (To avoid misunderstandings, we refer to the definitions and denotations given in the appendix: the curvature tensor of a manifold $\mathfrak{M}$ (or a tangent bundle) with a connection is denoted with Ω and its coefficients with Ω^m_{ilj}, whereas in the more special case of a Riemannian manifold the curvature tensor is denoted as R and its coefficients with R^m_{ilj}). On the right side, the second summand agrees formal with the left side of the relation (2.15a) for the simple case $*_1 = *_2$, that is, for pseudorin gsymmetry of first type. This formal similarity is growing in the case of a flat manifold. In this case it holds $\Omega^m_{ilj} = 0$ and if we choose a local coordinate system with the special symmetries (It can be shown, that coordinate systems with this symmetry exist and can be constructed explicitly) $\Gamma^m_{i[j,l]} = 0$, then the relation above reduces to $\Gamma^k_{i[j}\Gamma^m_{|k|l]} = 0$, which is formal identical with the symmetry (2.15a) of a pseudoring of first type. Therefore we see

> *that the curvature tensor prefers the symmetry of pseudoassociativity of first type and not the symmetry of associativity*. (Associativity follows only in the case of flat spaces with torsion zero, that is, if $\Gamma^j_{im} = \Gamma^j_{mi}$. In this case, Γ^j_{im} shows the symmetry of the structure constant tensor of a commutative algebra and according to the considerations above it holds: commutativity + pseudoringsymmetries = commutativity + associativity).

Since the pseudoring symmetry of first type seems geometric fundamental, we can expect very interesting symmetries of the corresponding $*$-analytic tensors. As remarked in the introduction of this section, we are interested especially in the partial derivatives, the covariant derivatives and the Lie derivatives of $*$-analytical tensor fields. All these derivatives can be written by means of the so called $*$-*derivative*, which will be defined now as follows.

Definition 2.9. Given $T \in \mathfrak{T}^p(W;*)_{(z_0)}$, $S \in \mathfrak{T}_q(W;*)_{(z_0)}$. If the following series expansions

$$\begin{aligned}
\left(\mathfrak{D}_{(*)}T\right)^{i_1\cdots i_p}(z) &:= \left\{A_{[1]} + \sum_{\alpha\geq 2} P^{[\alpha-1]}_{(*)}\left(\alpha A_{[\alpha]}, z-z_0\right)\right\}^{i_1\ldots i_p} \\
&= \left\{A_{[1]} + 2A_{[2]}\circledast_{(*)}(z-z_0) + \left(3A_{[3]}\circledast_{(*)}(z-z_0)\right)\circledast_{(*)}(z-z_0) + \cdots\right\}^{i_1\ldots i_p}, \\
\left(\mathfrak{D}_{(*)}S\right)_{i_1\cdots i_q}(z) &:= \left\{B_{[1]} + \sum_{\alpha\geq 2} P^{\mathrm{dual}[\alpha-1]}_{(*)}\left(\alpha B_{[\alpha]}, z-z_0\right)\right\}_{i_1\cdots i_q} \\
&= \left\{B_{[1]} + 2B_{[2]}\circledast^{\mathrm{dual}}_{(*)}(z-z_0) + \left(3B_{[3]}\circledast^{\mathrm{dual}}_{(*)}(z-z_0)\right)\circledast^{\mathrm{dual}}_{(*)}(z-z_0) + \cdots\right\}_{i_1\cdots i_q}
\end{aligned} \tag{2.19}$$

are convergent for $z \in W$, the uniquely defined elements $\mathfrak{D}_{(*)}T \in \mathfrak{T}^p(W;*)_{(z_0)}$ and $\mathfrak{D}_{(*)}S \in \mathfrak{T}_q(W;*)_{(z_0)}$ will be denoted as the $*$-*derivative of tensor* T and the $*$-*derivative of tensor* S.

We also will write sometimes T' instead of $\mathfrak{D}_{(*)}T$ and analog for tensor S. From this definition and relation (2.6c) from Proposition 2.2, it follows immediately that

$$\begin{aligned} T \text{ is (anti)symmetric} &\Longleftrightarrow \mathfrak{D}_{(*)}T \text{ is (anti)symmetric}, \\ S \text{ is (anti)symmetric} &\Longleftrightarrow \mathfrak{D}_{(*)}S \text{ is (anti)symmetric}. \end{aligned} \tag{2.20}$$

Of course it is not ensured that this formal concept of $*$-derivative has the properties of the classical derivatives as to be a "direction-independent" limit of a difference quotient, and so forth. The concrete circumstances sensitive depend on the structure of the given algebra. To get interesting symmetries (we have to define what is interesting!) of the $*$-derivative, we will specialise the algebra to a pseudoring of first type.

Theorem 2.10. *Given a pseudoring of first type* $(\mathbb{R}^n;*)$ *and a tensor* $T \in \mathfrak{T}^p(W;*)_{(z_0)}, W \subseteq \mathbb{R}^n, p \geq 1$. *Then the partial derivatives of* T *satisfy the symmetry*

$$T^{i_1\cdots i_p}_{,m} = (T')^{l\, i_2\cdots i_p} C^{i_1}_{(*)lm} + (T')^{i_1\, l\, i_3\cdots i_p} C^{i_2}_{(*)lm} + \cdots + (T')^{i_1\, i_2\cdots i_{p-1}\, l} C^{i_p}_{(*)lm}. \tag{2.21}$$

If there exists a right unit $e_{(*,R)}$ *on* $(\mathbb{R}^n;*)$ *an inverse relation holds:*

$$(T')^{i_1\cdots i_p} = \frac{1}{p} T^{i_1\cdots i_p}_{,m} e^m_{(*,R)}. \tag{2.22}$$

Proposition 2.11. *Let* $(\mathbb{C};*)$ *denote the field of complex numbers. Then the relations* (2.21) *take the form of the classical Cauchy-Riemannian differential equations.*

Therefore, our considerations generalize the circumstances of classical complex function theory (and also those of modern function theory built over fields or division algebras) as follows:

$$\begin{gathered} \text{field structure of } \mathbb{C} \longrightarrow \text{pseudoring structure of } (\mathbb{R}^n;*) \\ \text{dimension 2 of } \mathbb{R}^2 \longrightarrow \text{arbitrary dimension } n \\ \text{vector fields on } \mathbb{R}^2 \longrightarrow \text{tensor fields on } \mathbb{R}^n. \end{gathered} \tag{2.23}$$

As the most remarkable generalization seems the generalization of the field structure to the structure of a nonassociative noncommutative pseudoring structur with zero divisors, these circumstances legitimize the denotation of (2.21) as *Pseudo-Cauchy-Riemannian equations* or shorter as *PCRE*. The analogon of Theorem 2.10 for tensors $S \in \mathfrak{T}_q(W;*)_{(z_0)}$ is given as follows.

Theorem 2.12. *Given a pseudoring of first type* $(\mathbb{R}^n;*)$ *and a tensor* $S \in \mathfrak{T}_q(W;*)_{(z_0)}, W \subseteq \mathbb{R}^n, q \geq 1$, *then the partial derivatives of* S *satisfy the symmetry (The following equations also will be denoted as PCRE.)*

$$S_{i_1\cdots i_q,m} = (S')_{l\, i_2\cdots i_q} C^l_{(*)i_1 m} + (S')_{i_1\, l i_3\cdots i_q} C^l_{(*)i_2 m} + \cdots + (S)_{i_1\, i_2\cdots i_{q-1}\, l} C^l_{(*)i_q m}. \tag{2.24}$$

If there exists a right unit $e_{(*,R)}$ *of* $(\mathbb{R}^n;*)$, *an inverse relation holds:*

$$(S')_{i_1\cdots i_q} = \frac{1}{q} S_{i_1\cdots i_q,m} e^m_{(*,R)}. \tag{2.25}$$

Now we will study the covariant derivative of $*$-analytical tensor fields. For this it is necessary to formalize the PCRE. By considering the properties of the $*$-associated products $\circledast_{(*)}$ and $\circledast^{\text{dual}}_{(*)}$ from Proposition 2.2, we see the possibility of the following formulation:

$$\partial T = \mathfrak{D}_{(*)} T \circledast_{(*)}, \qquad \partial S = \mathfrak{D}_{(*)} S \circledast^{\text{dual}}_{(*)}. \tag{2.26}$$

This relations show that partial derivatives can be "transformed" onto "ordinary" $*$-derivatives. This will offer an interesting approach for solving partial differential equations by transforming these equations onto ordinary "$*$-differential equations", that is, ordinary algebra-differential equations.

For studying the covariant derivatives of $*$-analytical tensor fields we remember that for arbitrary tensors $T \in \mathfrak{T}^p(W), S \in \mathfrak{T}_q(W), W \subset \mathbb{R}^n$ the covariant derivatives in component form are defined by

$$\begin{aligned} T^{i_1\, i_2\cdots i_p}_{;j} &= T^{i_1\, i_2\cdots i_p}_{,j} + T^{k\, i_2\cdots i_p}\Gamma^{i_1}_{kj} + T^{i_1\, k i_3\cdots i_p}\Gamma^{i_2}_{kj} + \cdots + T^{i_1\, i_2\cdots i_{p-1}\, k}\Gamma^{i_p}_{kj}, \\ S_{i_1\cdots i_q;j} &= S_{i_1\cdots i_q,j} - S_{k i_2\cdots i_q}\Gamma^k_{i_1 j} - S_{i_1\, k i_3\cdots i_q}\Gamma^k_{i_2 j} - \cdots - S_{i_1\, i_2\cdots i_{q-1}\, k}\Gamma^k_{i_q j}, \end{aligned} \tag{2.27}$$

where Γ^i_{jk} denotes the Christoffel symbols of the chosen $\mathbb{R}^n$-coordinate system. Comparison with the relations of Proposition 2.2 shows a strong formal similarity which legitimates the following denotations:

$$\begin{aligned} \left(T \circledast_{(\Gamma)}\right)^{i_1\, i_2\cdots i_p} &:= T^{k\, i_2\cdots i_p}\Gamma^{i_1}_{kj} + T^{i_1\, k i_3\cdots i_p}\Gamma^{i_2}_{kj} + \cdots + T^{i_1\, i_2\cdots i_{p-1}\, k}\Gamma^{i_p}_{kj}, \\ \left(S \circledast^{\text{dual}}_{(\Gamma)}\right)_{i_1\cdots i_q} &:= S_{k i_2\cdots i_q}\Gamma^k_{i_1 j} + S_{i_1\, k i_3\cdots i_q}\Gamma^k_{i_2 j} + \cdots + S_{i_1\, i_2\cdots i_{q-1}\, k}\Gamma^k_{i_q j}. \end{aligned} \tag{2.28}$$

Proposition 2.13. *Given a pseudoring of first type* $(\mathbb{R}^n;*)$ *and tensors* $T \in \mathfrak{T}^p(W;*)_{(z_0)}, S \in \mathfrak{T}_q(W;*)_{(z_0)}, W \subset \mathbb{R}^n$, $p, q \geq 1$, *then the covariant derivatives* DT, DS *satisfy the following symmetries:*

$$DT = \mathfrak{D}_{(*)} T \circledast_{(*)} + T \circledast_{(\Gamma)}, \qquad DS = \mathfrak{D}_{(*)} S \circledast^{dual}_{(*)} - S \circledast^{dual}_{(\Gamma)}. \tag{2.29}$$

Now we will study the Lie-derivatives of $*$-analytical tensor fields. For this we remember on the Lie-derivative $\mathfrak{L}_v : \mathfrak{T}^p_q(W) \to \mathfrak{T}^p_q(W)$ of a tensor $T \in \mathfrak{T}^p_q(W)$ in respect to a vector field v:

$$\begin{aligned}(\mathfrak{L}_v T)^{i_1\cdots i_p}_{j_1\cdots j_q} = {}& T^{i_1\cdots i_p}_{j_1\cdots j_q,m} v^m - T^{m\, i_2\cdots i_p}_{j_1\cdots j_q} v^{i_1}_{,m} - T^{i_1 m\, i_3\cdots i_p}_{j_1\cdots j_q} v^{i_2}{}_{,m} - \cdots - T^{i_1\cdots i_{p-1} m}_{j_1\cdots j_q} v^{i_p}_{,m} \\ & + T^{i_1\cdots i_p}_{m\ \, j_2\cdots j_q} v^m_{,j_1} + T^{i_1\cdots i_p}_{j_1\ m j_3\cdots j_q}\ v^m_{,j_2} + \cdots + T^{i_1\cdots i_p}_{j_1\cdots j_{q-1} m} v^m_{,j_q}.\end{aligned} \tag{2.30}$$

Proposition 2.14. *Given two pseudorings of first type* $(\mathbb{R}^n;*)$ *and* $(\mathbb{R}^n;\circ)$*, furthermore the tensors* $T \in \mathfrak{T}^p(W;*)_{(z_0)}, S \in \mathfrak{T}_q(W;*)_{(z_0)}, W \subset \mathbb{R}^n, p, q \geq 1$ *and a vector* $v \in \mathfrak{T}^1(W;\circ)_{(z_0)}$*. Then the Lie-derivatives* $\mathfrak{L}_v T$ *and* $\mathfrak{L}_v S$ *satisfy the following symmetries (The following operation* $\circ^T$ *is defined by* $C^{\ \ k}_{(\circ^T)mj} = C^k_{(\circ)jm}$*, that is, by "transposition" of the corresponding quadratic matrix with fixed index* k*.):*

$$\mathfrak{L}_v T = \mathfrak{D}_{(*)} T \circledast_{(*)} v - T \circledast_{(\circ^T)} \mathfrak{D}_{(\circ)} v, \qquad \mathfrak{L}_v S = \mathfrak{D}_{(*)} S \circledast^{dual}_{(*)} v + S \circledast^{dual}_{(\circ^T)} \mathfrak{D}_{(\circ)} v. \tag{2.31}$$

For a pseudoring of first type $(\mathbb{R}^n;*)$ *with right unit* $e_{(*,R)}$ *and a tensor field, it holds that*

$$\mathfrak{D}_{(*)} T = \mathfrak{L}_{e_{(*,R)}} T, \quad T \in \mathfrak{T}^p(W;*)_{(z_0)}, \tag{2.32a}$$

$$(\mathfrak{L}_g f)' = \mathfrak{L}_g f' + \mathfrak{L}_{g'} f, \quad f, v \in \mathfrak{T}^1(W;*)_{(z_0)}. \tag{2.32b}$$

Relation (2.32a) *shows a very simple characterization of the $*$-derivative by the Lie-derivative, but it is only possible in a pseudoring with a right unit.*

After the partial derivatives, covariant derivatives and Lie-derivatives we will study the exterior derivatives of $*$-analytic forms. Let us denote the set of all in region $W \subseteq \mathbb{R}^n$ smooth q -forms with $\mathfrak{A}^q(W)$, and the subset of all in $W$$*$-analytical q-forms with $\mathfrak{A}^q(W;*)_{(z_0)}$.

Proposition 2.15. *Given a pseudoring* $(\mathbb{R}^n;*)$ *of first type, an arbitrary nontrivial* q*-form* $\omega \in \mathfrak{A}^q(W;*)_{(z_0)}$*, and arbitrary vector fields* $u, v \in \mathfrak{T}^1(W;*)_{(z_0)}, w \in \mathfrak{T}_1(W;*)_{(z_0)}, v$ *not constant ("Not constant" means of course not constant in the considered coordinate system), then it holds that*

$$\begin{gathered} d \wedge \omega = 0 \Longleftrightarrow (\mathbb{R}^n;*) \text{ is commutative}, \\ u * v \in \mathfrak{T}^1(W;*)_{(z_0)} \Longleftrightarrow (\mathbb{R}^n;*) \text{ is commutative}, \\ w *^{dual} v \in \mathfrak{T}_1(W;*)_{(z_0)} \Longleftrightarrow (\mathbb{R}^n;*) \text{ is commutative}, \\ u^{-1_{(*)}} \in \mathfrak{T}^1(W;*)_{(z_0)} \Longleftrightarrow (\mathbb{R}^n;*) \text{ is commutative}. \end{gathered} \tag{2.33}$$

This proposition shows that some fundamental symmetries with practical importance only exist in commutative pseudorings: the $*$-product of $*$-analytical vector fields is $*$-analytic only for commutative pseudorings, and an analogue result holds for the $*^{\text{dual}}$-product. Also the inverse element of a $*$-analytical vector field will be $*$-analytical only in a commutative pseudoring. Let us sum up the relations for the classical derivatives as follows:
Pseudo-Cauchy-Riemann equations:

$$\partial T = \mathfrak{D}_{(*)} T \circledast_{(*)}, \quad \partial S = \mathfrak{D}_{(*)} S \circledast^{\text{dual}}_{(*)}, \tag{2.34a}$$

Covariant derivatives:

$$DT = \mathfrak{D}_{(*)}T\circledast_{(*)} + T\circledast_{(\Gamma)}, \quad DS = \mathfrak{D}_{(*)}S\circledast^{\text{dual}}_{(*)} - S\circledast^{\text{dual}}_{(\Gamma)}, \tag{2.34b}$$

Lie-derivative:

$$\begin{gathered}\mathfrak{L}_v T = \mathfrak{D}_{(*)}T\circledast_{(*)}v - T\circledast_{(\circ^T)}\mathfrak{D}_{(\circ)}v, \quad \mathfrak{L}_v S = \mathfrak{D}_{(*)}S\circledast^{\text{dual}}_{(*)}v + S\circledast^{\text{dual}}_{(\circ^T)}\mathfrak{D}_{(\circ)}v, \\ \mathfrak{D}_{(*)}T = \mathfrak{L}_{e_{(*,R)}}T,\end{gathered} \tag{2.34c}$$

Exterior derivative:

$$d \wedge \omega = 0 \Longleftrightarrow (\mathbb{R}^n; *) \text{ is commutative}, \quad \omega \in \mathfrak{A}^q(W; *)_{(z_0)}, \tag{2.34d}$$

with $(\mathbb{R}^n; *), (\mathbb{R}^n; \circ)$ pseudorings of first type, $T \in \mathfrak{T}^p(W; *)_{(z_0)}, S \in \mathfrak{T}_q(W; *)_{(z_0)}, W \subset \mathbb{R}^n, v \in \mathfrak{T}^1(W; \circ)_{(z_0)}$. We sum up in the following paragraph.

Our aim was to study $*$-analytical tensor fields in the framework of pseudorings, which have been introduced here as a fundamental concept. It was shown that pseudoringsymmetry is an algebraic symmetry which also appears in the world of differential geometry, moreover: the geometrical world prefers the pseudoringsymmetry against the classical symmetry of associativity. For this the study of pseudorings is legitimized, and so the study of tensor fields built over pseudorings. In particular we have studied the partial derivatives of $*$-analytical tensor fields, their covariant derivatives, their Lie-derivatives and their exterior derivatives, all this in the frame of pseudoringsymmetry. It was shown, that all these fundamental derivatives can be written in terms of the so-called $*$-derivative, which has been introduced here as a fundamental concept, following from the symmetry of $*$-analyticity.

2.3. *Generalizations. The World of GAPs*

Until now we have dealt with PAk-structures $(\mathbb{R}^n; *_1, *_2)$ and pseudorings $(\mathbb{R}^n; *)$, that is, with algebraic structures defined by *maximal two* product operations. For some later applications, this framework will be too small and shall be generalized here (The most applications on mathematical physics only need some small parts of this section: the concept of a GAP, of a GAP characteristic, and of GAP Exponentials). Such an algebra generalization will lead us to the concept of GAPs (PAk-structures and pseudorings will be shown as the most simple GAP-structures) and will allow us to generalize the function theoretical concepts of $*$-analyticity, $*$-derivative, $*$-integration, and so forth, from the section above by replacing the underlying PAk-structures (pseudorings) by GAPs.

The aim of this section is to generalize the concepts of PAk-structures, $*$-analyticity, $*$-derivative, $*$-integration, Pseudo Cauchy Riemann equations, and so forth.

Let us denote $\mathfrak{X} = (\mathbb{R}^n; *_1, *_2, \ldots, *_M)$ as an algebraic structure with the operations $*_\alpha : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n, \alpha \in I_{1,M}, M$ an arbitrary natural number which also might be infinite.

To generalize the concept of $*$-analyticity we have to generalize the concept of the power function from Section 2.1. For this we introduce the maps $P^{[\alpha]}_{(*_1,*_2,\ldots,*_\alpha)} : (\mathbb{R}^n)^p \times (\mathbb{R}^n) \to (\mathbb{R}^n)^p$ and $P^{\mathrm{dual}[\alpha]}_{(*_1,*_2,\ldots,*_\alpha)} : (\mathbb{R}^n)_q \times (\mathbb{R}^n) \to (\mathbb{R}^n)_q$ as follows:

$$
\begin{aligned}
P^{[\alpha]}_{(*_1,*_2,\ldots,*_\alpha)}(A,z) &:= \left(\cdots\left(\left(A \circledast_{(*_\alpha)} z\right)\circledast_{(*_{\alpha-1})} z\right)\cdots \circledast_{(*_2)} z\right)\circledast_{(*_1)} z, \quad \alpha\text{-times}, \\
P^{\mathrm{dual}[\alpha]}_{(*_1,*_2,\ldots,*_\alpha)}(B,z) &:= \left(\cdots\left(\left(B \circledast^{\mathrm{dual}}_{(*_\alpha)} z\right)\circledast^{\mathrm{dual}}_{(*_{\alpha-1})} z\right)\cdots \circledast^{\mathrm{dual}}_{(*_2)} z\right)\circledast^{\mathrm{dual}}_{(*_1)} z, \quad \alpha\text{-times},
\end{aligned}
\tag{2.35}
$$

$A \in (\mathbb{R}^n)^p, B \in (\mathbb{R}^n)_q$ arbitrary constants, $z \in \mathbb{R}^n$. After these assumption we can introduce the concept of "chain-analyticity".

Definition 2.16. Given an arbitrary algebraic structure $\mathfrak{X} = (\mathbb{R}^n; *_1, *_2, \ldots, *_M)$ and arbitrary tensors $T \in \mathfrak{T}^p(W_1), S \in \mathfrak{T}_q(W_1)$, we denote T and S, respectively, as *chain-analytic*in region W_1, if there exists an element $z_0 \in W_1$, which allows the following representations of T, respectively, S:

$$
\begin{aligned}
T^{i_1\cdots i_p}(z) &= \left\{A_{[0]} + \sum_{\alpha=1}^{M} P^{[\alpha]}_{(*_1,*_2,\ldots,*_\alpha)}(A_{[\alpha]}, z - z_0)\right\}^{i_1\cdots i_p} := A_{[0]} + A_{[1]}\circledast_{(*_1)}(z - z_0) \\
&\quad + \left(A_{[2]}\circledast_{(*_2)}(z - z_0)\right)\circledast_{(*_1)}(z - z_0) \\
&\quad + \left(\left(A_{[3]}\circledast_{(*_3)}(z - z_0)\right)\circledast_{(*_2)}(z - z_0)\right)\circledast_{(*_1)}(z - z_0) + \cdots, \\
S_{i_1\cdots i_q}(z) &= \left\{B_{[0]} + \sum_{\alpha=1}^{M} P^{\mathrm{dual}[\alpha]}_{(*_1,*_2,\ldots,*_\alpha)}(B_{[\alpha]}, z - z_0)\right\}_{i_1\cdots i_q} \\
&:= B_{[0]} + B_{[1]}\circledast^{\mathrm{dual}}_{(*_1)}(z - z_0) + \left(B_{[2]}\circledast^{\mathrm{dual}}_{(*_2)}(z - z_0)\right)\circledast^{\mathrm{dual}}_{(*_1)}(z - z_0) \\
&\quad + \left(\left(B_{[3]}\circledast^{\mathrm{dual}}_{(*_3)}(z - z_0)\right)\circledast^{\mathrm{dual}}_{(*_2)}(z - z_0)\right)\circledast^{\mathrm{dual}}_{(*_1)}(z - z_0) + \cdots,
\end{aligned}
\tag{2.36}
$$

$z \in W_1, A_{[\alpha]}$ a constant of type $(p,0), B_{[\alpha]}$ a constant of type $(0,q)$. The set of all in region W_1 chain-analytical tensors T and S for fixed z_0 is denoted as $\mathfrak{T}^p(W_1; *_1, *_2, \ldots, *_M)_{(z_0)}$ and $\mathfrak{T}_q(W_1; *_1, *_2, \ldots, *_M)_{(z_0)}$.

Consider that in the series above an infinite value of M is allowed and an infinite set of $*_\alpha$-operations is allowed to contain an infinite subset of identical operations. In the case $*_1 = *_2 = \cdots = *_M$ (=finite or infinite), we will denote the structures above as $*$-analytical as in the earlier case of a single product operation. The denotation "chain-analytical" comes from the fact, that the series above are built by $*_\alpha$-operations like the "elements of a chain". We see, that for different $*_\alpha$-operations the structure of chain-analytical tensor fields is much more general than the structure of simple $*$-analytical tensor fields. Now we will generalize the concept of PAk-structures for vector spaces with different dimensions.

Now I will present a concept, which allows a short overview about all symmetries—PAk-symmetry as well as standard symmetries—of a given algebraic structure. For this we assign the algebraic structure $\mathfrak{X} = (\mathbb{R}^n; *_1, *_2, \ldots, *_M)$ a $M \times M$-matrix $G_{(\mathfrak{X})}$, denoted as the *symmetry characteristic of the algebra* $\mathfrak{X}$ as follows:

$$G_{(\mathfrak{X})\alpha\beta} := \begin{Bmatrix} k & \text{if the subalgebra } (\mathbb{R}^n; *_\alpha, *_\beta) \text{ is a PAk-structure,} \\ 0 & \text{otherwise.} \end{Bmatrix} \tag{2.37}$$

("0" means: "no PAk symmetry is assumed" and *not*: "no PAk-symmetry is allowed". With other words: In the case "0" it is possible that a PAk-symmetry exists, but it is not ensured.)

Definition 2.17. An algebraic structure $\mathfrak{X} = (\mathbb{R}^n; *_1, *_2, \ldots, *_M)$ is called a *GAP of order M*, if the symmetry characteristic $G_{(\mathfrak{X})}$ includes at least one value 1 or 2. In this case $G_{(\mathfrak{X})}$ is denoted as the *GAP characteristic* of the GAP $\mathfrak{X}$.

In other words a GAP is an algebraic structure which has at least one PA1-symmetry or one PA2 -symmetry. For further applications it will be advantageable, to generalize the GAP-characteristic by giving an overview not only about the PAk-symmetries but also about the "standard symmetries" of all algebraic structures $(\mathbb{R}^n; *_\alpha)$, that is, of its possible (anti)commutativity, Jacobi-symmetry, associativity, existence of a right unit $e_{(*,R)}$, existence of a left unit $e_{(*,L)}$, and so forth. Since these symmetries all are defined for a single algebra operation, they only will appear in the main diagonal of $G_{(\mathfrak{X})}$. Concretely we introduce the following denotations as subscripts of the elements $G_{(\mathfrak{X})\alpha\alpha}$:

$$\begin{gathered} c \sim \text{commutativity}, \\ \text{ac} \sim \text{anticommutativity}, \\ J \sim \text{Jacobi-symmetry}, \\ e_{(R)} \sim \text{existence of a right unit}, \\ e_{(L)} \sim \text{existence of a left unit}, \\ e \sim \text{existence of a unit}. \end{gathered} \tag{2.38}$$

As an example, we consider an algebraic structure $\mathfrak{X} = (\mathbb{R}^n; *_1, *_2, *_3, *_4)$ with the following symmetry characteristic:

$$G_{(\mathfrak{X})} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0_{e_{(R)}} & 1 & 0 \\ 0 & 0 & 0_{\text{ac},J} & 0 \\ 0 & 2 & 0 & 0 \end{pmatrix} \tag{2.39}$$

which means that the algebra $(\mathbb{R}^n; *_2)$ has a right unit, the algebra $(\mathbb{R}^n; *_3)$ is anticommutative and satisfies the Jacobi-symmetry (i.e., $(\mathbb{R}^n; *_3)$ is a Lie-algebra), the algebra $(\mathbb{R}^n; *_4)$ is commutative, the algebraic structure $(\mathbb{R}^n; *_1, *_2)$ is a PA1-structure, and the algebraic structure $(\mathbb{R}^n; *_4, *_2)$ is a PA2-structure. Per definition $\mathfrak{X}$ is a GAP.

Now some special types of GAPs will be introduced which allow a wide generalization of the functional theoretical concepts of $*$-analyticity, $*$-derivative, Pseudo Cauchy Riemann equations, $*$-integration, and so forth.

Definition 2.18. A PA1-chain is a GAP $\mathfrak{X} = (\mathbb{R}^n; *_1, *_2, \ldots, *_M)$ with $\mathrm{PAk}_{\alpha\,\alpha-1} = 1, \alpha \in I_{2,M}$. A PA1-chain is denoted as a *closed PA1-chain*, if additionally to the above it holds $\mathrm{PAk}_{1M} = 1$. A *closed PA1-chain of Lie-type*is defined as a closed PA1-chain of order 3 with the additional restriction

$$(x *_3 y) *_2 z + (y *_2 z) *_1 x + (z *_1 x) *_3 y = 0, \quad \forall x, y, z \in \mathbb{R}^n. \tag{2.40}$$

The concept of a PA1-chain allows a wide generalization of the earlier concept of the $*$-derivation, which will be presented now.

Definition 2.19. Given a PA1-chain $\mathfrak{X} = (\mathbb{R}^n; *_1, *_2, \ldots, *_M)$ and chain-analytic tensors $T \in \mathfrak{T}^p(W_1; *_1, \ldots, *_M)_{(z_0)}, S \in \mathfrak{T}_q(W_1; *_1, \ldots, *_M)_{(z_0)}$, then the following series expansions

$$\begin{aligned}
\left(\mathfrak{D}_{(*_1)} T\right)^{i_1\cdots i_p}(z) &:= \left\{ A_{[1]} + \sum_{\alpha\ge 2}^{M} P^{[\alpha-1]}_{(*_2,\ldots,*_\alpha)}\left(\alpha A_{[\alpha]}, z - z_0\right) \right\}^{i_1\cdots i_p} \\
&:= \left\{ A_{[1]} + 2A_{[2]} \circledast_{(*_2)} (z - z_0) + \left(3A_{[3]} \circledast_{(*_3)} (z - z_0)\right) \circledast_{(*_2)} (z - z_0) + \cdots \right\}^{i_1\cdots i_p}, \\
\left(\mathfrak{D}_{(*_1)} S\right)_{i_1\cdots i_q}(z) &:= \left\{ B_{[1]} + \sum_{\alpha\ge 2}^{M} P^{\mathrm{dual}[\alpha-1]}_{(*_2,\ldots,*_\alpha)}\left(\alpha B_{[\alpha]}, z - z_0\right) \right\}_{i_1\cdots i_q} \\
&:= \left\{ B_{[1]} + 2B_{[2]} \circledast^{\mathrm{dual}}_{(*_2)} (z - z_0) + \left(3B_{[3]} \circledast^{\mathrm{dual}}_{(*_3)} (z - z_0)\right) \circledast^{\mathrm{dual}}_{(*_2)} (z - z_0) + \cdots \right\}_{i_1\cdots i_q},
\end{aligned} \tag{2.41}$$

$z \in W_1$, uniquely define elements $\mathfrak{D}_{(*_1)} T \in \mathfrak{T}^p(W_1; *_2, \ldots, *_M)_{(z_0)}$ and $\mathfrak{D}_{(*_1)} S \in \mathfrak{T}_q(W_1; *_2, \ldots, *_M)_{(z_0)}$ which will be denoted as the $*_1$*-derivative of tensor* T and the $*_1$*-derivative of tensor* S.

We see that the tensors $\mathfrak{D}_{(*_1)} T$ and $\mathfrak{D}_{(*_1)} S$ do not contain the operation $*_1$ any longer and so we cannot define $\mathfrak{D}_{(*_1)}\mathfrak{D}_{(*_1)} T$ or $\mathfrak{D}_{(*_1)}\mathfrak{D}_{(*_1)} S$. Only $\mathfrak{D}_{(*_2)}\mathfrak{D}_{(*_1)} T$ and $\mathfrak{D}_{(*_2)}\mathfrak{D}_{(*_1)} S$ make sense, and also $\mathfrak{D}_{(*_3)}\mathfrak{D}_{(*_2)}\mathfrak{D}_{(*_1)} T$ and $\mathfrak{D}_{(*_3)}\mathfrak{D}_{(*_2)}\mathfrak{D}_{(*_1)} S$, and so forth. Thus we will define the operators

$$\mathfrak{D}^k_{(*_k, *_{k-1}, \ldots, *_1)} := \mathfrak{D}_{(*_k)}\mathfrak{D}_{(*_{k-1})} \cdots \mathfrak{D}_{(*_1)}, \quad k \in I_{1,M} \tag{2.42}$$

as the *chain derivatives of order k*. In the special case $*_1 = *_2 = \cdots = *_k := *$ we write simplifying $\mathfrak{D}^k_{(*)}$ instead of $\mathfrak{D}^k_{(*,*,\ldots,*)}$. From the definitions above it follows for $z \in W_1$:

$$\begin{aligned}
\left(\mathfrak{D}^k_{(*k, *k-1, \ldots, *_1)} T\right)^{i_1\cdots i_p}(z) &:= \left\{ A_{[k]} + \sum_{\alpha\ge k+1}^{M} P^{[\alpha-k]}_{(*_2,\ldots,*_\alpha)}\left(\alpha(\alpha-1)\cdots(\alpha-k) A_{[\alpha]}, z - z_0\right) \right\}^{i_1\cdots i_p}, \\
\left(\mathfrak{D}^k_{(*_k, *_{k-1}, \ldots, *_1)} S\right)_{i_1\cdots i_q}(z) &:= \left\{ B_{[k]} + \sum_{\alpha\ge k+1}^{M} P^{\mathrm{dual}[\alpha-k]}_{(*_2,\ldots,*_\alpha)}\left(\alpha(\alpha-1)\cdots(\alpha-k) B_{[\alpha]}, z - z_0\right) \right\}_{i_1\cdots i_q},
\end{aligned} \tag{2.43}$$

and we see that

$$\mathfrak{D}^k_{(*_k,*_{k-1},\dots,*_1)}T \in \mathfrak{T}^p(W_1;*_{k+1},\dots,*_M)_{(z_0)}, \quad \mathfrak{D}^k_{(*_k,*_{k-1},\dots,*_1)}S \in \mathfrak{T}_q(W_1;*_{k+1},\dots,*_M)_{(z_0)}. \tag{2.44}$$

Theorem 2.20. *Given a PA1-chain $\mathfrak{X} = (\mathbb{R}^n;*_1,*_2,\dots,*_M)$ and a chain-analytic tensor $T \in \mathfrak{T}^p(W_1;*_1,\dots,*_M)_{(z_0)}$, $p \geq 1$. Then the following symmetries hold:*

$$\begin{aligned}
&\left(\mathfrak{D}^k_{(*_k,*_{k-1},\dots,*_1)}T\right)^{i_1\cdots i_p}_{,\alpha} \\
&\quad= \left(\mathfrak{D}^{k+1}_{(*_{k+1},*_k,\dots,*_1)}T\right)^{li_2\cdots i_p} C^{i_1}_{(*_{k+1})l\alpha} \\
&\qquad+ \left(\mathfrak{D}^{k+1}_{(*_{k+1},*_k,\dots,*_1)}T\right)^{i_1li_3\cdots i_p} C^{i_2}_{(*_{k+1})l\alpha} + \cdots + \left(\mathfrak{D}^{k+1}_{(*_{k+1},*_k,\dots,*_1)}T\right)^{i_1i_2\cdots i_{p-1}l} C^{i_p}_{(*_{k+1})l\alpha},
\end{aligned} \tag{2.45}$$

for all $\alpha \in I_{1,n_1}, i_1,\dots,i_p \in I_{1,n_2}, k \in I_{0,M-1}$ (In the case $k = 0$ the left side must be interpreted as $T^{i_1\cdots i_p}_{,\alpha}$). An inverse relation exists, if the algebra $(\mathbb{R}^n;*_1)$ has a right unit $e_{(*_1,R)}$:

$$\left(D^{k+1}_{(*_{k+1},*_k,\dots,*_1)}T\right)^{i_1\,i_2i_3\cdots i_p} = \frac{1}{p}\left(\mathfrak{D}^k_{(*_k,*_{k-1},\dots,*_1)}T\right)^{i_1\cdots i_p}_{,m} e^m_{(*_1,R)}. \tag{2.46}$$

Theorem 2.21. *Given a PA1-chain $\mathfrak{X} = (\mathbb{R}^n;*_1,*_2,\dots,*_M)$ and a chain-analytic tensor $S \in \mathfrak{T}_q(W_1;*_1,\dots,*_M)_{(z_0)}$, $p \geq 1$. Then the following symmetries hold:*

$$\begin{aligned}
&\left(\mathfrak{D}^k_{(*_k,*_{k-1},\dots,*_1)}S\right)_{i_1\cdots i_q,\alpha} \\
&\quad= \left(\mathfrak{D}^{k+1}_{(*_{k+1},*_k,\dots,*_1)}S\right)_{li_2\cdots i_q} C^{l}_{(*_{k+1})i_1\alpha} \\
&\qquad+ \left(\mathfrak{D}^{k+1}_{(*_{k+1},*_k,\dots,*_1)}S\right)_{i_1li_3\cdots i_q} C^{l}_{(*_{k+1})i_2\alpha} + \cdots + \left(\mathfrak{D}^{k+1}_{(*_{k+1},*_k,\dots,*_1)}S\right)_{i_1i_2\cdots i_{q-1}l} C^{l}_{(*_{k+1})i_q\alpha},
\end{aligned} \tag{2.47}$$

for all $\alpha \in I_{1,n_1}, i_1,\dots,i_p \in I_{1,n_2}, k \in I_{1,M-1}$ (In the case $k = 0$ the left side must be interpreted as $S_{i_1\cdots i_q,\alpha}$). An inverse relation exists, if the algebra $(\mathbb{R}^n;*_1)$ has a right unit $e_{(*_1,R)}$:

$$\left(\mathfrak{D}^{k+1}_{(*_{k+1},*_k,\dots,*_1)}S\right)_{i_1i_2\cdots i_q} = \frac{1}{q}\left(\mathfrak{D}^k_{(*_k,*_{k-1},\dots,*_1)}S\right)_{i_1\cdots i_q,m} e^m_{(*_1,R)}. \tag{2.48}$$

The proofs of Theorems 2.20 and 2.21 run the same lines as the proofs of Theorems 2.10 and 2.12. Relations (2.45) and (2.47) will be denoted as *Pseudo Cauchy Riemann equations (PCRE)* as in the simple case of a single vector space and a single product operation. (The expression $\mathfrak{D}^0_{(*_0,*_{0-1},\dots,*_1)}T$ must be interpreted as T, and $\mathfrak{D}^0_{(*_0,*_{0-1},\dots,*_1)}S$ as S.) Now we will generalize the former concept of $*$-integration.

Definition 2.22. Given an arbitrary algebra $\mathfrak{X} = (\mathbb{R}^n; *_1, *_2, \ldots, *_M)$*and chain-analytic tensors* $T \in \mathfrak{T}^p(W_1; *_1, \ldots, *_M)_{(z_0)}, S \in \mathfrak{T}_q(W_1; *_1, \ldots, *_M)_{(z_0)}, p \geq 1$, and further let $(\mathbb{R}^n; *_0)$ an algebra with the property

$$(\mathbb{R}^n; *_1, *_0) \text{ is a PA1-structure,} \tag{2.49}$$

Then the series

$$\begin{aligned}
\left(\mathfrak{D}^{-1}_{(*_0)}T\right)^{i_1\cdots i_p}(z) &:= \left\{\sum_{\alpha\geq 0}^{M} P^{[\alpha+1]}_{(*_0,*_1,\ldots,*_\alpha)}\left(\frac{1}{\alpha+1}A_{[\alpha]}, z-z_0\right)\right\}^{i_1\cdots i_p} + K^{i_1\cdots i_p} \\
&:= \Big\{K + A_{[0]}\circledast_{(*_0)}(z-z_0) + \left(\frac{1}{2}A_{[1]}\circledast_{(*_1)}(z-z_0)\right)\circledast_{(*_0)}(z-z_0) \\
&\quad + \left(\left(\frac{1}{3}A_{[2]}\circledast_{(*_2)}(z-z_0)\right)\circledast_{(*_1)}(z-z_0)\right)\circledast_{(*_0)}(z-z_0) + \cdots\Big\}^{i_1\cdots i_p}, \\
\left(\mathfrak{D}^{-1}_{(*_0)}S\right)_{i_1\cdots i_q}(z) &:= \left\{\sum_{\alpha\geq 0}^{M} P^{\mathrm{dual}[\alpha+1]}_{(*_0,*_1,\ldots,*_\alpha)}\left(\frac{1}{\alpha+1}B_{[\alpha]}, z-z_0\right)\right\}_{i_1\cdots i_q} + L_{i_1\ldots i_q} \\
&:= \Big\{L + B_{[0]}\circledast^{\mathrm{dual}}_{(*_0)}(z-z_0) + \left(\frac{1}{2}B_{[1]}\circledast^{\mathrm{dual}}_{(*_1)}(z-z_0)\right)\circledast^{\mathrm{dual}}_{(*_0)}(z-z_0) \\
&\quad + \left(\left(\frac{1}{3}B_{[2]}\circledast^{\mathrm{dual}}_{(*_2)}(z-z_0)\right)\circledast^{\mathrm{dual}}_{(*_1)}(z-z_0)\right)\circledast^{\mathrm{dual}}_{(*_0)}(z-z_0) + \cdots\Big\}^{i_1\cdots i_p},
\end{aligned} \tag{2.50}$$

with $z \in W_1$, K and L arbitrary constants, are called the $*_0$-*integrals* or the $*_0$-*antiderivatives* of the tensors T and S.

The denotation $\mathfrak{D}^{-1}_{(*_0)}$ is well chosen because it is $\mathfrak{D}_{(*_0)}\mathfrak{D}^{-1}_{(*_0)}T = \mathfrak{D}^{-1}_{(*_0)}\mathfrak{D}_{(*_0)}T$ and $\mathfrak{D}_{(*_0)}\mathfrak{D}^{-1}_{(*_0)}S = \mathfrak{D}^{-1}_{(*_0)}\mathfrak{D}_{(*_0)}S$. We see, that

$$\begin{aligned}
T \in \mathfrak{T}^p(W_1; *_1, \ldots, *_M)_{(z_0)} &\Longrightarrow \mathfrak{D}^{-1}_{(*_0)}T \in \mathfrak{T}^p(W_1; *_0, *_1, \ldots, *_M)_{(z_0)}, \\
S \in \mathfrak{T}_q(W_1; *_1, \ldots, *_M)_{(z_0)} &\Longrightarrow \mathfrak{D}^{-1}_{(*_0)}S \in \mathfrak{T}_q(W_1; *_0, *_1, \ldots, *_M)_{(z_0)}.
\end{aligned} \tag{2.51}$$

Proposition 2.23. *If* $\mathfrak{X}$ *is a PA1-chain, then the* $*_0$*-integrals* $\mathfrak{D}^{-1}_{(*0)}T$ *and* $\mathfrak{D}^{-1}_{(*0)}S$ *can be expressed by the classical integral conception as follows:*

$$\mathfrak{D}^{-1}_{(*_0)}T(z) = \int T(z)\circledast_{(*_0)}dz, \qquad \mathfrak{D}^{-1}_{(*_0)}S(z) = \int S(z)\circledast^{dual}_{(*_0)}dz. \tag{2.52}$$

Theorem 2.24. $\mathfrak{X}$ *a PA1-chain and* $\mathfrak{L}$ *a smooth closed curve in region* W_1*. Then it is*

$$\oint_{\mathfrak{L}} T(z)\circledast_{(*_0)}dz = 0, \qquad \oint_{\mathfrak{L}} S(z)\circledast^{dual}_{(*_0)}dz = 0. \tag{2.53}$$

This Theorem generalizes the well-known classical Cauchy Integral Theorem for GAPs and is denoted as *Pseudo Cauchy Integral Theorem (PCIT)*. For the next step we remember on the classical denotation $y' = dy/dx$ for a scalar differentiable function $y(x)$, or formally $y'dx = dy$. It shall be shown now, that for the $*$-derivative of vector-valued functions a similar result holds:

Proposition 2.25. *Given a PA1-chain $\mathfrak{X} = (\mathbb{R}^n; *_0, *_1, \ldots, *_M)$ and vectors $U \in \mathfrak{T}^1(W; *_0, *_1, \ldots, *_M)_{(z_0)}, V \in \mathfrak{T}_1(W; *_0, *_1, \ldots, *_M)_{(z_0)}$. Then it holds*

$$dU(z) = \mathfrak{D}_{(*_0)} U \circledast_{(*_0)} dz, \qquad dV(z) = \mathfrak{D}_{(*_0)} V \circledast^{dual}_{(*_0)} dz. \tag{2.54}$$

Proof. We start from the relation (2.52), with $T \in \mathfrak{T}^1(W_1; *_1, \ldots, *_M)_{(z_0)}$ and $S \in \mathfrak{T}_1(W_1; *_1, \ldots, *_M)_{(z_0)}$. Now we define $U(z) = \mathfrak{D}^{-1}_{(*_0)} T(z), V(z) = \mathfrak{D}^{-1}_{(*_0)} S(z)$, from which follows $U \in \mathfrak{T}^1(W; *_0, *_1, \ldots, *_M)_{(z_0)}, V \in \mathfrak{T}_1(W; *_0, *_1, \ldots, *_M)_{(z_0)}$ as it is assumed above. Inserting U and V in (2.52) we obtain $U(z) = \int \mathfrak{D}_{(*_0)} U \circledast_{(*_0)} dz, \Leftrightarrow dU(z) = \mathfrak{D}_{(*_0)} U \circledast_{(*_0)} dz$, and analog:$V(z) = \int \mathfrak{D}_{(*_0)} V \circledast^{\mathrm{dual}}_{(*_0)} dz, \Leftrightarrow dV(z) = \mathfrak{D}_{(*_0)} V \circledast^{\mathrm{dual}}_{(*_0)} dz$. □

We sum up in the following paragraph

In this section, the main earlier results have been generalized,along with the algebraic structures as well as the tensor fields built over these structures. In particular the PAk-structures have been generalized by the concept of the so-called "GAPs". We have investigated special GAPs like PA1 chains and their specializations (closed PA1-chains and closed PA1-chains of Lie-type), and have shown that the main results of the classical complex function theory (Cauchy Riemannian differential equations, Cauchy Integral Theorem) also hold in PA1-chains.

2.4. The Explicit Construction of GAPs

The aim of this section is the explicit construction of a wide variety of GAPs.

In other words, we have to construct structure constant tensors of pseudorings, PAk-structures, and PA1-chains explicitly. This problem makes sense, because without knowledge of concrete GAPs, the results of the sections above would be not applicable in practice. In the following, we introduce the maps $H_n : \mathbb{Z} \to]0, n]$, $n \in \mathbb{N}$ ($\mathbb{Z}$ the set of integers, $\mathbb{N}$ the set of natural numbers) with $n \geq 1$ by

$$H_n(x) = \begin{cases} x - mn, & \text{if } mn < x \leq (m+1)n, \\ x & \text{if } 0 < x \leq n, \\ x + mn & \text{if } -mn < x \leq (-m+1)n, \end{cases} \tag{2.55}$$

where $m \in \mathbb{N}$ denotes the smallest possible value, defined uniquely by the variable x and the conditions above (For example, let us consider the case $n = 3, x = 3.7$. Then m is uniquely defined by $m = 1$, since it is $mn < x \leq (m+1)n$ only for $m = 1$. Then it is $H_n(x) = x - mn = 0.7$). From this definition it follows immediately $H_n(x + y) = H_n(y + x)$, and $H_n(x + H_n(y + z)) = H_n(y + H_n(z + x)) = H_n(z + H_n(x + y)) = H_n(x + y + z)$.

Proposition 2.26. *Given the constants $P \in \mathbb{N}, M \in I_{1,P}$, and $A^M \in \mathbb{R}, a^M, \sigma \in \mathbb{Z}$. Then PAk-structures $(\mathbb{R}^n; *_1, *_2)$ are given by the following structure constant tensors:*

$$\text{case } k = 1 : C^k_{(*_1)ij} = \sum_{M=1}^{P} A^M \delta^k_{H_n(a^M i+\sigma j)}, \quad C^m_{(*_2)kl} = \delta^m_{H_n(k+\sigma l)}, \tag{2.56a}$$

$$\text{case } k = 2 : C^k_{(*_1)ij} = \delta^k_{H_n(\sigma i+j)}, \quad C^m_{(*_2)kl} = \sum_{M=1}^{P} A^M \delta^m_{H_n(a^M k+\sigma l)}. \tag{2.56b}$$

Proof. We show that the algebra (2.56a) satisfies the PA1-condition $C^k_{(*_1)i[j} C^m_{(*_2)|k|l]} = 0$. Writing all sums explicitely we get $\sum_{k=1}^n C^k_{(*_1)ij} C^m_{(*_2)kl} = \sum_{k=1}^n \sum_{M=1}^P A^M \delta^k_{H_n(a^M i+\sigma j)} \delta^m_{H_n(k+\sigma l)} = \sum_{M=1}^P A^M \delta^m_{H_n(H_n(a^M i+\sigma j)+\sigma l)}$, because $\delta^k_{H^n(a^M i+j)}$ only delivers a nontrivial value for the index value $k = H^n(a^M i + j)$ and this index k must be inserted in the term $\delta^m_{H_n(k+\sigma l)}$. Since it is $H_n(H_n(a^M i + \sigma j) + \sigma l) = H_n(a^M i + \sigma j + \sigma l)$, the right side above is symmetrical in indices j and l. Thus the left side is symmetrical in indices j and l, that is, the PA1-symmetry holds and statement (2.56a) has been proved. Now we have to show that algebra (2.56b) satisfies the PA2-condition $C^k_{(*_2)i[j} C^m_{(*_1)l]k} = 0 : \sum_{k=1}^n C^k_{(*_2)ij} C^m_{(*_1)lk} = \sum_{k=1}^n \sum_{M=1}^P A^M \delta^k_{H^n(a^M i+\sigma j)} \delta^m_{H^n(\sigma l+k)} = \sum_{M=1}^P A^M \delta^m_{H^n(\sigma l+H^n(a^M i+\sigma j))} = \delta^m_{H^n(\sigma l+a^M i+\sigma j)}$. The right side is symmetrical in indices j and l, and so the left side is, that is, the PA2-symmetry holds and statement (2.56b) has been proved. □

Proposition 2.27. *Let be $n \in \mathbb{N}, \sigma \in \mathbb{Z}$ arbitrary. Then the structure constant tensor*

$$C^k_{(*)ij} = \delta^k_{H_n(i+\sigma j)}, \quad i, j, k \in I_{1,n} \tag{2.57}$$

defines a pseudoring of first type $(\mathbb{R}^n; *)$. In the case $\sigma = 1$, a unit element exists, given by $e^j_{(*)} = \delta^j_n$, and furthermore it hold s that

$$\exists \lambda \in \mathbb{R}^n, \quad \lambda \neq 0 \quad \textit{with } \nexists \lambda^{-1_{(*)}}, \tag{2.58a}$$

$$\textit{A non existing } \lambda^{-1_{(*)}} \textit{ does not imply } \lambda^2 = 0. \tag{2.58b}$$

Because the PAk-structures above are defined by Kronecker-symbols, we will denote them as *PAk-structures of Kronecker-type* or shorter as *Kronecker PAk-structures*. According to the property (2.58a) a commutative Kronecker pseudoring is not a division algebra. It is easy to show, that this property also holds for noncommutative Kronecker pseudorings. Also the property (2.58a) can be generalized for noncommutative Kronecker pseudorings, where $\lambda^{-1_{(*)}}$ must be replaced by $\lambda^{-1_{(*,R,R)}}$, and so forth. (see the definition of $\lambda^{-1_{(*,R,R)}}, \lambda^{-1_{(*,R,L)}}, \lambda^{-1_{(*,L,R)}}, \lambda^{-1_{(*,L,L)}}$ in (2.2)). We see that a Kronecker algebra has a very simple structure, because the values of the structure constants are given by only two reals: 0 and 1. Now another type of PAk-structures will be presented which is more subtle than the Kronecker type.

Proposition 2.28. *Given an algebra $(\mathbb{R}^n; *_1, *_2)$ as follows:*

$$\begin{gathered} C^{\,h}_{(*_\alpha)ij} = d_i \delta^h_j + b_{(*_\alpha)} d_j \delta^h_i + d_i d_j h^h_{(*_\alpha)}, \quad \alpha = 1, 2, \\ \left(\textit{in formal denotation} : C_{(*_\alpha)} = d \otimes I + b_{(*_\alpha)} I \otimes d + d \otimes d \otimes h_{(*_\alpha)}\right) \end{gathered} \tag{2.59}$$

with $d, h_{(*_1)}, h_{(*_2)} \in \mathbb{R}^n$, $d \neq 0, b_{(*_1)}, b_{(*_2)} \in \mathbb{R}$ *arbitrary nonzero constants. Then* $(\mathbb{R}^n; *_1, *_2)$ *is a PAα-structure if and only if*

$$\left\langle d, h_{(*_\alpha)} \right\rangle = -1 + \sigma(\alpha)\left\{ b_{(*_2)} - b_{(*_1)}^{\sigma(\alpha)} \right\} \quad \text{with } \sigma(\alpha) = (-1)^{\alpha-1}, \alpha = 1, 2. \tag{2.60}$$

$(\mathbb{R}^n; *_\alpha), \alpha = 1, 2$*has a right unit* $e_{(*_\alpha,R)}$ *if and only if*

$$\left\langle d, h_{(*_\alpha)} \right\rangle = -1, \text{Then it is } e_{(*_\alpha,R)} = -b_{(*_\alpha)}^{-1} h_{(*_\alpha)}. \tag{2.61}$$

The PAk-structures presented above are built by combinations of rank 2-tensors and rank 1-tensors. Therefore we will denote these structures as *PAk-structures of splitting type* or shorter as *splitting PAk-structures.* From this follows immediately the structure of *splitting pseudorings*. It is easy to show, that Kronecker PAk-structures generally cannot be transformed onto splitting PAk-structures by basistransformations in $\mathbb{R}^n$, that is, these both structures are really different.

Splitting pseudorings have a large advantage to Kronecker pseudorings because it allow the explicit calculation of $\sqrt[p]{z}, z^{-1}$, and so forth.

Proposition 2.29. *Given the structure constant tensor*

$$\begin{gathered} C_{(*)ij}^k = d_i \delta_j^k + b d_j \delta_i^k + d_i d_j h^k \text{ with } \ h^k d_k = -1, \\ \left(\text{in formal denotation} : C_{(*_\alpha)} = d \otimes I + bI \otimes d + d \otimes d \otimes h, \text{ with } \langle h, d \rangle = -1\right) \end{gathered} \tag{2.62}$$

$i, j, k \in I_{1,n}, b \in \mathbb{R}$ *only restricted by* $b \neq 0$. *Then the following statements hold:*

Statement 1. The algebra $(\mathbb{R}^n; *)$ *defines an n-dimensional pseudoring of first type with right unit* $e_{(*,R)} = -b^{-1} h$.

Statement 2. In this algebra a vector z has a right invers element $z^{-1_{(*,R,R)}}$ *in respect of the right unit, if and only if* $\langle z, d \rangle \neq 0$.

Statement 3. In $(\mathbb{R}^n; *)$ *the power functions, roots and inverses of vectors are calculable explicitly as follows:*

$$z^{[P]} = b^{P-2}\left\{ (b + P - 1)\langle d, z\rangle^{P-1} z + (P-1)\langle d, z\rangle^P h \right\}, \quad P \in \mathbb{N}, \tag{2.63}$$

$$z^{-1_{(*,R,R)}} = -b^{-2}\left\{ b\langle d, z\rangle^{-2} z + (1+b)\langle d, z\rangle^{-1} h \right\}, \tag{2.64}$$

where $z^{[P]}$ is defined by $z^{[P]} = (\cdots((z * z) * z)\cdots) * z$. In the case $b = 1$ the relation (2.63) also holds for arbitrary rational numbers.

Statement 4. For $b = 1$ it holds

$$
\begin{gathered}
A \text{ non existing } \lambda^{-1_{(*)}} \text{ implies } \lambda^2 = 0, \\
U \in \mathfrak{T}^1(W;*)_{(z_0)} \Longrightarrow \lambda < d, \quad U > \in \mathfrak{T}^1(W;*)_{(z_0)} \quad \text{iff } \nexists \lambda^{-1_{(*)}}, \\
U \in \mathfrak{T}_1(W;*)_{(z_0)} \Longrightarrow d < \lambda, \quad U > \in \mathfrak{T}_1(W;*)_{(z_0)} \quad \text{iff } \nexists \lambda^{-1_{(*)}}.
\end{gathered} \tag{2.65}
$$

Proposition 2.29 shows, that in the framework of splitting pseudorings the inverse elements as well as the roots of vectors are calculable explicitly, that is, without using numerical methods. In Kronecker algebras this is not possible. Now we will show, that PA1-chains and furthermore PA1-chains of Lie-type can be constructed explicitly, proving, that the world of the PAk-structures is compatible with the world of generalized Lie-symmetries:

Proposition 2.30. *The following algebraic structures define PA1-chains $\mathfrak{X} = (\mathbb{R}^n; *_1, *_2, \ldots, *_M)$:Kronecker PA1-chains:*

$$
C^k_{(*_\alpha)ij} = \delta^k_{H_n(i+\sigma j)}, \quad \alpha = 1, \ldots, M, \ \sigma \in \mathbb{Z} \text{ arbitrary}, \tag{2.66a}
$$

Splitting PA1-chains:

$$
\begin{gathered}
C_{(*_\alpha)ij}^{\ k} = d_i \delta^k_j + b_{(*_\alpha)} d_j \delta^k_i + d_i d_j h^k_{(*_\alpha)}, \quad \alpha \in I_{1,M}, \\
\text{with } 1 + b_{(*_\alpha)} + \langle d, h_{(*_\alpha)} \rangle = b_{(*_{\alpha-1})}, \quad \alpha \in I_{2,M}.
\end{gathered} \tag{2.66b}
$$

Modified Splitting

$$
\begin{gathered}
C^k_{(*_\alpha)ij} = d_{(*_\alpha)i} A^k_{(*_\alpha)j} + g_j B^k_{(*_\alpha)i}, \quad \alpha \in I_{1,M}, \\
\text{with the restrictions}
\end{gathered} \tag{2.66c}
$$

PA1-chains:

$$
\begin{gathered}
A^k_{(*_{\alpha+1})j} d_{(*_\alpha)k} = 0, \quad B^k_{(*_{\alpha+1})j} d_{(*_\alpha)k} = u d_{(*_{\alpha+1})j}, \\
A^k_{(*_{\alpha+1})j} B^m_{(*_\alpha)k} = u A^m_{(*_\alpha)j}, \quad u \in \mathbb{R} \text{ arbitrary}.
\end{gathered} \tag{2.66d}
$$

Proposition 2.31. *The following algebraic structure $(\mathbb{R}^n; *_1, *_2, *_3)$ defines a closed PA1-chain from Lie type:*

$$
\begin{gathered}
C_{(*_\alpha)ij}^{\ k} = d_i \delta^k_j + b_{(*_\alpha)} d_j \delta^k_i + d_i d_j h^k_{(*_\alpha)}, \quad \alpha \in I_{1,3}, \quad \text{with} \\
b_{(*_1)} = \frac{1}{3} \left(\langle h_{(*_2)}, d \rangle - \langle h_{(*_1)}, d \rangle \right), \\
b_{(*_2)} = -1 + \frac{2}{3} \langle h_{(*_2)}, d \rangle + \frac{1}{3} \langle h_{(*_1)}, d \rangle, \\
b_{(*_3)} = 1 + \langle h_{(*_1)}, d \rangle, \\
h^m_{(*_1)} b_{(*_2)} + h^m_{(*_2)} b_{(*_3)} + h^m_{(*_3)} b_{(*_1)} = 0.
\end{gathered} \tag{2.67}
$$

Until now we have delt with special PAk -structures (Kronecker structures and splitting structures). Now we will analyze the problem, if PAk-structures can be combined in a way, that we come to PAk-structures again. There are two ways to analyze this problem. The first way is to try a construction of new n-dimensional PAk-structure by combining n-dimensional PAk-structures, the second way is to try a construction of new N-dimensional PAk-structure by combining n-dimensional PAk-structures, $N > n$. We will give answers to both problems by the following both propositions.

Proposition 2.32. *Given a GAP* $\mathfrak{X} = (\mathbb{R}^n; *_1, *_2, *_3, *_4)$ *with the characteristic*

$$G_{(\mathfrak{X})} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{2.68}$$

Then the algebra $\mathfrak{Y} = (\mathbb{R}^n; \circledast_1, \circledast_2)$ *defined by*

$$C^k_{(\circledast_1)ij} := C^k_{(*_1)ij} + C^k_{(*_2)ij}, \quad C^k_{(\circledast_2)ij} := C^k_{(*_3)ij} + C^k_{(*_4)ij} \tag{2.69}$$

is a PA1-structure, if and only if

$$C^k_{(*_2)i[j} C^m_{(*_3)|k|l]} = -C^k_{(*_1)i[j} C^m_{(*_4)|k|l]}. \tag{2.70}$$

Given a GAP $\mathfrak{X} = (\mathbb{R}^n; *_1, *_2, *_3, *_4)$ *with the characteristic:*

$$G_{(\mathfrak{X})} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{2.71}$$

Then the algebra $\mathfrak{Z} = (\mathbb{R}^n; \circledcirc_1, \circledcirc_2)$ *defined by*

$$C^k_{(\circledcirc_1)ij} := C^s_{(*_1)ij} C^k_{(*_2)sp} u^p, \quad C^k_{(\circledcirc_2)ij} := C^{\ s}_{(*_3)ij} C^k_{(*_4)sp} v^p, \quad u, v \in \mathbb{R}^n \text{ arbitrary} \tag{2.72}$$

is a PA1-structure.

Proof. To prove the first statement we calculate

$$\begin{aligned}
C^{k}_{(\circledast_1)ij}C^{m}_{(\circledast_2)kl} &= \left(C_{(*_1)ij}^{\ k} + C_{(*_2)ij}^{\ k}\right)\left(C^{m}_{(*_3)kl} + C^{m}_{(*_4)kl}\right) \\
&= C_{(*_1)ij}^{\ k}C^{m}_{(*_3)kl} + C_{(*_1)ij}^{\ k}C^{m}_{(*_4)kl} + C_{(*_2)ij}^{\ k}C^{m}_{(*_3)kl} + C_{(*_2)ij}^{\ k}C^{m}_{(*_4)kl}, \\
&\Longrightarrow C^{k}_{(\circledast_1)i[j}C^{m}_{(\circledast_2)|k|l]} \\
&= C_{(*_1)i[j}^{\ k}C^{m}_{(*_3)|k|l]} + C_{(*_1)i[j}^{\ k}C^{m}_{(*_4)|k|l]} + C_{(*_2)i[j}^{\ k}C^{m}_{(*_3)|k|l]} + C_{(*_2)i[j}^{\ k}C^{m}_{(*_4)|k|l]} \\
&\overset{\text{GAP-symmetries(2.69)}}{=} C_{(*_1)i[j}^{\ k}C^{m}_{(*_4)|k|l]} + C_{(*_2)i[j}^{\ k}C^{m}_{(*_3)|k|l]}.
\end{aligned} \tag{2.73}$$

We see, that the PA1 -symmetry $C^{k}_{(\circledast_1)i[j}C^{m}_{(\circledast_2)|k|l]} = 0$ holds if and only if $C_{(*_1)i[j}^{\ k}C^{m}_{(*_4)|k|l]} + C_{(*_2)i[j}^{\ k}C^{m}_{(*_3)|k|l]} = 0$, according to (2.70). To prove the second statement we calculate

$$\begin{aligned}
C^{k}_{(\circledcirc_1)ij}C^{m}_{(\circledcirc_2)kl} &= C^{s}_{(*_1)ij}C^{k}_{(*_2)sp}u^{p}C_{(*_3)kl}^{\ r}C^{m}_{(*_4)rq}v^{q}, \\
&\Longrightarrow C^{k}_{(\circledcirc_1)i[j}C^{m}_{(\circledcirc_2)|k|l]} = C_{(*_1)i[j}^{\ s}C^{k}_{(*_2)|sp}u^{p}C^{r}_{(*_3)k|l]}C^{m}_{(*_4)rq}v^{q} \\
&\overset{\text{GAP-symmetries (2.72)}}{=} C_{(*_1)i[j}^{\ s}C^{k}_{(*_2)|s|l]}u^{p}C^{r}_{(*_3)kp}C^{m}_{(*_4)rq}v^{q} \overset{\text{GAP-symmetries (2.72)}}{=} 0.
\end{aligned} \tag{2.74}$$

□

Proposition 2.33. *Given a PA1-structure* $\mathfrak{X} = (\mathbb{R}^{n}; *_1, *_2)$. *Then the following algebra* $\mathfrak{Y} = (\mathbb{R}^{2n}; \circledast)$ *is a pseudoring of first type:*

$$C^{k}_{(\circledast)ij} = \begin{cases} C_{(*_1)ij}^{\ k}, & \text{if } i,j,k \in I_{1,n}, \\ C_{(*_2)i-nj-n}^{\ k-n}, & \text{if } i,j,k \in I_{n+1,2n}, \\ 0, & \text{in all other cases}. \end{cases} \tag{2.75}$$

Proof. Per defnition it is

$$C^{k}_{(\circledast)ij}C^{m}_{(\circledast)kl} = \sum_{k=1}^{n} C^{k}_{(\circledast)ij}C^{m}_{(\circledast)kl} + \sum_{k=n+1}^{2n} C^{k}_{(\circledast)ij}C^{m}_{(\circledast)kl} \tag{2.76}$$

for arbitrary indices $i, j, l, m \in I_{1,2n}$. Let us consider first the case that $i \in I_{1,n}$. Then nontrivial elements of $C^{k}_{(\circledast)ij}$ only exist in the case $j, k \in I_{1,n}$, defined by $C^{k}_{(\circledast)ij} = C_{(*_1)ij}^{\ k}$, and the term above takes the form

$$C^{k}_{(\circledast)ij}C^{m}_{(\circledast)kl} = \sum_{k=1}^{n} C^{k}_{(\circledast)ij}C^{m}_{(\circledast)kl} = \sum_{k=1}^{n} C^{k}_{(*_1)ij}C^{m}_{(\circledast)kl}$$
$$= \begin{cases} 0, & \text{if } l > n \text{ or } m > n, \\ \displaystyle\sum_{k=1}^{n} C^{k}_{(*_1)ij}C^{\;m}_{(*_2)kl}, & \text{if } l, m \in I_{1,n}. \end{cases} \tag{2.77}$$

Thus it follows for $i \in I_{1,n}$:

$$C^{k}_{(\circledast)i[j}C^{m}_{(\circledast)|k|l]} = \left\{ \begin{matrix} 0, & \text{if } l > n \text{ or } m > n \\ \displaystyle\sum_{k=1}^{n} C^{\;k}_{(*_1)i[j}C^{m}_{(*_2)|k|l]}, & \text{if } l, m \in I_{1,n} \end{matrix} \right\} = 0 \tag{2.78}$$

in every case, because the algebra $(\mathbb{R}^n; *_1, *_2)$ is assumed as a PA1-structure, that is, it is $\sum_{k=1}^{n} C^{k}_{(*_1)i[j}C^{m}_{(*_2)|k|l]} = 0$ for $i \in I_{1,n}$.

Let us consider now the case that $i \in I_{n+1,2n}$. Then nontrivial elements of $C^{k}_{(\circledast)ij}$ only exist in the case $j, k \in I_{n+1,2n}$, defined by $C^{k}_{(\circledast)ij} = C^{\;k-n}_{(*_2)i-nj-n}$, and we get

$$C^{k}_{(\circledast)ij}C^{m}_{(\circledast)kl} = \sum_{k=n+1}^{2n} C^{k}_{(\circledast)ij}C^{m}_{(\circledast)kl} = \sum_{k=n+1}^{2n} C^{\;k-n}_{(*_1)i-nj-n}C^{m}_{(\circledast)kl}$$
$$= \begin{cases} 0, & \text{if } l \leq n \text{ or } m \leq n, \\ \displaystyle\sum_{k=n+1}^{2n} C^{k-n}_{(*_1)i-nj-n}C^{\;m-n}_{(*_2)k-nl-n}, & \text{if } l, m \in I_{n+1,2n}. \end{cases} \tag{2.79}$$

Thus it follows with the denotations $\bar{\imath} = i - n, \bar{\jmath} = j - n, \overline{k} = k - n, \bar{l} = l - n, \overline{m} = m - n$:

$$C^{k}_{(\circledast)i[j}C^{m}_{(\circledast)|k|l]} = \left\{ \begin{matrix} 0 & \text{if } l \leq n \text{ or } m \leq n, \\ \displaystyle\sum_{\bar{k}=1}^{n} C^{\bar{k}}_{(*_1)\bar{\imath}[\bar{\jmath}}C^{\overline{m}}_{(*_2)|\bar{k}|\bar{l}]} & \text{in all other cases,} \end{matrix} \right\} = 0. \tag{2.80}$$

in every case, because the indices $\bar{\imath}, \bar{\jmath}, \overline{k}, \bar{l}, \overline{m}$ move in the region $I_{1,n}$, and the algebra $(\mathbb{R}^n; *_1, *_2)$ is assumed as a PA1-structure. $\square$

In this section a wide variety of GAPs has been constructed explicitly. In particular two different types of explicit calculable PAk-structures have been characterized: Kronecker PAk-structures and (generalized) Splitting PAk-structures. Also PA1-chains and even PA1-chains of Lie-type can be constructed explicitly.

3. Applications of GAP-Theory on Solving Partial Differential Equations. New Explicit Solutions of Einstein Equations and Navier-Stokes Equations

In this chapter we will show, that GAP-Theory allows interesting applications on Theoretical Physics by calculating new explicit solutions of Einstein's field equations from General Relativity Theory and new explicit solutions of Navier-Stokes equations.

In some cases, the gained solutions are presented not only in the (compact and elegant) notation of GAP-Theory but also in the more circumstantial classical notation. This allows the checking of the results by readers without knowledges in GAP-Theory (Assumed are only the definitions of the vector spaces $\mathfrak{T}^1(W;*)_{(\xi_0)}$ and $\mathfrak{T}_1(W;*)_{(\xi_0)}$ from Section 2.1.) by inserting the presented solutions into the corresponding differential equations.

3.1. New Explicit Solutions of Einstein Equations

We start with Einsteins field equations

$$R_{ij} - \frac{1}{2}Rg_{ij} = -\kappa T_{ij}. \tag{3.1}$$

Here g_{ij} denotes the metric tensor of an n-dimensional Riemannian manifold $\mathfrak{M}$ with relativistic signature, R_{ij} the Ricci tensor defined by

$$\begin{gathered} R_{ij} = g^{mk}R_{mijk} = \frac{1}{2}g^{mk}\left(g_{mk,ij} + g_{ij,mk} - g_{ik,jm} - g_{jm,ik}\right) + g^{mk}g^{hl}\left(\Gamma_{hij}\Gamma_{lmk} - \Gamma_{hik}\Gamma_{lmj}\right), \\ \Gamma_{hij} = \frac{1}{2}\left(g_{hi,j} - g_{ij,h} + g_{jh,i}\right), \end{gathered} \tag{3.2}$$

Γ_{hij} the Christoffel symbols of first kind, and $R := g^{ij}R_{ij}$ the curvature scalar. The right side of Einstein equations is defined by the relativistic gravitational constant $\varkappa = 8\pi\gamma/c^4 = 1.86 \cdot 10^{-27}\,\mathrm{cm/g}$ (γ the classical gravitational constant) and the energy momentum tensor T_{ij} of the considered matter.

Einstein equations represent a system of nonlinear PDEs of second order to calculate the covariant components g_{ij} of the metric. The left side of these equations is linear in the second derivatives but nonlinear in the first derivatives of the covariant metric g_{ij} and also nonlinear in the covariant metric itself, because g^{mk} is a nonlinear function of g_{ij}. Generally also the right side of Einstein equations is nonlinear in the metric and the type of nonlinearity depends on the concrete physically situation.

In the past a lot of explicit solutions of Einstein equations have been constructed, but most of them are restricted on the very special case $R_{ij} = \lambda g_{ij}$ (Einstein spaces). Among these solutions only a small set is of physical importance because most of them have too special mathematical symmetries for describing physically gravitational fields. By contrast we will develope a variety of explicit solutions in the framework of GAP-Theory without assuming Einstein spaces.

Algebraic tools for solving Einstein equations have been used in [18, 19] (associative and commutative algebras) and [4] (nonassociative and noncommutative algebras; see also the overview given in [4]), discussions of certain generalizations of Einstein equations with algebra-valued metrics. Unlike we will deal here with the classical Einstein equations, that is, with real-valued, four dimensional metrics, and special GAPs are used for describing certain inherent symmetries of the explicit solutions. These GAPs differ from the algebras used in the papers cited above.

Finally some remarks to the denotions in this chapter are presented: η_{ij} is the covariante Minkowski metric, η^{jk} defined by $\eta_{ij}\eta^{jk} = \delta_j^k$. In relativistic theories, the indices run from 0 to 3 or—for generalizations—from 0 to n. In the framework of algebra theory, however, the left index range is not 0 but 1 and the relativistic denotion would lead to some problems and misunderstandings. For this we always will use the index range $1,\ldots,4$ or—for generalizations—from 1 to n. Furthermore, sometimes we will write $\langle u, v\rangle$ for $g_{ij}u^i v^j$.

Theorem 3.1 (in GAP-notation). *Given a four-dimensional splitting pseudoring $(R^4;*)$ with the splitting parameters b, d, h and the restrictions*

$$b = 1 \quad \text{or} \quad b = -3, \qquad d_i d_j \eta^{ij} = 0, \tag{3.3}$$

and further given an arbitrary vector field $a \in \mathfrak{T}^1(W;*)_{(\xi_0)}$, $W \subseteq \mathfrak{M}$, then all metric tensors of the form

$$g_{ij}(\xi) = a^m(\xi) C^l_{(*)m\{i}\eta_{j\}l} \tag{3.4}$$

($C_{(*)}$ the corresponding structure constant tensor, $C^l_{(*)m\{i}\eta_{j\}l} := C^l_{(*)mi}\eta_{jl} + C^l_{(*)mj}\eta_{il}$) are solutions of the four-dimensional vacuum Einstein equations in a local coordinate system (ξ).

Theorem 3.2 (in classical notation). *Given the reals b, d_i, h^i, $i = 1,\ldots,4$ with the restrictions*

$$b = 1 \quad \text{or} \quad b = -3, \qquad d_i d_j \eta^{ij} = 0, \qquad d_i h^i = -1, \tag{3.5}$$

And further given an arbitrary vector field $a \in \mathfrak{T}^1(W;*)_{(\xi_0)}$, $W \subseteq \mathfrak{M}$, then all metric tensors of the form

$$g_{ij}(\xi) = 2a^m(\xi) d_m \eta_{ij} + b\left[d_i a^m(\xi)\eta_{mj} + d_j a^m(\xi)\eta_{mi}\right] + (a^m(\xi) d_m)\left[d_i h^l \eta_{lj} + d_j h^l \eta_{li}\right] \tag{3.6}$$

are solutions of the four-dimensional vacuum Einstein equations in a local coordinate system (ξ).

The proof is omitted for space reason, the interested reader may insert the solution into the vacuum Einstein equations $R_{ij} = 0$. It is remarkable that vacuum solutions are realizable in the case of commutative algebras ($b = 1$) as well as in the case of noncommutative algebras ($b = -3$).

Now we will show that the vacuum metric above is a new unknown metric and not only a certain representation of the well-known Kerr metric or if the metric in [17]. For this we remember on the structure of the Kerr metric, which is given in Boyer-Lindquist coordinates t, r, ϕ, θ as follows:

$$ds^2 = \left(1 - \frac{r_s r}{\rho^2}\right) c^2 dt^2 + \frac{2 r_s r a \sin^2\theta}{\rho^2} c\, dt\, d\phi - \frac{\rho^2}{\Lambda^2} dr^2 - \rho^2 d\theta^2 - \left(r^2 + a^2 + \frac{r_s r a^2}{\rho^2} \sin^2\theta\right) d\phi^2, \tag{3.7}$$

r_s the Schwarzschild radius, a the Kerr parameters and $\rho^2 := r^2 + a^2 \cos^2\theta$, $\Lambda^2 := r^2 - r_s r + a^2$. In the following matrix form:

$$(g_{ij}) = \begin{pmatrix} \left(1 - \frac{r_s r}{\rho^2}\right) & 0 & \frac{2 r_s r a \sin^2\theta}{\rho^2} c & 0 \\ \cdots & -\frac{\rho^2}{\Lambda^2} & 0 & 0 \\ \cdots & \cdots & -\left(r^2 + a^2 + \frac{r_s r a^2}{\rho^2} \sin^2\theta\right) & 0 \\ \cdots & \cdots & \cdots & -\rho^2 \end{pmatrix}, \tag{3.8}$$

remember that the two fundamental symmetries of the Kerr metric—which describes an uncharged rotating black hole—are given by $(\partial/\partial t) g_{ij} = 0$ (time-independence) and $(\partial/\partial\phi) g_{ij} = 0$ (axial symmetry).

The matrix form of our metric (3.6) is given by

$$(g_{ij}) = \begin{pmatrix} 2(\psi + d_1 w_1) & d_1 w_2 + d_2 w_1 & d_1 w_3 + d_3 w_1 & d_1 w_4 + d_4 w_1 \\ \cdots & 2(-\psi + d_2 w_2) & d_2 w_3 + d_3 w_2 & d_2 w_4 + d_4 w_2 \\ \cdots & \cdots & 2(-\psi + d_3 w_3) & d_3 w_4 + d_4 w_3 \\ \cdots & \cdots & \cdots & 2(-\psi + d_4 w_4) \end{pmatrix}, \tag{3.9}$$

with $w_j(\xi) := \eta_{jk}(b a^k + h^k)$, $\psi(\xi) := a^k d_k$, $a \in \mathfrak{T}^1(W; *)_{(\xi_0)}$ an arbitrary field. It is easy to see that there does not exist any coordinate transformation such that the metric (3.9) takes the structure of the Kerr metric (3.8). The zeros in (3.8) cannot be realized by preserving the structure of the nontrivial elements, that is, *our vacuum metric is not from Kerr type*; the Kerr symmetries of time-independence and axial symmetry are replaced by the symmetry of $*$-analyticity.

It remains to show that the metric is not from the type presented in [17]. For this we compare the following:

$$\begin{aligned} &\text{Metric in [17]}: \; g_{ij}(\xi) = \eta_{ij} + C^l_{(*)\{ij\}} A_l(\xi) = \eta_{ij} + \left(C^l_{(*)ij} + C^l_{(*)ji}\right) A_l(\xi), \\ &\text{Metric above}: \; g_{ij}(\xi) = a^m(\xi) \left(C^l_{(*)m\{i}\eta_{j\}l} = a^m(\xi)\left(C^l_{(*)mi}\eta_{jl} + C^l_{(*)mj}\eta_{il}\right)\right. \end{aligned} \tag{3.10}$$

with $A \in \mathfrak{T}_1(W; *)_{(\xi_0)}$, $a \in \mathfrak{T}^1(W; *)_{(\xi_0)}$. It is easy to see that the metrics are of different types.

(i) The intrinsic symmetries of the vector fields $A \in \mathfrak{T}_1(W;*)_{(\xi_0)}$ and $a \in \mathfrak{T}^1(W;*)_{(\xi_0)}$ are of different types ($\mathfrak{T}_1(W;*)_{(\xi_0)} \neq \mathfrak{T}^1(W;*)_{(\xi_0)}$).

(ii) The formal structure of the metrics differs, because the structure constant tensor appears in different manners: ($C^l_{(*)ij} A_l(\xi) \neq a^m(\xi) C^l_{(*)mi}$, $C^l_{(*)\{ij\}} \neq C^l_{(*)m\{i}\eta_{j\}l}$).

(iii) The metric in [17] cannot describe nontrivial vacuum solutions (i.e., vacuum solutions for nonflat spaces; see [17]), whereas our new metric describes nontrivial vacuum solutions.

For these reasons, we recognize the metric presented here as a new, unknown vacuum metric. Now we will solve Einstein equations for a nonvanishing stress energy tensor.

Theorem 3.3 (in GAP-notation). *Given an n-dimensional splitting pseudoring $(\mathbb{R}^n;*)$ with the splitting parameters b, d, h and the restrictions*

$$d_i d_j \eta^{ij} = 0, \tag{3.11}$$

and further given the n-dimensional Einstein equations in a local coordinate system (ξ) in the form

$$\begin{gathered} R_{ij} - \frac{1}{2} R g_{ij} = -\varkappa \varepsilon d_i d_j \quad \text{with} \\ \varepsilon(\xi) = -\varkappa^{-1} b^2 (n-2) \frac{\left\langle w, \mathfrak{D}^2_{(*)} f \right\rangle - b \langle w, \mathfrak{D}_{(*)} f \rangle \langle d, \mathfrak{D}_{(*)} a \rangle}{2\left(1 + \langle w, \mathfrak{D}_{(*)} f \rangle\right)}, \qquad w^l d_l = 0, \end{gathered} \tag{3.12}$$

$a, f \in \mathfrak{T}_1(W;*)_{(\xi_0)}$, $W \subseteq \mathfrak{M}$ arbitrary vector fields, restricted only by the condition $a_i \neq \gamma d_i$, γ a scalar field, w^l constants, then these equations can be solved explicitly by

$$g_{ij}(\xi) = \left(1 + \langle w, \mathfrak{D}_{(*)} f \rangle\right) \eta_{ij} + \left(C^h_{(*)ij} + C^h_{(*)ji}\right) a_h(\xi), \tag{3.13}$$

$C_{(*)}$ the structure constant tensor of the splitting pseudoring.

Theorem 3.4 (in classical notation). *Given the reals b, d_i, h^i, w^i, $i = 1, \ldots, n$ with the restrictions*

$$d_i d_j \eta^{ij} = 0, \qquad h^i d_i = -1, \qquad w^l d_l = 0, \tag{3.14}$$

and further given the n-dimensional Einstein equations in a local coordinate system (ξ) in the form

$$R_{ij} - \frac{1}{2} R g_{ij} = -\varkappa \varepsilon(\xi) d_i d_j, \tag{3.15}$$

ε a scalar field, these equations can be solved explicitly by

$$g_{ij}(\xi) = \left(1 + w^m \left(\mathfrak{D}_{(*)} f\right)_m\right) \eta_{ij} + (b+1)\left[d_i a_j(\xi) + d_j a_i(\xi)\right] + 2 d_i d_j h^m a_m(\xi), \tag{3.16}$$

where $C_{(*)}$ *is the structure constant tensor of the splitting pseudoring,* $a, f \in \mathfrak{T}_1(W;*)_{(\xi_0)}$*,* $W \subseteq \mathfrak{M}$ *arbitrary vector fields, restricted only by the condition* $a_i \neq \gamma d_i$*,* γ *a scalar field. The concrete structure of the vector fields* a, f *defines the structure of the scalar field* ε*.*

The proof is omitted for space reason, the interested reader may insert the solution above into the Einstein equations. Theorem 3.4 is less general than Theorem 3.3 because nothing is told about the concrete structure of the scalar field ε. (The classical notation of ε would be very circumstantial.) We see that solutions of the inhomogeneous n-dimensional Einstein equations are realizable in the case of commutative algebras ($b = 1$) as well as in the case of noncommutative algebras ($b \neq 1$), and that a noncommutative algebra with $b = -1$ exists, where the solutions take the most simple structure.

Consider, furthermore, that Theorem 3.3 is not a simple generalization of Theorem 3.1 by generalizing the dimension and introducing a nonvanishing stress energy tensor. Theorem 3.1 works in the framework of $\mathfrak{T}^1(W;*)_{(\xi_0)}$, whereas Theorem 3.3 works in the framework of $\mathfrak{T}_1(W;*)_{(\xi_0)}$.

It remains to show that the metric in Theorem 3.3 differs from the metric presented in [17]. For this we compare the following:

$$\begin{aligned}
&\text{Metric in [17]}: \ g_{ij}(\xi) = \eta_{ij} + \left(C^l_{(*)ij} + C^l_{(*)ji}\right) A_l(\xi),\\
&\text{with arbitrary splitting parameters}\\
&\text{Metric above}: \ g_{ij}(\xi) = \left(1 + \langle w, \mathfrak{D}_{(*)} f\rangle\right)\eta_{ij} + \left(C^l_{(*)ij} + C^l_{(*)ji}\right) a_l(\xi)\\
&\text{with restricted splitting parameters } d_i d_j \eta^{ij} = 0,
\end{aligned} \tag{3.17}$$

with $A, a, f \in \mathfrak{T}_1(W;*)_{(\xi_0)}$. The metrics become identical if we restrict the algebraic framework of the first metric by $d_i d_j \eta^{ij} = 0$ and the second metric by $\mathfrak{D}_{(*)} f = 0$; in all other cases the metrics are different.

Finally some remarks to the physical importance of the stress energy tensor above are presented. For this we have a look on the following overview:

$$\begin{array}{rl}
\text{stress energy tensor} & \text{physical characterization}\\
T_{ij} = (p+\varepsilon)u_i u_j + p g_{ij} & \text{macroskopic matter}
\end{array} \tag{3.18a}$$

$$\begin{array}{rl}
T_{ij} = \dfrac{1}{4\pi}\left(F_{im}F^m_j - \dfrac{1}{4}F_{pq}F^{pq}g_{ij}\right) & \\
\text{with } F_{im} := A_{m;i} - A_{i;m} & \text{free electromagnetic stress energy tensor}
\end{array} \tag{3.18b}$$

$$T_{ij} = \mu_0 u_i u_j + \frac{1}{4\pi}\left(F_{im}F^m_j - \frac{1}{4}F_{pq}F^{pq}g_{ij}\right) \quad \text{incoherent matter in the electromagnetic field.} \tag{3.18c}$$

(i) The case (3.18a) describes macroscopic matter with velocity u_i, pressure p, and energy density ε.

(ii) The case (3.18b) describes the interchanging of the space-time structure with the electromagnetic field, where F denotes the antisymmetric electromagnetic field tensor, defined by the potential A.

(iii) The case in (3.18c) describes the interchanging of the space-time structure with incoherent matter in the electromagnetic field.

The stress energy tensor considered in Theorem 3.3 is defined by $T_{ij} = \varepsilon d_i d_j$, which allows a physical interpretation as macroscopic matter with velocity d_i (constancy by introducing a special coordinate system), energy density ε, and pressure $p = 0$. It can be shown that also special electromagnetic fields lead to the stress energy tensor above.

3.2. *New Explicit Solutions of Navier-Stokes Equations*

"Although these [Navier-Stokes] equations were written down in the 19th century, our understanding of them remains minimal. The challenge is to make substantial progress toward a mathematical theory which will unlock the secrets hidden in the Navier-Stokes equations."

With these words, the Clay Mathematic Institute describes a problem which is denoted as one of the seven "Millennium Problems" (see at http://www.claymath.org/). In the official formulation (The complete formulation can be found for instance in http://www.claymath.org./) by [20], the Navier-Stokes equations are presented as follows.

"The Euler and Navier-Stokes equations describe the motion of a fluid in $\mathbb{R}^n$ ($n = 2$ or 3). These equations are to be solved for an unknown velocity vector $u(x,t) = (u_i(x,t))_{1\le i\le n} \in \mathbb{R}^n$ and pressure $p(x,t) \in \mathbb{R}$, defined for position $x \in \mathbb{R}^n$ and time $t \ge 0$. We restrict attention here to incompressible fluids filling all of $\mathbb{R}^n$. The Navier-Stokes equations are then given by

$$\frac{\partial}{\partial t}u_i + \sum_{j=1}^{n} u_j \frac{\partial u_i}{\partial x_j} = \nu \Delta u_i - \frac{\partial p}{\partial x_i} + f_i(x,t), \quad x \in \mathbb{R}^n,\ t \ge 0, \tag{3.19a}$$

$$\operatorname{div} u = \sum_{j=1}^{n} \frac{\partial u_i}{\partial x_i} = 0, \quad x \in \mathbb{R}^n,\ t \ge 0, \tag{3.19b}$$

with initial conditions

$$u(x,0) = u^0(x), \quad x \in \mathbb{R}^n. \tag{3.20}$$

Here, $u^0(x)$ is a given, C^∞ divergence-free vector field on $\mathbb{R}^n$, $f_i(x,t)$ are the components of a given, externally applied force (e.g., gravity), ν is a positive coefficient (the viscosity), and $\Delta = \sum_{j=1}^{n}(\partial^2/\partial x_j^2)$ is the Laplacian in the space variables..."

Before presenting our new GAP-solution method, we remember some standard methods and main results of Navier-Stokes solution theory. Leray [21] has introduced the program of analyzing weak solutions of Navier-Stokes equations and he was able to show that in the three-dimensional case always exists a weak solution with suitable growth properties. Lerays method was taken and amended by Scheffer [22], Caffarelli et al. [23], Lin [24], Shnirelman [25], and others with the main result that the growth conditions of weak solutions can be classified quantitative [23, 24].

But at time it is not possible to perform the step from weak solutions to the physically interesting smooth solutions; we do not know if smooth initial conditions always induce smooth solutions or not. Also uniqueness of weak solutions is not known. Furthermore, the concrete construction of Navier-Stokes-solutions is a very hard problem. Standard methods for a nonnumerical construction are group methods, also the classical method of separation is used in a modified way, but actually the gained results are very small, which is not a surprise; all nonnumerical methods deal with very symmetric structures, and we cannot expect that such special tools delivers very general results.

In the official formulation [20] of the "Navier-Stokes Millenium Problem", these difficulties are summarized as follows:

"$\cdots$ *our understand [of the Navier-Stokes equations] is a very primitive level. Standard methods from PDE appear inadequate to settle the problem.*"

In the following, we will rewrite the Navier-Stokes system (3.19a) and (3.19b). With (In the following considerations, always a cartesian coordinate system is assumed. Thus the contravariant components u^i of the vector field u can be identified with the covariant components u_i from [20] in (3.19a) and (3.19b)). $u^i_{,j} := \partial u^i / \partial x^j$, $q^i := -p^{,i} + f^i$, the Einstein summation convention, and the generalization of $\mathbb{R}^n$ to a region $W \subseteq \mathbb{R}^n$ (in the case $W \neq \mathbb{R}^n$, adequate boundary conditions must be assumed), we get

$$u^i_{,t} - \nu \Delta u^i + u^j u^i_{,j} = q^i, \tag{3.21a}$$

$$u^i_{,i} = 0, \quad \text{for } x \in W,\ t \geq 0. \tag{3.21b}$$

We will solve these equations under the assumptions $u(x,t) = U(z(x,t))$, $q(x,t) = Q(z(x,t))$, where the n-dimensional vector fields $U(z)$ and $Q(z)$ are assumed as $*$-analytic in the n-dimensional vector z of the following structure:

$$z(x,t) = \lambda * x + \mu t, \quad W \subseteq \mathbb{R}^n,\ t \in [0, \infty[,\ z \in W_z. \tag{3.22}$$

Here $\lambda, \mu \in \mathbb{R}^n$ denote arbitrary fixed constants and $W_z \subseteq \mathbb{R}^n$ denotes the region of z.

Theorem 3.5. *Given a commutative pseudoring* $(\mathbb{R}^n; *)$ *with structure constants* $C^k_{(*)ij}$, $C^m_{(*)[3]} := C^{im}_{(*)i}$, $C_{(*)[1]m} := C^i_{(*)mi}$, *and further given two arbitrary vectors* $\lambda, \mu \in \mathbb{R}^n$, *the* n*-dimensional Navier-Stokes system* (3.21a) *and* (3.21b) *with viscosity coefficient*$\nu \in \mathbb{R}$ *and an arbitrary inhomogeneity of the symmetry*

$$\begin{gathered} q(x,t) = Q(z(x,t)), \\ Q \in \mathfrak{T}^1(W_z; *)_{(z_0)} \quad \text{an arbitrary field} \end{gathered} \tag{3.23}$$

with $z(x,t)$ *from* (3.22), *then the following statements hold.*

Statement 1. A solution $u(x,t)$ of the Navier-Stokes differential equations (3.21a) is given by

$$\begin{aligned}
&\text{(a)} \quad u(x,t) = U(z(x,t)),\\
&\text{(b)} \quad U(z) = -2\nu\lambda * C_{(*)[3]} * Y^{-1_{(*)}}(z) * \mathfrak{D}_{(*)}Y,\\
&\qquad Y(z) \quad \text{defined by the linear differential equation,}\\
&\text{(c)} \quad \nu^2\left(C_{(*)[3]}\right)^2 * \lambda^3 * \mathfrak{D}^2_{(*)}Y - \nu C_{(*)[3]} * \lambda * \mu * \mathfrak{D}_{(*)}Y - \frac{1}{2}\left(\mathfrak{D}^{-1}_{(*)}Q\right) * Y = 0.
\end{aligned} \tag{3.24}$$

Statement 2. If the inhomogeneity Q satisfies the condition

$$C^{[jk]}_{(*)i}\left(\mathfrak{D}_{(*)}Q\right)^i = 0, \tag{3.25}$$

it becomes interpretable as the pressure gradient $Q = \operatorname{grad} p$, and therefore the pressure is explicitly given by

$$p(x,t) = \int Q_i(z(x,t))dx^i. \tag{3.26}$$

Statement 3. If furthermore

$$C_{(*)[1]i}C^k_{(*)ij}\lambda^j = 0, \tag{3.27}$$

also the divergence condition (3.21b) is satisfied.

Before considering the proof structure (the extensive proof is omitted for space reason), we will give some hints to the statements above from a physical point of view as well as from a mathematical point of view. We begin by remembering that the entire Navier-Stokes-problem as formulated in (3.19a) and (3.19b) can be split up to three subproblems:

(1) the problem of solving the Navier-Stokes differential equation (3.19a) for an arbitrary given inhomogeneity ($-\partial p/\partial x_i + f_i$ must not be a gradient),

(2) the problem of restricting (3.19a) solutions by demanding inhomogeneities of gradient form.

(3) the problem of restricting (3.19a)-solutions (for gradient inhomogenities as well as for arbitrary inhomogeneities) by demanding the divergence condition (3.19b).

The solution of problem 1 is interesting from a mathematical point of view and the solution of problem 2 is interesting not only from a mathematical but also from a physical point of view, because for many practical applications, gradient inhomogeneities (pressure gradient) are assumed. Finally the solution of problem 3 (for gradient inhomogeneities) is interesting for getting a prize (see the "Millennium Problem").

Our statements in the Theorem 3.5 follow this classification: Statement 1 solves the problem 1 (without any restriction on the used algebraic framework), Statement 2 solves the problem 2 (by restricting the algebraic framework according to (3.25)), and Statement 3 gives a condition for solving the remaining problem 3 (by a further restriction of the algebraic framework according to (3.27)). It can be shown explicitly that algebras exist, such that the problems 1 and 2 can be solved explicitly (Kronecker-algebras, see the Section 2), but at time we do not know an algebra such that all three problems become solvable at the same time (for a new, unknown solution set).

Therefore, the statements of Theorem 3.5 are interesting from a mathematical point of view (Statement 1) as well as from a physical point of view (Statement 2), but—of course—the Millennium Problem remains unsolved.

Now some remarks to the solution structure (3.21a) and (3.21b) are presented.

(i) The solution structure holds for arbitrary commutative pseudorings; we are not restricted on splitting pseudorings as in the case of the Einstein solutions above.

(ii) The vector $Y^{-1_{(*)}}(z)$ appears, that is, the solution $U(z)$ generally will have singularities. In division algebras, these singularities are described entirely by values z_α, $\alpha = 1, 2, \ldots$ with the property $Y^{-1_{(*)}}(z_\alpha) = 0$, but in nondivision algebras there exist a larger set of singularities (see Section 2 and also the following considerations). In other words, the solution structure above allows an explicit description of the singularities depending on the chosen algebra.

(iii) It must be mentioned that our considerations here were restricted on the simple cases of commutative pseudorings and the special $\mathfrak{T}^1$-symmetry. Noncommutative pseudorings and generalizations of $\mathfrak{T}^1$-symmetry (or $\mathfrak{T}_1$-symmetry) lead to very complex Navier-Stokes solutions which cannot be presented in this framework.

Finally some hints to the proof-structure are presented.

(1) It is shown that in the case of a commutative pseudoring (which is assumed) the assumed symmetry $Q(z) \in \mathfrak{T}^1(W_z; *)_{(z_0)}$ of the inhomogeneity leads to the symmetry $u(x,t) = U(z) \in \mathfrak{T}^1(W_z; *)_{(z_0)}$.

(2) This property of U now allows the transformation of the Navier-Stokes differential equation (3.21a) onto a "*GAP-Riccati differential equation*" (i.e., an algebra valued Riccati equation).

(3) The assumed commutativity of the pseudoring now allows to solve this GAP-Riccati equation in the same way as the classical scalar Riccati-equation by transforming this equation onto a linear (algebra-valued) second-order differential equation (see (3.24)(c)).

(4) The explicit solution of this linear differential equation by comparison of coefficients is a simple problem, which only depends on the concrete structure of $\mathfrak{D}^{-1}_{(*)}Q$, that is, on the structure of the given inhomogeneity Q.

(5) After calculating the vector field $Y(z)$ from (3.24)(c), we get the vector field $U(z)$ according to (3.24)(b). The inverse vector field $Y^{-1_{(*)}}(z)$ is calculable explicitly for the physical interesting dimensions $n \leq 3$, in the case of a splitting pseudoring (see Section 2) also for arbitrary dimensions n, and therefore, $U(z)$ is calculable explicitly.

(6) After calculating the vector field $U(z)$, we get the Navier-Stokes solution $u(x,t)$ according to (3.24)(a) by inserting $z(x,t) = \lambda * x + \mu t$ (see (3.22)).

3.3. The Necessity of an Embedding Formalism

The results above show that GAP-Theory allows the explicit solution of important nonlinear partial differential equations from Theoretical Physics. For space reason, we have restricted these applications on Einstein field equations and Navier-Stokes equations, but GAP-Theory also allows applications on other famous differential equations as the Yang-Mills equation, the Korteweg-de Vries equation, and so forth.

The fundamental symmetry-framework of our considerations was the symmetry of $*$-analyticity (GAP-symmetry) which has turned out as "compatible" with the considered partial differential equations in the sense that it allows an explicit solution formalism. From the mathematical point of view, $*$-analyticity seems to be a useful tool to look inside the structure of covariant differential equations, but from the physical point of view $*$-analyticity represents a strong restriction: in many cases problem defining physically vector fields (tensor fields) will be not be $*$-analytical but only smooth.

To demonstrate the concrete problems, we have a look on the Theorem 3.5 for solving the Navier-Stokes system (3.21a) and (3.21b) in the framework of $*$-analyticity: Theorem 3.5 shows that a $*$-analytical inhomogeneity q allows an explicit solution of the Navier-Stokes differential equation (3.21a) for arbitrary commutative pseudorings, and that furthermore for a special set of these pseudorings also the divergence condition (3.21b) holds. But for many applications, the inhomogeneity q will not be $*$-analytical but a gradient, and in this case Theorem 3.5 is applicable only on $*$-analytical gradients, which represents a strong restriction.

Similar considerations hold for the Einstein equations: In Theorem 3.3 $*$-analyticity appears in the stress energy tensor (in a hidden form) as well as in the metric tensor, but a lot of physical interesting stress energy tensors will not be $*$-analytical.

These considerations boil down to the following question: is it possible to embed an arbitrary n-dimensional smooth vector field f (tensor field) into an N-dimensional $*$-analytical vector field F (tensor field) for a sufficient large dimension $N \geq n$ in the sense, that the first n components of F allow a representation of the n components of f? (An exact definition of "embedding" will be given in Section 4.)

In the case of a positive answer, *we can embed a n-dimensional nonlinear partial differential equation defined by only smooth vector fields onto a N-dimensional nonlinear partial differential equation defined by $*$-analytical vector fields, and this high-dimensional differential equation can be solved in the framework of GAP-Theory as demonstrated in the sections above*.

Finally the first n components of the obtained N-dimensional solution vector field (tensor field) represent the solution of the n-dimensional start problem.

Section 4 will show that such an Embedding formalism of smoothness into $*$-analyticity (or more general: chain analyticity) exists.

4. Embedding Problems for GAPs. New Products

Embedding problems (also denoted as Immersion problems) are important in mathematical physics as well as in nonlinear system theory [14, 16, 26, 27] and can be formulated in various ways. Roughly spoken immersion theory asks if an n-dimensional structure with small symmetry can be seen as a "projection" of an N-dimensional structure with high symmetry, $n \leq N$. Of course the answer depends on the concretely given symmetry classes as well as on the dimensions n and N.

In the following, we are interested on a special class of embedding problems, defined by the question if smoothness (which is a very general symmetry) can be embedded into chain-analyticity (which is a rather special symmetry).

Definition 4.1. Given an n-dimensional smooth vector field $f \in \mathfrak{T}^1(W^n)$, $W^n \subseteq \mathbb{R}^n$, and an N-dimensional chain-analytic vector field $F \in \mathfrak{T}^1(W^N; *_1, \ldots, *_M)_{(z_0)}$, $\mathfrak{X} = (\mathbb{R}^N; *_1, \ldots, *_M)$ an arbitrary PA1-chain, then f is denoted as embaddable into F if it holds that

$$f^\gamma(z) = F^\gamma(Z)|_{Z^{n+1}=Z^{n+2}\cdots=Z^N=0}, \quad \text{with } Z^\gamma = z^\gamma,\ \gamma \in I_{1,n}. \tag{4.1}$$

If this relation holds for arbitrary $f \in \mathfrak{T}^1(W^n)$, then $\mathfrak{T}^1(W^n)$ is denoted as embaddable into $\mathfrak{T}^1(W^N; *_1, \ldots, *_M)_{(z_0)}$.

Of course the possibility of embedding $\mathfrak{T}^1(W^n)$ into $\mathfrak{T}^1(W^N; *_1, \ldots, *_M)_{(z_0)}$ will depend on the concrete form of the given PA1-chain $\mathfrak{X}$ as well as on the proportion of n to N. Is $\mathfrak{T}^1(W^n)$ embaddable for N-dimensional Kronecker PA1-chains, N finite and/or for N-dimensional Splitting PA1-chains, N finite? Or must N be infinite? We will answer these questions by specializing on the subset $\mathfrak{T}^1_{[P]}(W^n) \subseteq \mathfrak{T}^1(W^n)$ defined by all polynomials of order $p \leq P \in \mathbb{N}$, and by specializing PA1-chains to pseudorings.

Theorem 4.2. *Given the following classes of N-dimensional pseudorings $\mathfrak{T}^1(\mathbb{R}^N; *)_{(z_0)}$:*

$$C^k_{(*)ij}(\mu) = \delta^k_{H_N(i+H_N(u_\mu(j)))}, \qquad u_\mu(j) := j^\mu, \qquad \mu > {}^2\log(n+1), \quad \mu \in \mathbb{N},\ i,j,k \in I_{1,N}, \tag{4.2}$$

with the index function H_N from (2.55)*, then the following statements hold.*

Statement 1. For every given pair $n, P \in \mathbb{N}$ there exist finite numbers $N(n, P, \mu) \in \mathbb{N}$ such that $\mathfrak{T}^1_{[P]}(W^n)$ is embaddable into $\mathfrak{T}^1_{[P]}(W^N; *)_{(0)}$, $0 \in W^n \subset W^N$.

Statement 2. A sufficient condition for embedding is given by

$$N \geq N_{\min}(n, P, \mu), \quad \text{with } N_{\min}(n, P, \mu) := n + n^\mu P. \tag{4.3}$$

Statement 3. Let $a^\gamma_{[p]\beta_1\beta_2\cdots\beta_p}$ with $\gamma, \beta_1, \ldots, \beta_p \in I_{1,n}$, $p \in I_{0,P}$ denote the coefficients of an arbitrary element $f \in \mathfrak{T}^1_{[P]}(W^n)$. Then under the assumption (4.3) the coefficients $A^i_{[p]}$, $p \in I_{0,P}$, $i \in I_{1,N}$ of the corresponding element $F \in \mathfrak{T}^1_{[P]}(W^N; *)_{(0)}$ are defined by

$$A^\gamma_{[0]} = a^\gamma_{[0]}, \qquad A^{\gamma+\sum_{k=1}^p (\beta_k)^\mu}_{[p]} = a^\gamma_{[p]\beta_1\beta_2\cdots\beta_p}, \quad p \in I_{1,P}. \tag{4.4}$$

Proof

Step 1. The pseudoring symmetry of $\mathfrak{T}^1(\mathbb{R}^N; *)_{(z_0)}$: for a given μ, it is

$$
\begin{aligned}
C^l_{(*)ij}(\mu)C^m_{(*)lk}(\mu) &= \delta^l_{H_N(i+H_N(u(j)))}\delta^m_{H_N(l+H_N(u(k)))} = \delta^m_{H_N(H_N(i+H_N(u(j)))+H_N(u(k)))} \\
&= \delta^m_{H_N(i+H_N(u(j))+u(k))} = \delta^m_{H_N(i+u(j)+u(k))}, \\
&\Longrightarrow C^l_{(*)i[j}(\mu)C^m_{(*)|l|k]}(\mu) = 0,
\end{aligned}
\tag{4.5}
$$

where we have used the property $H_N(H_N(k)+m) = H_N(k+m)$, and so forth.

Step 2. The conditions of embaddability $\mathfrak{T}^1_{[P]}(W^n)$ into $\mathfrak{T}^1_{[P]}(W^N;*_1,\ldots,*_P)_{(0)}$ in general PA1-chains: By definition every element $f \in \mathfrak{T}^1_{[P]}(W^n)$, $0 \in W^n$ has a representation (In the following we use Greek indices for the range $I_{1,n}$, Latin indices for the ranges $I_{1,N}$ and $I_{0,P}$.)

$$
f^\gamma(z) = a^\gamma_{[0]} + a^\gamma_{[1]\beta_1}z^{\beta_1} + a^\gamma_{[2]\beta_1\beta_2}z^{\beta_1}z^{\beta_2} + \cdots + a^\gamma_{[P]\beta_1\beta_2\cdots\beta_P}z^{\beta_1}z^{\beta_2}\cdots z^{\beta_P},
\tag{4.6}
$$

$z \in W^n$, where the coefficients $a^\gamma_{[p]\beta_1\cdots\beta_p} \in \mathbb{R}$ for $\gamma,\beta_1,\ldots,\beta_p \in I_{1,n}$, $p \in I_{0,P}$ can be assumed as total symmetric in the indices $\beta_1,\ldots,\beta_p$. Furthermore, for a general given PA1-chain $\mathfrak{X} = (\mathbb{R}^N;*_1,\ldots,*_P)$ every element $F \in \mathfrak{T}^1_{[P]}(W^N;*_1,\ldots,*_P)_{(0)}$ has a representation

$$
\begin{aligned}
F(Z) &= A_{[0]} + A_{[1]}*_1Z + (A_{[2]}*_2Z)*_1Z + ((A_{[3]}*_3Z)*_2Z)*_1Z \\
&\quad + \cdots + (\cdots((A_{[M]}*_PZ)*_{P-1}Z)*_{P-2}\cdots)*_1Z, \quad Z \in W^N
\end{aligned}
\tag{4.7}
$$

with certain vector-valued coefficients $A_{[p]} \in \mathbb{R}^N$, $p \in I_{0,P}$. In component representation,

$$
\begin{aligned}
F^k(X) &= A^k_{[0]} + A^i_{[1]}C^k_{(*_1)ij}Z^j + A^i_{[2]}C^{l_1}_{(*_2)ij_1}Z^{j_1}C^k_{(*_1)l_1j_2}Z^{j_2} \\
&\quad + \cdots + A^i_{[P]}C^{l_1}_{(*_P)ij_1}Z^{j_1}C^{l_2}_{(*_{P-1})l_1j_2}Z^{j_2}\cdots C^k_{(*_1)l_{P-1}j_P}Z^{j_P} \\
&\overset{\text{rewritten}}{=} A^k_{[0]} + A^i_{[1]}C^k_{(*_1)ij}Z^j + A^i_{[2]}C^{l_1}_{(*_2)ij_1}C^k_{(*_1)l_1j_2}Z^{j_1}Z^{j_2} \\
&\quad + \cdots + A^i_{[P]}C^{l_1}_{(*_P)ij_1}C^{l_2}_{(*_{P-1})l_1j_2}\cdots C^k_{(*_1)l_{P-1}j_P}Z^{j_1}Z^{j_2}{}^{j_P}.
\end{aligned}
\tag{4.8}
$$

The restriction on the subset $W^n \subseteq W^N$ by $Z^{n+1} = Z^{n+2}\cdots = Z^N = 0,\ Z^\alpha = z^\alpha,\ \alpha \in I_{1,n}$ leads to

$$
\begin{aligned}
F^\gamma(Z)|_{Z^{n+1}=Z^{n+2}\cdots=Z^N=0} &= A^\gamma_{[0]} + A^i_{[1]}C^\gamma_{(*_1)i\beta_1}z^{\beta_1} + A^i_{[2]}C^{l_1}_{(*_2)i\beta_1}C^\gamma_{(*_1)l_1\beta_2}z^{\beta_1}z^{\beta_2} \\
&\quad + \cdots + A^i_{[P]}C^{l_1}_{(*_P)i\beta_1}C^{l_2}_{(*_{P-1})l_1\beta_2}\cdots C^\gamma_{(*_1)l_{P-1}\beta_P}z^{\beta_1}z^{\beta_2}\cdots z^{\beta_P}, \quad \gamma \in I_{1,n},
\end{aligned}
\tag{4.9}
$$

and the embedding condition (4.1) takes the form

$$\begin{aligned} a^{\gamma}_{[0]} &+ a^{\gamma}_{[1]\beta_1} z^{\beta_1} + a^{\gamma}_{[2]\beta_1\beta_2} z^{\beta_1} z^{\beta_2} + \cdots + a^{\gamma}_{[P]\beta_1\beta_2\cdots\beta_P} z^{\beta_1} z^{\beta_2} \cdots z^{\beta_P} \\ &= A^{\gamma}_{[0]} + A^{i}_{[1]} C^{\gamma}_{(*_1)i\beta_1} z^{\beta_1} + A^{i}_{[2]} C^{l_1}_{(*_2)i\beta_1} C^{\gamma}_{(*_1)l_1\beta_2} z^{\beta_1} z^{\beta_2} \\ &\quad + \cdots + A^{i}_{[P]} C^{l_1}_{(*_P)i\beta_1} C^{l_2}_{(*_{P-1})l_1\beta_2} C^{\gamma}_{(*_1)l_{P-1}\beta_P} z^{\beta_1} z^{\beta_2} \cdots z^{\beta_P}, \quad \gamma \in I_{1,n}. \end{aligned} \tag{4.10}$$

Comparison of the components leads to the following equation system:

$$A^{\gamma}_{[0]} = a^{\gamma}_{[0]}, \qquad A^{i}_{[p]} C^{l_1}_{(*_p)i\beta_1} C^{l_2}_{(*_{p-1})l_1\beta_2} \cdots C^{\gamma}_{(*_1)l_{p-1}\beta_p} = a^{\gamma}_{[p]\beta_1\beta_2\cdots\beta_p}, \text{(b)}, \tag{4.11}$$

$\gamma, \beta_1, \ldots, \beta_p \in I_{1,n}$, $p \in I_{1,P}$, or in detailed denotion:

$$\begin{gathered} A^{\gamma}_{[0]} = a^{\gamma}_{[0]}, \\ A^{i}_{[1]} C^{\gamma}_{(*_1)i\beta_1} = a^{\gamma}_{[1]\beta_1}, \\ A^{i}_{[2]} C^{l_1}_{(*_2)i\beta_1} C^{\gamma}_{(*_1)l_1\beta_2} = a^{\gamma}_{[2]\beta_1\beta_2}, \\ A^{i}_{[3]} C^{l_1}_{(*_3)i\beta_1} C^{l_2}_{(*_2)l_1\beta_2} C^{\gamma}_{(*_1)l_2\beta_3} = a^{\gamma}_{[3]\beta_1\beta_2\beta_3}, \\ \vdots \\ A^{i}_{[P]} C^{l_1}_{(*_P)i\beta_1} C^{l_2}_{(*_{P-1})l_1\beta_2} \cdots C^{\gamma}_{(*_1)l_{P-1}\beta_P} = a^{\gamma}_{[P]\beta_1\beta_2\cdots\beta_P}, \end{gathered} \tag{4.12}$$

$\gamma, \beta_1, \ldots, \beta_p \in I_{1,n}$. We see that the total symmetry in the covariant indices of the right side is compatible with the total symmetry in the covariant indices of the left side which follows immediately from the PA1-chain symmetry. The question is now if this system is solvable for arbitrary given n, P and components $a^{\gamma}_{[P]\beta_1\cdots\beta_p}$, $\gamma, \beta_1, \ldots, \beta_p \in I_{1,n}$, $p \in I_{0,P}$, that is, if we can find an adequate dimension $N(n,P)$, adequate coefficients $A^{i}_{[\alpha]}$, $\alpha \in I_{0,P}$, $i \in I_{1,N}$, and an adequate PA1-chain with structure constants $C^{k}_{(*_\alpha)ij}$ such that (4.12) get solvable. Since not only the coefficients $A^{i}_{[\alpha]}$ but also the structure constants are unknown, the equation system is nonlinear.

Step 3. The conditions of embaddability $\mathfrak{T}^1_{[P]}(W^n)$ into $\mathfrak{T}^1_{[P]}(W^N;*)_{(0)}$ in the special PA1-chain (4.2): now we have to insert the pseudoring structure (4.2) in the equation system above.

For this we calculate

$$\begin{aligned}
C^{l_1}_{(*_2)i\beta_1} C^{\gamma}_{(*_1)l_1\beta_2} &= \delta^{l_1}_{H_N(i+H_N(u_\mu(\beta_1)))} \delta^{\gamma}_{H_N(l_1+H_N(u_\mu(\beta_2)))} \\
&= \delta^{\gamma}_{H_N(H_N(i+H_N(u_\mu(\beta_1)))+H_N(u_\mu(\beta_2)))} \\
&\overset{\text{see step 1 above}}{=} \delta^{\gamma}_{H_N(i+u_\mu(\beta_1)+u_\mu(\beta_2))}, \\
C^{l_1}_{(*_3)i\beta_1} C^{l_2}_{(*_2)l_1\beta_2} C^{\gamma}_{(*_1)l_2\beta_3} &= \delta^{l_1}_{H_N(i+H_N(u_\mu(\beta_1)))} \delta^{\gamma}_{H_N(l_1+u_\mu(\beta_2)+u_\mu(\beta_3))} \\
&= \delta^{\gamma}_{H_N(H_N(i+H_N(u_\mu(\beta_1)))+u_\mu(\beta_2)+u_\mu(\beta_3))} \\
&\overset{\text{see step 1 above}}{=} \delta^{\gamma}_{H_N(i+u_\mu(\beta_1)+u_\mu(\beta_2)+u_\mu(\beta_3))},
\end{aligned} \tag{4.13}$$

and we obtain for the general case

$$\begin{aligned}
C^{l_1}_{(*_p)i\beta_1} C^{l_2}_{(*_{p-1})l_1\beta_2} \cdots C^{\gamma}_{(*_1)l_{p-1}\beta_p} &= \delta^{\gamma}_{H_N(i+u_\mu(\beta_1)+u_\mu(\beta_2)+\cdots+u_\mu(\beta_p))} \\
&\overset{\text{inserting } u_\mu(\beta_m)}{=} \delta^{\gamma}_{H_N(i-(\beta_1)^\mu-(\beta_2)^\mu-\cdots-(\beta_p)^\mu)} \\
&= \delta^{\gamma}_{H_N(i-\sum_{k=1}^{p}(\beta_k)^\mu)}, \quad p \in I_{1,P}.
\end{aligned} \tag{4.14}$$

Inserting in the equation system (4.11), we get

$$A^{\gamma}_{[0]} = a^{\gamma}_{[0]}, \tag{4.15a}$$

$$A^{H_N(\gamma+\sum_{k=1}^{p}(\beta_k)^\mu)}_{[p]} = a^{\gamma}_{[p]\beta_1\beta_2\cdots\beta_p}, \tag{4.15b}$$

$\gamma, \beta_1, \beta_2, \ldots, \beta_p \in I_{1,n}$, $p \in I_{1,P}$. Now we have to show that for arbitrary given n, P and arbitrary components $a^{\gamma}_{[P]\beta_1\cdots\beta_p}$, there exist natural numbers $\mu(n,P)$ and $N(n,P,\mu)$ and coefficients $A^{i}_{[\alpha]}$, $\alpha \in I_{0,P}$, $i \in I_{1,N}$ with the property above. The positive answer will be given in the next step.

Step 4. The solvability of the equation system (4.15a) *and* (4.15b)*:* for the following considerations, we define the set

$$\overline{I}^{[p+1]}_{1,n} := \{(\gamma, \beta_1, \beta_2, \ldots, \beta_p) \quad \text{with } \gamma, \beta_i \in I_{1,n},\ \beta_{i+1} \geq \beta_i\}, \quad p \in I_{1,P}, \tag{4.16}$$

that is, $\overline{I}^{[p+1]}_{1,n}$ is a subset of the set $I^{[p+1]}_{1,n} := I_{1,n} \times I_{1,n} \times \cdots \times I_{1,n}$, ($p+1$ times) defined by the additional property $\beta_{i+1} \geq \beta_i$. We see that $\overline{I}^{[p+1]}_{1,n}$ is the definition region of the index field $(\gamma, \beta_1, \beta_2, \ldots, \beta_p)$ from the coefficients $a^{\gamma}_{[p]\beta_1\beta_2\cdots\beta_p}$ in (4.15b), since for these coefficients the total symmetry in the covariant indices hold by assumption. For the following, it is advantageous to write β instead of $(\beta_1, \beta_2, \ldots, \beta_p)$ and (γ, β) instead of $(\gamma, \beta_1, \beta_2, \ldots, \beta_p)$. With these denotions, we can formulate a necessary and sufficient condition for the solvability of (4.15a) and (4.15b) for arbitrary given n, P, and for arbitrary given coefficients $a^{\gamma}_{[p]\beta_1\beta_2\cdots\beta_p}$ as follows:

$$H_N\left(\gamma + \sum_{k=1}^{p} (\beta_k)^{\mu}\right) \neq H_N\left(\overline{\gamma} + \sum_{k=1}^{p} \left(\overline{\beta}_k\right)^{\mu}\right) \quad \text{for } (\gamma, \beta) \neq \left(\overline{\gamma}, \overline{\beta}\right), \tag{4.17}$$

with $(\gamma, \beta), (\overline{\gamma}, \overline{\beta}) \in \overline{I}^{[p+1]}_{1,n}$, $p \in I_{1,P}$ (if the following relation is not satisfied, that is, if there exists for a fixed p at least one pair of $(p+1)$-tuples $(\gamma, \beta) \neq (\overline{\gamma}, \overline{\beta})$ such that $H_N(\gamma + \sum_{k=1}^{p} (\beta_k)^{\mu}) = H_N(\overline{\gamma} + \sum_{k=1}^{p} (\overline{\beta}_k)^{\mu})$, the condition (4.15b) would imply for the fixed p: $a^{\gamma}_{[p]\beta_1\beta_2\cdots\beta_p} = a^{\overline{\gamma}}_{[p]\overline{\beta}_1\overline{\beta}_2\cdots\overline{\beta}_p}$, which means a specialization by the coefficients $a^{\gamma}_{[p]\beta_1\beta_2\cdots\beta_p}$). We have to show that this relation holds under our assumptions of $\mu(n, P)$ and $N(n, P, \mu) \geq N_{\min}(n, P, \mu)$ as stated in (4.2), (4.3). In the case $N(n, P, \mu) \geq N_{\min}(n, P, \mu) = n + n^{\mu}P$, it is $\gamma + \sum_{k=1}^{p} (\beta_k)^{\mu} \leq N_{\min}(n, P, \mu)$ for $p \in I_{1,P}$ which implies $H_N(\gamma + \sum_{k=1}^{p} (\beta_k)^{\mu}) = \gamma + \sum_{k=1}^{p} (\beta_k)^{\mu}$ for $p \in I_{1,P}$. Inserting in (4.15a) and (4.15b), we obtain the statement (4.4). The solvability condition (4.17) takes the form $\gamma + \sum_{k=1}^{p} (\beta_k)^{\mu} \neq \overline{\gamma} + \sum_{k=1}^{p} (\overline{\beta}_k)^{\mu}$ or rewritten as

$$\sum_{k=1}^{p} (\beta_k)^{\mu} - \sum_{k=1}^{p} (\beta_k)^{\mu} \neq \overline{\gamma} - \gamma, \quad \text{for } (\gamma, \beta) \neq \left(\overline{\gamma}, \overline{\beta}\right), \tag{4.18}$$

$(\gamma, \beta), (\overline{\gamma}, \overline{\beta}) \in \overline{I}^{[p+1]}_{1,n}$, $p \in I_{1,P}$. From the considerations above, we see that this condition is sufficient for the solvability of the the system (4.15b) for general given $n, P, a^{\gamma}_{[p]\beta_1\beta_2\cdots\beta_p}$. Since $\overline{\gamma} - \gamma$ moves in the region $[-n, n]$, the condition is certainly satisfied by demanding

$$\left| \sum_{k=1}^{p} (\overline{\beta}_k)^{\mu} - \sum_{k=1}^{p} (\overline{\beta}_k)^{\mu} \right| > n, \quad \text{for } \beta \neq \overline{\beta}, \tag{4.19}$$

$\beta, \overline{\beta} \in \overline{I}^{[p+1]}_{1,n}$, $p \in I_{1,P}$. The worst case of this new condition is given if only one component of $\overline{\beta}$ differs from one component of β. Let k_0 denote the corresponding index, then we obtain from above the worst case in the form $|(\overline{\beta}_{k_0})^{\mu} - (\overline{\beta}_{k_0})^{\mu}| > n$, for $\beta_{k_0} \neq \overline{\beta}_{k_0}$. The worst case of this scenario is given by the minimal possible difference, that is, by $\beta_{k_0} = 1$, $\overline{\beta}_{k_0} = 2$. In this case μ is defined by the sufficient condition $|1 - 2^{\mu}| > n$ or $2^{\mu} > n + 1$ which is satisfied by $\mu > {}^{2}\log(n+1)$ as stated in (4.2).

I will denote this Theorem as the *Pseudoring Embedding Theorem*, and the pseudoring (4.2) as *embedding pseudoring*. The Theorem shows that a non $*$-analytic vector field always can be embedded into a higher-dimensional $*$-analytic vector field. Furthermore, this Theorem not only characterizes the possibility of embedding but also allows an explicit description of the practical embedding process. Some remarks for a deeper understanding are as follows.

(i) Since the map u_μ in the embedding pseudoring is not a linear map, the embedding pseudoring is neither commutative nor associative: embedding is not compatible with commutativity and associativity.

(ii) The embedding pseudoring is a Kronecker algebra, which leads to the question if also splitting pseudorings can be used for embedding. It can be shown that this is not possible in a general framework, that is, for embedding problems Kronecker algebras are more important than splitting algebras. Otherwise, the applications in [17] show that in the framework of solving partial differential equations the splitting algebras seem to be more important than Kronecker algebras. The upshot is that both algebra-types are important.

(iii) The embedding condition (4.3) on the embedding dimension $N \geq N_{\min}$ is a sufficient condition. It is possible that there exist cases where a smaller dimension $N \leq N_{\min}$ is purposeful.

(iv) Until now our embedding analyse was restricted on polynomials. It can be generalized on arbitrary smooth vector fields, but in this case the embedding dimension N will not remain finite, according to the embedding relation (4.3). Thus it would be interesting to investigate if (4.3) is not only sufficient but also necessary.

(v) Relation (4.4) shows that the coefficients $A^i_{[p]}$ of the vector field F are not defined entirely by the coefficients $a^\gamma_{[p]\beta_1\beta_2\cdots\beta_p}$ of the vector field f: a lot of coefficients $A^i_{[p]}$ remain unrestricted. In other words, embedding from f into F does not define F entirely but only partially.

(vi) Until now our investigation was fixed on the case of vector-field embedding problems. More general is the case of tensor field embedding problems, which can be solved in an analogue way. An analyse is omitted here for space reason.

As a simple example, we consider the case of embedding an $(n = 2)$-dimensional polynomial of order $P = 3$. Then the parameter $\mu \in \mathbb{N}$ is restricted by the condition $\mu > {}^2\log(n+1)$, and the smallest natural solution is given by $\mu = 2$. The embedding dimension N is restricted by the condition $N \geq N_{\min}(n, P, \mu) = n + n^\mu P = 2 + 2^2 \cdot 3 = 14$, and we take the smallest value $N = 14$. According to (4.4), the coefficients $A^i_{[p]}$ of the embedding polynomial F take the form

$$\begin{aligned}
p = 0 :\ & \begin{aligned} &\gamma = 1 : A^1_{[0]} = a^1_{[0]}, \\ &\gamma = 2 : A^2_{[0]} = a^2_{[0]}, \end{aligned} \\
p = 1 :\ & \begin{aligned} &\gamma = 1 : A^2_{[1]} = a^1_{[1]1}, A^5_{[1]} = a^1_{[1]2}, \\ &\gamma = 2 : A^3_{[1]} = a^2_{[1]1}, A^6_{[1]} = a^2_{[1]2}, \end{aligned} \\
p = 2 :\ & \begin{aligned} &\gamma = 1 : A^3_{[2]} = a^1_{[2]11}, A^6_{[2]} = a^1_{[2]12}, A^9_{[2]} = a^1_{[2]22}, \\ &\gamma = 2 : A^4_{[2]} = a^2_{[2]11}, A^7_{[2]} = a^2_{[2]12}, A^{10}_{[2]} = a^2_{[2]22}, \end{aligned} \\
p = 3 :\ & \begin{aligned} &\gamma = 1 : A^4_{[3]} = a^1_{[3]111}, A^7_{[3]} = a^1_{[3]112}, A^{10}_{[3]} = a^1_{[3]122}, A^{13}_{[3]} = a^1_{[3]222}, \\ &\gamma = 2 : A^5_{[3]} = a^2_{[3]111}, A^8_{[3]} = a^2_{[3]112}, A^{11}_{[3]} = a^2_{[3]122}, A^{14}_{[3]} = a^2_{[3]222}, \end{aligned}
\end{aligned} \tag{4.20}$$

and according to the remarks above the remaining coefficients $A^i_{[p]}$ can be chosen to zero, that is, $A^3_{[0]} = A^4_{[0]} = \cdots = A^{14}_{[0]} = 0$, $A^1_{[1]} = A^4_{[1]} = A^7_{[1]} = A^8_{[1]} = \cdots = A^{14}_{[0]} = 0$, and so forth.

Finally we will discuss a certain extension of our pseudoring-theory, which is interesting for certain embedding problems. We remember that until now we only have dealed with products $\circledast_{(*)} : V^p \times V \to V^p$ and $\circledast^{\text{dual}}_{(*)} : V_q \times V \to V_q$, which leads to the question if "similar" products $V^p \times V^p \to V^p$ and $V_q \times V_q \to V_q$ can be defined with similar properties. This will be done now.

Definition 4.3. Given an arbitrary algebra $(V;*)$ with structure constant tensor $C_{(*)}$, then the $*$-associated product of order p: $\boxtimes_{(*)} : V^p \times V^p \to V^p$ is defined by

$$\left(X \boxtimes_{(*)} Y\right)^{k_1 k_2 \cdots k_p} := X^{i_1 i_2 \cdots i_p} C^{k_1}_{(*) i_1 j_1} C^{k_2}_{(*) i_2 j_2} \cdots C^{k_p}_{(*) i_p j_p} Y^{j_1 j_2 \cdots j_p}, \tag{4.21}$$

and the dual-associated product of order q: $\boxtimes^{\text{dual}}_{(*)} : V_q \times V^q \to V_q$ is defined by

$$\left(X \boxtimes^{\text{dual}}_{(*)} Y\right)_{k_1 k_2 \cdots k_q} := X_{i_1 i_2 \cdots i_q} C^{i_1}_{(*) k_1 j_1} C^{i_2}_{(*) k_2 j_2} \cdots C^{i_q}_{(*) k_q j_q} Y^{j_1 j_2 \cdots j_q}. \tag{4.22}$$

Proposition 4.4. *Given a PA1-structure $(V; *_1, *_2)$, then the following symmetry holds:*

$$\left(X \boxtimes_{(*_1)} Y\right) \boxtimes_{(*_2)} Z = \left(X \boxtimes_{(*_1)} Z\right) \boxtimes_{(*_2)} Y, \quad \forall X, Y, Z \in V^p. \tag{4.23}$$

Proof. It is

$$\begin{aligned}
&\left\{\left(X \boxtimes_{(*_1)} Y\right) \boxtimes_{(*_2)} Z\right\}^{m_1 m_2 \cdots m_p} \\
&\quad = \left(X \boxtimes_{(*_1)} Y\right)^{k_1 k_2 \cdots k_p} C^{m_1}_{(*_2) k_1 l_1} C^{m_2}_{(*_2) k_2 l_2} \cdots C^{m_p}_{(*_2) k_p l_p} Z^{l_1 l_2 \cdots l_p} \\
&\quad = X^{i_1 i_2 \cdots i_p} C^{k_1}_{(*_1) i_1 j_1} C^{k_2}_{(*_1) i_2 j_2} \cdots C^{k_p}_{(*_1) i_p j_p} Y^{j_1 j_2 \cdots j_p} C^{m_1}_{(*_2) k_1 l_1} C^{m_2}_{(*_2) k_2 l_2} \cdots C^{m_p}_{(*_2) k_p l_p} Z^{l_1 l_2 \cdots l_p} \\
&\quad \overset{\text{rewritten}}{=} X^{i_1 i_2 \cdots i_p} Y^{j_1 j_2 \cdots j_p} Z^{l_1 l_2 \cdots l_p} C^{k_1}_{(*_1) i_1 j_1} C^{k_2}_{(*_1) i_2 j_2} \cdots C^{k_p}_{(*_1) i_p j_p} C^{m_1}_{(*_2) k_1 l_1} C^{m_2}_{(*_2) k_2 l_2} \cdots C^{m_p}_{(*_2) k_p l_p}.
\end{aligned} \tag{4.24}$$

Since $C^{k_\alpha}_{(*_1) i_\alpha [l_\alpha} C^{m_\alpha}_{(*_2) |k_\alpha| j_\alpha]} = 0$, $\alpha \in I_{1,p}$, the term above can be written as

$$\begin{aligned}
&X^{i_1 i_2 \cdots i_p} Y^{j_1 j_2 \cdots j_p} Z^{l_1 l_2 \cdots l_p} C^{k_1}_{(*_1) i_1 l_1} C^{k_2}_{(*_1) i_2 l_2} \cdots C^{k_p}_{(*_1) i_p l_p} C^{m_1}_{(*_2) k_1 j_1} C^{m_2}_{(*_2) k_2 j_2} \cdots C^{m_p}_{(*_2) k_p j_p} \\
&\quad \overset{\text{rewritten}}{=} X^{i_1 i_2 \cdots i_p} C^{k_1}_{(*_1) i_1 l_1} C^{k_2}_{(*_1) i_2 l_2} \cdots C^{k_p}_{(*_1) i_p l_p} Z^{l_1 l_2 \cdots l_p} C^{m_1}_{(*_2) k_1 j_1} C^{m_2}_{(*_2) k_2 j_2} \cdots C^{m_p}_{(*_2) k_p j_p} Y^{j_1 j_2 \cdots j_p} \\
&\quad \overset{\text{rewritten}}{=} \left(X \boxtimes_{(*_1)} Z\right)^{k_1 k_2 \cdots k_p} C^{m_1}_{(*_2) k_1 j_1} C^{m_2}_{(*_2) k_2 j_2} \cdots C^{m_p}_{(*_2) k_p j_p} Y^{j_1 j_2 \cdots j_p} \\
&\quad = \left\{\left(X \boxtimes_{(*_1)} Z\right) \boxtimes_{(*_2)} Y\right\}^{m_1 m_2 \cdots m_p}.
\end{aligned} \tag{4.25}$$

□

In other words, the PA1-symmetry is induced from the operations $*_1, *_2$ to the operations $\boxtimes_{(*_1)}, \boxtimes_{(*_2)}$. The next question is if the property of $*$-analyticity is preserved, that is, if $X, Y \in \mathfrak{T}^2(W;*)_{(\xi_0)}$ induces $(X\boxtimes_{(*)}Y) \in \mathfrak{T}^2(W;*)_{(\xi_0)}$, and if $X \in \mathfrak{T}_2(W;*)_{(\xi_0)}$ and $Y \in \mathfrak{T}^2(W;*)_{(\xi_0)}$ induce $(X\boxtimes^{\mathrm{dual}}_{(*)}Y) \in \mathfrak{T}_2(W;*)_{(\xi_0)}$. The next proposition gives a positive answer.

Proposition 4.5. *For a pseudoring $(\mathbb{R}^n;*)$, the following symmetries hold:*

$$\begin{gathered}
X, Y \in \mathfrak{T}^2(W;*)_{(\xi_0)} \Longrightarrow \left(X\boxtimes_{(*)}Y\right) \in \mathfrak{T}^2(W;*)_{(\xi_0)} \\
\text{if } (\mathbb{R}^n;*) \text{ is commutative or the vector field } Y \text{ is a constant,}
\end{gathered} \tag{4.26}$$

$$\begin{gathered}
X \in \mathfrak{T}_2(W;*)_{(\xi_0)}, \quad Y \in \mathfrak{T}^2(W;*)_{(\xi_0)} \Longrightarrow \left(X\boxtimes^{dual}_{(*)}Y\right) \in \mathfrak{T}_2(W;*)_{(\xi_0)} \\
\text{if } (\mathbb{R}^n;*) \text{ is commutative or vector field } Y \text{ is a constant.}
\end{gathered} \tag{4.27}$$

Proof. To show (4.26) we have to show that for the product $(X\boxtimes_{(*)}Y)$ the PCRE hold under the given assumptions, that is,

$$\left(X\boxtimes_{(*)}Y\right)^{kl}_{,m} = \left[\mathfrak{D}_{(*)}\left(X\boxtimes_{(*)}Y\right)\right]^{sl}C^k_{(*)sm} + \left[\mathfrak{D}_{(*)}\left(X\boxtimes_{(*)}Y\right)\right]^{ks}C^l_{(*)sm} \tag{4.28}$$

with adequate $\mathfrak{D}_{(*)}(X\boxtimes_{(*)}Y)$. With $Z^{kl} := (X\boxtimes_{(*)}Y)^{kl} = X^{ij}C^k_{(*)ip}C^l_{(*)jq}Y^{pq}$, we calculate

$$\begin{aligned}
Z^{kl}_{,m} &= X^{ij}_{,m}C^k_{(*)ip}C^l_{(*)jq}Y^{pq} + X^{ij}C^k_{(*)ip}C^l_{(*)jq}Y^{pq}_{,m} \\
&\overset{\text{PCRE}}{=} \left[\left(\mathfrak{D}_{(*)}X\right)^{sj}C^i_{(*)sm} + \left(\mathfrak{D}_{(*)}X\right)^{is}C^j_{(*)sm}\right]C^k_{(*)ip}C^l_{(*)jq}Y^{pq} \\
&\quad + X^{ij}C^k_{(*)ip}C^l_{(*)jq}\left[\left(\mathfrak{D}_{(*)}Y\right)^{sq}C^p_{(*)sm} + \left(\mathfrak{D}_{(*)}Y\right)^{ps}C^q_{(*)sm}\right].
\end{aligned} \tag{4.29}$$

In the case of a constant Y, we obtain

$$\begin{aligned}
Z^{kl}_{,m} &= \left[\left(\mathfrak{D}_{(*)}X\right)^{sj}C^i_{(*)sm}C^k_{(*)ip}C^l_{(*)jq} + \left(\mathfrak{D}_{(*)}X\right)^{is}C^j_{(*)sm}C^k_{(*)ip}C^l_{(*)jq}\right]Y^{pq} \\
&\overset{\text{pseudoring symmetry}}{=} \left[\left(\mathfrak{D}_{(*)}X\right)^{sj}C^i_{(*)sp}C^k_{(*)im}C^l_{(*)jq} + \left(\mathfrak{D}_{(*)}X\right)^{is}C^j_{(*)sq}C^k_{(*)ip}C^l_{(*)jm}\right]Y^{pq} \\
&= \left[\left(\mathfrak{D}_{(*)}Z\right)^{il}C^k_{(*)im} + \left(\mathfrak{D}_{(*)}Z\right)^{ki}C^l_{(*)im}\right]Y^{pq}, \quad \text{with} \\
\left(\mathfrak{D}_{(*)}Z\right)^{il} &:= \left(\mathfrak{D}_{(*)}X\right)^{sj}C^i_{(*)sp}\; C^l_{(*)jq}Y^{pq},
\end{aligned} \tag{4.30}$$

and the second statement of (4.26) has been proved. To show the first statement, we start from the partial derivations above.

$$
\begin{aligned}
Z^{kl}_{,m} &= \left[(\mathfrak{D}_{(*)}X)^{sj}C^{i}_{(*)sm}C^{k}_{(*)ip}C^{l}_{(*)jq} + (\mathfrak{D}_{(*)}X)^{is}C^{j}_{(*)sm}C^{k}_{(*)ip}C^{l}_{(*)jq}\right]Y^{pq} \\
&\quad + X^{ij}\left[(\mathfrak{D}_{(*)}Y)^{sq}C^{p}_{(*)sm}C^{k}_{(*)ip}C^{l}_{(*)jq} + (\mathfrak{D}_{(*)}Y)^{ps}C^{q}_{(*)sm}C^{k}_{(*)ip}C^{l}_{(*)jq}\right] \\
&\overset{\text{comm. pseudoring}}{=} \left[(\mathfrak{D}_{(*)}X)^{sj}C^{i}_{(*)sp}C^{k}_{(*)im}C^{l}_{(*)jq} + (\mathfrak{D}_{(*)}X)^{is}C^{j}_{(*)sq}C^{k}_{(*)ip}C^{l}_{(*)jm}\right]Y^{pq} \\
&\quad + X^{ij}\left[(\mathfrak{D}_{(*)}Y)^{sq}C^{p}_{(*)si}C^{k}_{(*)mp}C^{l}_{(*)jq} + (\mathfrak{D}_{(*)}Y)^{ps}C^{q}_{(*)sj}C^{k}_{(*)ip}C^{l}_{(*)mq}\right] \\
&\overset{\text{rena min g indices}}{=} \left[(\mathfrak{D}_{(*)}X)^{ij}C^{s}_{(*)ip}C^{k}_{(*)sm}C^{l}_{(*)jq} + (\mathfrak{D}_{(*)}X)^{ij}C^{s}_{(*)jq}C^{k}_{(*)ip}C^{l}_{(*)sm}\right]Y^{pq} \\
&\quad + X^{ij}\left[(\mathfrak{D}_{(*)}Y)^{pq}C^{s}_{(*)pi}C^{k}_{(*)ms}C^{l}_{(*)jq} + (\mathfrak{D}_{(*)}Y)^{pq}C^{s}_{(*)qj}C^{k}_{(*)ip}C^{l}_{(*)ms}\right] \\
&= C^{k}_{(*)sm}\left[(\mathfrak{D}_{(*)}X)^{ij}C^{s}_{(*)ip}C^{l}_{(*)jq}Y^{pq} + X^{ij}(\mathfrak{D}_{(*)}Y)^{pq}C^{s}_{(*)pi}C^{l}_{(*)jq}\right] \\
&\quad + C^{l}_{(*)sm}\left[(\mathfrak{D}_{(*)}X)^{ij}C^{s}_{(*)jq}C^{k}_{(*)ip}Y^{pq} + X^{ij}(\mathfrak{D}_{(*)}Y)^{pq}C^{s}_{(*)qj}C^{k}_{(*)ip}\right] \\
&= C^{k}_{(*)sm}(\mathfrak{D}_{(*)}Z)^{sl} + C^{l}_{(*)sm}(\mathfrak{D}_{(*)}X)^{ks}, \quad \text{with} \\
(\mathfrak{D}_{(*)}Z)^{sl} &:= (\mathfrak{D}_{(*)}X)^{ij}C^{s}_{(*)ip}C^{l}_{(*)jq}Y^{pq} + X^{ij}(\mathfrak{D}_{(*)}Y)^{pq}C^{s}_{(*)pi}C^{l}_{(*)jq} \\
&= \left[(\mathfrak{D}_{(*)}X)\boxtimes_{(*)}Y + X\boxtimes_{(*)}(\mathfrak{D}_{(*)}Y)\right]^{sl}.
\end{aligned}
\tag{4.31}
$$

The proof of the statement (4.27) runs the same lines. □

We see that the property of $*$-analyticity generally will not be preserved by the operations $\boxtimes_{(*)}$ and $\boxtimes^{\text{dual}}_{(*)}$, only for simple underlying structures (commutative pseudorings) or for a simple operand (a constant). Thus it makes sense to introduce the concept of *weak $*$-analytical fields* as $\boxtimes_{(*)}$-products and $\boxtimes^{\text{dual}}_{(*)}$-products of $*$-analytical fields, which allows applications in general relativity as well as in system theory, but will not be discussed here for space reason.

In this chapter, a class of Embedding Problems has been analyzed with the result that polynomial n-dimensional vector fields always can be embedded into polynomial N-dimensional, $*$-analytic vector fields, $N \geq n$ finite. In the case $N \rightarrow \infty$, the results hold for arbitrary smooth vector fields. Furthermore, not only the existence of embedding was shown, but the embedding process was described explicitly. Finally new product operations with interesting properties have been analyzed.

5. Conclusion and Outlook

The paper presents and solves a new type of Embedding Problems which are important for solving nonlinear partial differential equations. The most important results are as follows.

(i) A set of algebras was found (so-called "PA1-chains" or "GAPs"), such that for the corresponding function theories a generalization of the Cauchy-Riemannian differential equations exists as well as a generalization of the classical Cauchy Integral Theorem.

(ii) A wide variety of GAPs can be characterized explicitly by presenting their structure constant tensors explicitly.

(iii) GAP-Theory endows new explicit solutions of Einsteins field equations from General Relativity Theory and new explicit solutions of the n-dimensional Navier-Stokes equations.

(iv) A new Embedding Theorem was formulated, which allows a wide generalization of the results above: it was shown explicitly that an arbitrary n-dimensional smooth vector field always can be embedded into a special N-Dimensional Smooth Vector Field ($N \geq n$) restricted by PA1-symmetries. In other words, the world of PA1-chains is wide enough to allow embedding for rather general ("smooth") structures.

(v) Some consequences of the Embedding Theorem for the explicit solution of nonlinear partial differential equations have been discussed.

(vi) A simple example for the Embedding Theorem has been given.

The following investigations and generalizations would be useful, the construction of more general PA1-chains which would generalize our Embedding Theorem.

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