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Research Article

Higher-Order Equations of the KdV Type are Integrable

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We show that a nonlinear equation that represents third-order approximation of long wavelength, small amplitude waves of inviscid and incompressible fluids is integrable for a particular choice of its parameters, since in this case it is equivalent with an integrable equation which has recently appeared in the literature. We also discuss the integrability of both second- and third-order approximations of additional cases.

1. Introduction

The one-dimensional motion of solitary waves of inviscid and incompressible fluids has been the subject of research for more than a century [1]. Probably, one of the most important results in the above study was the derivation of the famous Korteweg-de Vries (KdV) equation [2]

$$u_t + u_x + \alpha u u_x + \beta u_{xxx} = 0. \tag{1.1}$$

At first, the equation was difficult to be examined due to the nonlinearity. The first important step was made with the numerical discovery of soliton solutions by Zabusky and Kruskal [3]. Soon thereafter great progress was made by the discovery of the Lax Pair representation [4] and the Inverse Scattering Transform [5]. The laters results led to a new notion of integrability. More specifically, according to this notion, a partial differential equation is said to be "completely integrable" if it is linearizable through a Lax Pair; thus, it is solvable via the Inverse Scattering Transform (see, e.g., [6]).

Equation (1.1) represents a first-order approximation in the study of long wavelength, small amplitude waves of inviscid and incompressible fluids. Allowing the appearance of higher-order terms in α and β, one can obtain more complicated equations. Two such equations, including second- and third-order terms, were proposed in [7, 8] and have, respectively, the forms

$$u_t + u_x + \alpha u u_x + \beta u_{xxx} + \alpha^2 \rho_1 u^2 u_x + \alpha\beta\left(\rho_2 u u_{xxx} + \rho_3 u_x u_{xx}\right) = 0, \tag{1.2}$$

$$\begin{aligned} &u_t + u_x + \alpha u u_x + \beta u_{xxx} + \alpha^2 \rho_1 u^2 u_x + \alpha\beta\left(\rho_2 u u_{xxx} + \rho_3 u_x u_{xx}\right) \\ &\quad + \alpha^3 \rho_4 u^3 u_x + \alpha^2 \beta\left(\rho_5 u^2 u_{xxx} + \rho_6 u u_x u_{xx} + \rho_7 u_x^3\right) = 0. \end{aligned} \tag{1.3}$$

Equation (1.2) was first examined both analytically and numerically in [9]. The violation of the Painlevé property in many cases, together with a numerical study of the reduction $u = u(x)$ in the complex x-plane, gave strong indications that, in general, this equation is not integrable. Consequently, (1.2) was examined in [10, 11] and it was found that it possesses solitary wave solutions, which, for small values of the parameters α and β, behave like solitons. New wave solutions of both (1.2) and (1.3) were also examined numerically in [12] and were also found to behave like solitons.

Equation (1.2) was further examined in [13–20], while (1.3) was examined in [18, 21, 22]. Although an enormous amount of new solutions was presented, no progress has been made regarding the integrability of these equations.

In this paper we show that, for arbitrary ρ_1 and

$$\rho_2 = 4\rho_1, \qquad \rho_3 = 2\rho_1, \qquad \rho_4 = 0, \qquad \rho_5 = \rho_6 = 4\rho_1^2, \qquad \rho_7 = -\frac{8\rho_1^2}{9}, \tag{1.4}$$

equation (1.3) is equivalent to an integrable equation recently proposed by Qiao and Liu [23]. We, thus, reveal an integrable case of (1.3) itself. We also discuss the existence of additional integrable cases for both (1.2) and (1.3).

2. An Integrable Case of (1.3)

Recently, Qiao and Liu [23] proposed a new integrable equation, namely,

$$m_t = \frac{1}{2}\left(\frac{1}{m^2}\right)_{xxx} - \frac{1}{2}\left(\frac{1}{m^2}\right)_x. \tag{2.1}$$

The integrability follows directly from the fact that the equation admits a Lax Pair; thus, as mentioned above, is solvable by the Inverse Scattering Transform.

It is quite easy to prove that (2.1) is actually a subcase of (1.3). More specifically, we first set

$$m = v^{-2/3}, \tag{2.2}$$

thus, (2.1) becomes

$$v_t - v^2 v_x + v^2 v_{xxx} + v v_x v_{xx} - \frac{2}{9} v_x^3 = 0. \tag{2.3}$$

We then set

$$u = a_1 v(X, t) + a_2, \qquad X = a_3 x + a_4 t, \tag{2.4}$$

where a_i are constants; substitute in (1.3), divide by a_1, and write x instead of X. We thus obtain

$$\begin{aligned} &v_t + \left[a_3\left(1 + \alpha a_2 + \alpha^2 a_2^2 \rho_1 + \alpha^3 a_2^3 \rho_4\right) + a_4\right] v_x + \alpha a_1 a_3 \left(1 + 2\alpha a_2 \rho_1 + 3\alpha^2 a_2^2 \rho_4\right) v v_x \\ &\quad + \beta a_3^3 \left(1 + \alpha a_2 \rho_2 + \alpha^2 a_2^2 \rho_5\right) v_{xxx} + \alpha^2 a_1^2 a_3 (\rho_1 + 3\alpha a_2 \rho_4) v^2 v_x \\ &\quad + \alpha\beta a_1 a_3^3 \left[(\rho_2 + 2\alpha a_2 \rho_5) v v_{xxx} + (\rho_3 + \alpha a_2 \rho_6) v_x v_{xx}\right] + \alpha^3 a_1^3 a_3 \rho_4 v^3 v_x \\ &\quad + \alpha^2 \beta a_1^2 a_3^3 \left(\rho_5 v^2 v_{xxx} + \rho_6 v v_x v_{xx} + \rho_7 v_x^3\right) = 0. \end{aligned} \tag{2.5}$$

Clearly, (2.3) and (2.5) are equivalent if the following terms vanish:

$$\begin{aligned} A_1 &= a_3\left(1 + \alpha a_2 + \alpha^2 a_2^2 \rho_1 + \alpha^3 a_2^3 \rho_4\right) + a_4, \\ A_2 &= \alpha a_1 a_3 \left(1 + 2\alpha a_2 \rho_1 + 3\alpha^2 a_2^2 \rho_4\right), \\ A_3 &= \beta a_3^3 \left(1 + \alpha a_2 \rho_2 + \alpha^2 a_2^2 \rho_5\right), \\ A_4 &= \alpha^2 a_1^2 a_3 (\rho_1 + 3\alpha a_2 \rho_4) + 1, \\ A_5 &= \alpha\beta a_1 a_3^3 (\rho_2 + 2\alpha a_2 \rho_5), \\ A_6 &= \alpha\beta a_1 a_3^3 (\rho_3 + \alpha a_2 \rho_6), \\ A_7 &= \alpha^3 a_1^3 a_3 \rho_4, \\ A_8 &= \alpha^2 \beta a_1^2 a_3^3 \rho_5 - 1, \\ A_9 &= \alpha^2 \beta a_1^2 a_3^3 \rho_6 - 1, \\ A_{10} &= \alpha^2 \beta a_1^2 a_3^3 \rho_7 + \frac{2}{9}. \end{aligned} \tag{2.6}$$

Since $\alpha\beta a_1 a_3 \neq 0$, relations $A_7 = 0$, $A_2 = 0$, and $A_6 = 0$ imply, respectively, that

$$\rho_4 = 0, \qquad a_2 = -\frac{1}{2\alpha\rho_1}, \qquad \rho_6 = 2\rho_1\rho_3, \tag{2.7}$$

while relations $A_3 = 0$ and $A_5 = 0$ imply that

$$\rho_2 = 4\rho_1, \qquad \rho_5 = 4\rho_1^2. \tag{2.8}$$

Then, relations $A_4 = 0$, $A_1 = 0$, $A_9 = 0$, and $A_{10} = 0$ yield, respectively,

$$a_3 = -\frac{1}{\alpha^2 a_1^2 \rho_1}, \qquad a_4 = \frac{4\rho_1 - 1}{4\alpha^2 a_1^2 \rho_1^2}, \qquad \rho_3 = -\frac{\alpha^4 a_1^4 \rho_1^2}{2\beta}, \qquad \rho_7 = \frac{2\alpha^4 a_1^4 \rho_1^3}{9\beta}. \tag{2.9}$$

Finally,

$$A_8 = 0 \Longrightarrow a_1^4 = -\frac{4\beta}{\alpha^4 \rho_1}. \tag{2.10}$$

We, thus, conclude with relations (1.4), while ρ_1 remains arbitrary.

3. Discussion

In [9] it was shown that (1.2) does not pass the classical Painlevé test [24, 25] for any combination of the ρ_i parameters, except of course for the cases $\rho_1 = \rho_2 = \rho_3 = 0$ and $\rho_2 = \rho_3 = 0$ in which it reduces to KdV and modified KdV, respectively. However, as also stated in [9], there are infinitely many cases for which the equation has only algebraic singularities, that is, it admits the so-called weak-Painlevé property [26]. Although this property does not constitute a strong indication for integrability, there are still many integrable equations admitting only algebraic singularities (see, e.g., [27]).

On the other hand, at least to our knowledge, no results regarding integrable cases of (1.3) have appeared in the bibliography before. Equation (1.3) is highly nonlinear and most probably it is not integrable in general. However, as shown in the previous section, there is at least one nontrivial combination of the ρ_i parameters, for which it is completely integrable.

Based on the above statements, we believe that it would be interesting to study whether there are any integrable cases for (1.2) or any additional integrable cases for (1.3). We hope to present results in this direction in a future publication.

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