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Research Article

Local Analyticity in the Time and Space Variables and the Smoothing Effect for the Fifth-Order KdV-Type Equation

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We consider the initial value problem for the reduced fifth-order KdV-type equation: $\partial_t u - \partial_x^5 u - 10\partial_x(u^3) + 10\partial_x(\partial_x u)^2 = 0$, $t, x \in \mathbb{R}$, $u(0,x) = \phi(x)$, $x \in \mathbb{R}$. This equation is obtained by removing the nonlinear term $10u\partial_x^3 u$ from the fifth-order KdV equation. We show the existence of the local solution which is real analytic in both time and space variables if the initial data $\phi \in H^s(\mathbb{R})$ $(s > 1/8)$ satisfies the condition $\sum_{k=0}^{\infty} (A_0^k/k!)\|(x\partial_x)^k \phi\|_{H^s} < \infty$, for some constant A_0 $(0 < A_0 < 1)$. Moreover, the smoothing effect for this equation is obtained. The proof of our main result is based on the contraction principle and the bootstrap argument used in the third-order KdV equation (K. Kato and Ogawa 2000). The key of the proof is to obtain the estimate of $\partial_x(\partial_x u)^2$ on the Bourgain space, which is accomplished by improving Kenig et al.'s method used in (Kenig et al. 1996).

1. Introduction

The KdV hierarchy is well known as the series of the Lax pair formulation [1, 2], which are presented as

$$\text{1st-order KdV} \quad \partial_t u - \partial_x u = 0, \tag{1.1$_0$}$$

$$\text{3rd-order KdV} \quad \partial_t u + \partial_x^3 u - 6u\partial_x u = 0, \tag{1.1$_1$}$$

$$\text{5th-order KdV} \quad \partial_t u - \partial_x^5 u - 10\partial_x\left(u^3\right) + 10\partial_x(\partial_x u)^2 + 10u\partial_x^3 u = 0. \tag{1.1$_2$}$$

$$\vdots$$

These equations describe mathematical models of some water waves [3, 4]. We are interested in the existence theory of the analytic solution and the smoothing effect of the KdV hierarchy. There are some results concerning the analyticity for the third-order KdV equation $(1.1)_1$. To. Kato and Masuda [5] considered the initial value problem of the following equation:

$$\partial_t u + \partial_x^3 u + a(u)\partial_x u = 0, \quad t, x \in \mathbb{R}, \tag{1.2}$$

where $a(\lambda)$ is the real analytic and the no growth rate function in $\lambda \in \mathbb{R}$. They showed that if the initial data is real analytic, then, the global solution of (1.2) is real analytic in the space variable. Hayashi [6] also considered (1.2) in which $a(\lambda)$ is the polynomial. He showed that if the initial data is analytic and has an analytic continuation to a strip containing the real axis, then, the local solution also has the same property. When $a(u) = -6u$, (1.2) becomes the third-order KdV equation $(1.1)_1$. K. Kato and Ogawa [7] proved that $(1.1)_1$ has not only the real analytic solution in both time and space variables but also the smoothing effect.

Recently, it is shown that the nonlinear dispersive equations including the KdV hierarchy have the local analytic solution in the space variable (see [8]). However, neither the existence of the real analytic solution in both time and space variables nor the smoothing effect is obtained for $(1.1)_j$ with $j \geq 2$, because the bilinear estimate of $u\partial_x^{2j-1}u$ with $j \geq 2$ cannot be obtained by their method used in [9].

On the other hand, we may expect that the method used in [7] can work for the reduced equation given by removing $u\partial_x^{2j-1}u$ from the higher-order KdV equations $(1.1)_0$ with $j \geq 2$. In this paper, as a starting point for this attempt, we consider the following initial value problem of the reduced fifth-order KdV-type equation:

$$\begin{gathered} \partial_t u - \partial_x^5 u = \partial_x\left(u^3\right) + \partial_x(\partial_x u)^2, \quad t, x \in \mathbb{R}, \\ u(0, x) = \phi(x), \quad x \in \mathbb{R}, \end{gathered} \tag{1.3}$$

where we may take all coefficients of the nonlinear terms to be equal to 1 without loss of generality. This equation is obtained by removing the nonlinear term $10u\partial_x^3 u$ from the original fifth-order KdV equation $(1.1)_2$. Our main purpose is to prove not only the existence of a local real analytic solution of (1.3) in both time and space variables but also the smoothing effect.

Before stating the main result precisely, we introduce the function space introduced by Bourgain (see [10]): for $s, b \in \mathbb{R}$, define that

$$X_b^s = \left\{ f \in \mathcal{S}'\left(\mathbb{R}^2\right); \|f\|_{X_b^s} < \infty \right\}, \tag{1.4}$$

where

$$\|f\|_{X_b^s}^2 = \iint_{\mathbb{R}^2} \left(1 + \left|\tau - \xi^5\right|\right)^{2b} (1 + |\xi|)^{2s} \left|\mathcal{F}_{t,x} f(\tau, \xi)\right|^2 d\tau\, d\xi, \tag{1.5}$$

and $\mathcal{F}_{t,x} f$ is the Fourier transform of f in both x and t variables; that is,

$$\mathcal{F}_{t,x} f(\tau, \xi) = (2\pi)^{-1} \iint_{\mathbb{R}^2} f(t, x) e^{-it\tau - ix\xi} dt\, dx. \tag{1.6}$$

Our main result is the following theorem.

Theorem 1.1. *Let $s > 1/8$ and let $b \in (1/2, 23/40)$. Then, for any $\phi(x) \in H^s(\mathbb{R})$ such that*

$$(x\partial_x)^k \phi(x) \in H^s(\mathbb{R}) \quad (k = 0, 1, 2, \ldots),$$
$$\sum_{k=0}^{\infty} \frac{A_0^k}{k!} \left\| (x\partial_x)^k \phi \right\|_{H^s} < \infty, \quad \text{for some } 0 < A_0 < 1, \tag{1.7}$$

there exist a constant $T = T(\phi) > 0$ and a unique solution $u \in C((-T, T), H^s) \cap X_b^s$ of (1.3) satisfying

$$P^k u \in C((-T,\ T),\ H^s) \cap X_b^s,$$
$$\sum_{k=0}^{\infty} \frac{A_0^k}{k!} \left\| P^k u \right\|_{X_b^s} < \infty, \tag{1.8}$$

where $P = 5t\partial_t + x\partial_x$ is the generator of dilation for the linear part of the equation of (1.3).

Moreover this solution becomes real analytic in both time and space variables; that is, there exist the positive constants C and A_1 such that

$$\left| \partial_t^m \partial_x^l u(t, x) \right| \le C A_1^{m+l} (m + l)! \tag{1.9}$$

holds for all $(t, x) \in (-T, 0) \cup (0, T) \times \mathbb{R}$ and $l, m = 0, 1, 2, \ldots$.

Remark 1.2. The initial data $\phi(x)$ has to be analytic except for $x = 0$ but is allowed to have H^s-singularity at $x = 0$. It follows from (1.9) that the singularity of $\phi(x)$ disappears after time passes and the regularity of the local solution of (1.3) reaches real analyticity in both time and space variables; that is, the fifth-order KdV-type equation has the smoothing effect.

Remark 1.3. A typical example of the initial data satisfying the condition (1.7) is given by

$$|x|^{\gamma} e^{-x^2} \quad \text{with } \gamma > -\frac{3}{8}. \tag{1.10}$$

The existence results of the higher-order KdV equation are studied by many authors. Saut [11] and Schwarz [12] proved that each equation of the KdV hierarchy has a unique global solution in the spatially periodic Sobolev space. Kenig, Ponce, and Vega studied the initial value problem of the higher-order dispersive equation

$$\partial_t u + \partial_x^{2j+1} u + P\left(u, \partial_x u, \ldots, \partial_x^{2j-1} u\right) = 0, \tag{1.11}$$

where $j \ge 1$ and $P(\cdot)$ is a polynomial having no constant or linear part. They proved the local well-posedness in the weighted Sobolev space [13, 14]. Recently, Kwon [15] studied the simplified fifth-order KdV-type equation

$$\partial_t u + \partial_x^5 u + \partial_x u \partial_x^2 u + u \partial_x^3 u = 0, \tag{1.12}$$

which is obtained by removing the term $10\partial_x(u^3)$ from $(1.1)_2$. He showed the local well-posedness for the IVP of this equation in $H^s(\mathbb{R})$ with $s > 5/2$. On the other hand, Ta. Kato [16] proved the following result for $(1.1)_2$.

Well-Posedness Theorem (Ta. Kato)

(1) Let

$$s > -\frac{1}{4}, \quad s \geq -2a - 2, \quad \text{where } -\frac{3}{2} < a \leq -\frac{1}{4}. \tag{1.13}$$

Then, the local well-posedness for the IVP of $(1.1)_2$ holds in $H^{s,a}(\mathbb{R})$, where

$$H^{s,a}(\mathbb{R}) \equiv \left\{ f \in \mathcal{S}'(\mathbb{R}); \|f\|_{H^{s,a}} \equiv \left\| \langle \xi \rangle^{s-a} |\xi|^a \widehat{f} \right\|_{L^2_\xi} < \infty \right\}. \tag{1.14}$$

(2) Let

$$s \geq 1, \quad -\frac{1}{2} \leq a \leq -\frac{1}{4}. \tag{1.15}$$

Then, the global well-posedness for the IVP of $(1.1)_2$ holds in $H^{s,a}(\mathbb{R})$.

The plan of this paper is as follows. In Section 2, we give the existence and uniqueness of the local solution of (1.3), which is shown by the contraction argument consisting of Lemmas 2.3–2.5. In Section 3, we prove Lemma 2.4 which gives the bilinear estimate for $\partial_x(\partial_x u)^2$ in the Bourgain space. Kenig et al. [9] showed the bilinear estimate for $u\partial_x u$ of $(1.1)_1$ by estimating the potential which appears in an expression of the Bourgain norm of this term via duality. They divided the domain of integration of the potential into 17 subregions. However, their method of the domain decomposition is consistent with $(1.1)_2$, but not with the fifth-order KdV-type equation. We divide this domain into 30 subregions to derive the bilinear estimate for $\partial_x(\partial_x u)^2$. In Section 4, we show the analyticity of the solution stated in Theorem 1.1 by the bootstrap argument. The result of this paper is announced in Proceedings of the Japan Academy [17].

Notation. Let $\mathcal{F}_x$ be the Fourier transform in the x variable, and let $\mathcal{F}_\xi^{-1}$ and $\mathcal{F}_{\tau,\xi}^{-1}$ be the Fourier inverse transform in the ξ and (τ, ξ) variables, respectively. The Riesz operator D_x and its fractional derivative $\langle D_x \rangle^s$ are defined by

$$D_x = \mathcal{F}_\xi^{-1} |\xi| \mathcal{F}_x, \qquad \langle D_x \rangle^s = \mathcal{F}_\xi^{-1} \langle \xi \rangle^s \mathcal{F}_x, \tag{1.16}$$

respectively, where $\langle \cdot \rangle = (1 + |\cdot|)$. Similarly, $\langle D_{t,x} \rangle^s$ is defined by

$$\langle D_{t,x} \rangle^s = \mathcal{F}_{\tau,\xi}^{-1} \langle |\tau| + |\xi| \rangle^s \mathcal{F}_{t,x}. \tag{1.17}$$

$[A, B]$ denotes the commutator relation of two operators given by $AB - BA$. $L_t^p L_x^q$ denotes the space $L^p(\mathbb{R}_t; L^q(\mathbb{R}_x))$ for $1 \le p, q \le \infty$ with the norm

$$\|f\|_{L_t^p L_x^q} = \left(\int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} |f(t,x)|^p dt \right)^{q/p} dx \right)^{1/q}. \tag{1.18}$$

We use Sobolev spaces with both time and space variables

$$H_{t,x}^s\left(\mathbb{R}^2\right) = \left\{ u \in \mathcal{S}'\left(\mathbb{R}^2\right) : \langle D_{t,x} \rangle^s u \in L_t^2 L_x^2 \right\}, \tag{1.19}$$

with the norm $\|\cdot\|_{H_{t,x}^s(\mathbb{R}^2)} = \|\langle D_{t,x}\rangle^s \cdot\|_{L_t^2 L_x^2}$. Moreover, $L_t^2(\mathbb{R}; H_x^s)$ denotes the space $L^2(\mathbb{R}_t; H^s(\mathbb{R}_x))$ with the norm $\|\cdot\|_{L_t^2(\mathbb{R}; H_x^s)} = \|\langle D_x\rangle^s \cdot\|_{L_t^2 L_x^2}$. The dual coupling is expressed as $\langle f, g\rangle$. The convolution of f and g with both space and time variables is denoted by $f * g$. For the constant A_0 appearing in Theorem 1.1, we put

$$\mathcal{A}_{A_0}\left(X_b^s\right) = \left\{ \mathbf{f} = \left(f_0, f_1, \ldots\right); f_k \in X_b^s \quad (k = 0, 1, \ldots), |\mathbf{f}|_{\mathcal{A}_{A_0}(X_b^s)} < \infty \right\}, \tag{1.20}$$

where

$$|||\mathbf{f}|||_{\mathcal{A}_{A_0}(X_b^s)} \equiv \sum_{k=0}^{\infty} \frac{A_0^k}{k!} \|f_k\|_{X_b^s}. \tag{1.21}$$

For simplicity we make use of three notations:

$$\sum_{\mathbf{k}} = \sum_{k=k_1+k_2+k_3+k_4}, \qquad \sum_{\mathbf{l}} = \sum_{l=l_1+l_2+l_3}, \qquad \sum_{\mathbf{m}} = \sum_{m=m_1+m_2+m_3}. \tag{1.22}$$

2. Existence and Uniqueness of the Solution

In this section, we give the proof of the existence and uniqueness of the solution of (1.3). Let $u_k = P^k u$ and $\phi_k(x) = (x\partial_x)^k \phi(x)$, and we derive the equation which u_k and $\phi_k(x)$ satisfy. Since $[x\partial_x, \partial_x] = -\partial_x$, it follows that

$$(P + l)^k \partial_x = \partial_x (P + (l-1))^k, \quad k, l = 0, 1, 2, \ldots. \tag{2.1}$$

Using (2.1) and the following relations

$$\begin{gathered} \left[\partial_t - \partial_x^5, P\right] = 5\left(\partial_t - \partial_x^5\right), \\ \left(\partial_t - \partial_x^5\right) P^k = (P+5)^k \left(\partial_t - \partial_x^5\right), \end{gathered} \tag{2.2}$$

we have from (1.3)

$$\begin{aligned} &\partial_t u_k - \partial_x^5 u_k = \mathcal{N}_k(u), \quad t, x \in \mathbb{R}, \\ &u_k(0,x) = \phi_k(x), \quad x \in \mathbb{R}, \end{aligned} \qquad k = 0, 1, 2, \ldots, \tag{2.3}$$

where

$$\mathcal{N}_k(u) = \partial_x (P+4)^k \left(u^3\right) + \partial_x (P+4)^k (\partial_x u)^2. \tag{2.4}$$

Using the Leibniz rule and (2.1), we can see that

$$\begin{aligned} \mathcal{N}_k(u) &= \partial_x \sum_{l=0}^{k} \binom{k}{l} 4^{k-l} P^l \left(u^3\right) + \partial_x \sum_{l=0}^{k} \binom{k}{l} 3^{k-l} (P+1)^l (\partial_x u)^2 \\ &= \sum_{\mathbf{k}} \frac{k!}{k_1! k_2! k_3! k_4!} 4^{k_4} \partial_x (u_{k_1} u_{k_2} u_{k_3}) + \sum_{\mathbf{k}} \frac{k!}{k_1! k_2! k_3! k_4!} 3^{k_4} (-1)^{k_3} \partial_x ((\partial_x u_{k_1})(\partial_x u_{k_2})). \end{aligned} \tag{2.5}$$

We will show the existence and uniqueness of the solution of (2.3).

Proposition 2.1. *Let*

$$s > -\frac{1}{4}, \qquad b \in \left(\frac{1}{2}, \frac{1}{2} + \sigma\right), \tag{2.6}$$

where $\sigma = \min\{s/5 + 1/20, 3/16\}$. *Then, for any* $\phi \equiv (\phi_0, \phi_1, \ldots)$ *such that* $\phi_k \in H^s(\mathbb{R})$ $(k = 0, 1, \ldots)$ *and*

$$|||\phi|||_{\mathcal{A}_{A_0}(H^s)} < \infty, \tag{2.7}$$

there exist a constant $T = T(\phi) > 0$ *and a unique solution* $u_k \in C((-T,T),\ H^s) \cap X_b^s$ *of* (2.3) *satisfying*

$$|||\mathbf{u}|||_{\mathcal{A}_{A_0}(X_b^s)} < \infty, \quad \mathbf{u} \equiv (u_0, u_1, \ldots). \tag{2.8}$$

Remark 2.2. The uniqueness of the solution of (2.3) yields $u_k = P^k u$ for $k = 0, 1, 2, \ldots$. Moreover, u_0 becomes a solution of (1.3), the uniqueness of which also follows.

To prove this proposition we prepare three lemmas (Lemmas 2.3, 2.4 and 2.5), which play an important role in applying the contraction principle to the following system of the integral equations:

$$\psi(t) u_k = \psi(t) e^{t\partial_x^5} \phi_k + \psi(t) \int_0^t e^{(t-t')\partial_x^5} \psi_T(t') \mathcal{N}_k(u)(t') dt', \tag{2.9}$$

where

$$e^{t\partial_x^5} f \equiv \mathcal{F}_\xi^{-1}\left(e^{i\xi^5 t}\hat{f}(\xi)\right), \tag{2.10}$$

$\psi(t)$ denotes a cut-off function in $C_0^\infty(\mathbb{R})$ satisfying

$$\psi(t) = \begin{cases} 1, & \text{if } |t| \le 1, \\ 0, & \text{if } |t| > 2, \end{cases} \tag{2.11}$$

and $\psi_T(t) = \psi(t/T)$.

Lemma 2.3. *Let* $0 < T < 1$ *and let*

$$s \in \mathbb{R}, \qquad b \in \left(\frac{1}{2}, 1\right), \qquad a', a \in \left(0, \frac{1}{2}\right) \quad (a' < a). \tag{2.12}$$

Then

$$\left\|\psi(t)e^{t\partial_x^5}\phi(x)\right\|_{X_b^s} \le C_{0,s,b}\|\phi\|_{H^s}, \tag{2.13}$$

$$\left\|\psi(t)\int_0^t e^{(t-t')\partial_x^5}h(t')\,dt'\right\|_{X_b^s} \le C_{1,s,b}\|h\|_{X_{b-1}^s}, \tag{2.14}$$

$$\|\psi_T h\|_{X_{-a}^s} \le C_{2,s,-a,-a'}T^{(a-a')/4(1-a')}\|h\|_{X_{-a'}^s}, \tag{2.15}$$

where $C_{0,s,b}$, $C_{1,s,b}$, *and* $C_{2,s,-a,-a'}$ *are constants depending on* s, b, a, *and* a'.

Proof. Equations (2.13) and (2.14) are obtained by replacing the generator $e^{-t\partial_x^3}$ by $e^{t\partial_x^5}$ in the argument given by Kenig et al. [18] (see [19]). For the proof of (2.15), we refer to Lemma 2.5 in the study by Ginibre-Tsutsumi-Velo [20]. □

Lemma 2.4. *Let*

$$s > -\frac{1}{4}, \qquad b, b' \in \left(\frac{1}{2}, \frac{1}{2} + \sigma\right) \quad (b \le b'), \tag{2.16}$$

where $\sigma = \min\{s/5 + 1/20, 3/16\}$. *Then*

$$\|\partial_x((\partial_x u)(\partial_x v))\|_{X_{b'-1}^s} \le C_{3,s,b,b'}\|u\|_{X_b^s}\|v\|_{X_b^s}, \tag{2.17}$$

where $C_{3,s,b,b'}$ *is a constant depending on* s, b, *and* b'.

We give for the proof of this lemma, in Section 3.

Lemma 2.5. *Let*

$$s > -\frac{1}{4}, \qquad b, b' \in \left(\frac{1}{2}, \frac{3}{4}\right) \quad (b \leq b'). \tag{2.18}$$

Then

$$\|\partial_x (uvw)\|_{X^s_{b'-1}} \leq C_{4,s,b,b'} \|u\|_{X^s_b} \|v\|_{X^s_b} \|w\|_{X^s_b}, \tag{2.19}$$

where $C_{4,s,b,b'}$ is a constant depending on s, b, and b'.

Proof. This lemma is proved by improving Chen et al.'s argument used in the case where $b = b' \in (1/2, 3/4)$ [21]. □

Proof of Proposition 2.1. We define

$$X_{M_0} = \left\{ \mathbf{f} \in \mathcal{A}_{A_0}(X^s_b); |||\mathbf{f}|||_{\mathcal{A}_{A_0}(X^s_b)} \leq 2C_0 M_0 \right\}, \tag{2.20}$$

where

$$C_0 = C_{0,s,b}, \qquad M_0 = \left|\left|\left|\phi\right|\right|\right|_{\mathcal{A}_{A_0}(H^s)}. \tag{2.21}$$

We define a map $\Phi : X_{M_0} \to X_{M_0}$ by $\Phi(u) = (\Phi_0(u), \Phi_1(u), \ldots)$ and

$$\Phi_k(u) = \psi(t) e^{t\partial_x^5} \phi_k + \psi(t) \int_0^t e^{(t-t')\partial_x^5} \psi_T(t') \mathcal{N}_k(u)(t') dt'. \tag{2.22}$$

Let b' and T be positive constants satisfying $b < b' < 1/2 + \sigma$ and

$$T < min \left\{ 1, \left(24 C_0^2 C_5 e^{4A_0} M_0^2 + 8 C_0 C_6 e^{4A_0} M_0 \right)^{-1/\mu} \right\}, \tag{2.23}$$

respectively, where

$$C_5 = C_{1,s,b} C_{2,s,b-1,b'-1} C_{4,s,b,b'} \qquad C_6 = C_{1,s,b} C_{2,s,b-1,b'-1} C_{3,s,b,b'}. \tag{2.24}$$

We now show that Φ is a contraction mapping from X_{M_0} to itself. According to Lemmas 2.3, 2.4, and 2.5, we have for $u \in \mathcal{A}_{A_0}(X^s_b)$

$$\begin{aligned} \|\Phi_k(u)\|_{X^s_b} \leq{}& C_0 \|\Phi_k\|_{H^s} + C_5 T^\mu \sum_{\mathbf{k}} \frac{k!}{k_1! k_2! k_3! k_4!} 4^{k_4} \|u_{k_1}\|_{X^s_b} \|u_{k_2}\|_{X^s_b} \|u_{k_3}\|_{X^s_b} \\ &+ C_6 T^\mu \sum_{\mathbf{k}} \frac{k!}{k_1! k_2! k_3! k_4!} 3^{k_4} \|u_{k_1}\|_{X^s_b} \|u_{k_2}\|_{X^s_b}, \end{aligned} \tag{2.25}$$

for any $k \geq 0$. Here $\mu = (b' - b)/4b' > 0$. By taking a sum over k, we have

$$\begin{aligned} |||\Phi(u)|||_{\mathcal{A}_{A_0}(X_b^s)} &\equiv \sum_{k=0}^{\infty} \frac{{A_0}^k}{k!} \|\Phi_k(u)\|_{X_b^s} \leq C_0 \big| \|\phi\| \big|_{\mathcal{A}_{A_0}(H^s)} \\ &\quad + C_5 T^{\mu} \sum_{k_4=0}^{\infty} \frac{(4A_0)^{k_4}}{k_4!} \sum_{k_1=0}^{\infty} \frac{A_0^{k_1}}{k_1!} \|u_{k_1}\|_{X_b^s} \sum_{k_2=0}^{\infty} \frac{A_0^{k_2}}{k_2!} \|u_{k_2}\|_{X_b^s} \sum_{k_3=0}^{\infty} \frac{A_0^{k_3}}{k_3!} \|u_{k_3}\|_{X_b^s} \\ &\quad + C_6 T^{\mu} \sum_{k_4=0}^{\infty} \frac{(3A_0)^{k_4}}{k_4!} \sum_{k_3=0}^{\infty} \frac{A_0^{k_3}}{k_3!} \sum_{k_1=0}^{\infty} \frac{A_0^{k_1}}{k_1!} \|u_{k_1}\|_{X_b^s} \sum_{k_2=0}^{\infty} \frac{A_0^{k_2}}{k_2!} \|u_{k_2}\|_{X_b^s}. \end{aligned} \tag{2.26}$$

Since $\mathbf{u} \in X_{M_0}$, we have from (2.23)

$$\begin{aligned} |||\Phi(u)|||_{\mathcal{A}_{A_0}(X_b^s)} &\leq C_0 \big| \|\phi\| \big|_{\mathcal{A}_{A_0}(H^s)} + C_5 e^{4A_0} T^{\mu} |||\mathbf{u}|||^3_{\mathcal{A}_{A_0}(X_b^s)} + C_6 e^{4A_0} T^{\mu} |||\mathbf{u}|||^2_{\mathcal{A}_{A_0}(X_b^s)} \\ &\leq C_0 M_0 + 8C_0^3 C_5 e^{4A_0} T^{\mu} M_0^3 + 4C_0^2 C_6 e^{4A_0} T^{\mu} M_0^2 \\ &\leq \frac{3}{2} C_0 M_0, \end{aligned} \tag{2.27}$$

which implies $\Phi(u) \in X_{M_0}$. Similarly, we have for u and $\tilde{u} \in \mathcal{A}_{A_0}(X_b^s)$

$$\begin{aligned} &|||\Phi(u) - \Phi(\tilde{u})|||_{\mathcal{A}_{A_0}(X_b^s)} \\ &\leq C_5 e^{4A_0} T^{\mu} \\ &\quad \times \left(|||\mathbf{u}|||^2_{\mathcal{A}_{A_0}(X_b^s)} + |||\mathbf{u}|||_{\mathcal{A}_{A_0}(X_b^s)} |||\tilde{\mathbf{u}}|||_{\mathcal{A}_{A_0}(X_b^s)} + |||\tilde{\mathbf{u}}|||^2_{\mathcal{A}_{A_0}(X_b^s)} \right) |||\mathbf{u} - \tilde{\mathbf{u}}|||_{\mathcal{A}_{A_0}(X_b^s)} \\ &\quad + C_6 e^{4A_0} T^{\mu} \left(|||\mathbf{u}|||_{\mathcal{A}_{A_0}(X_b^s)} + |||\tilde{\mathbf{u}}|||_{\mathcal{A}_{A_0}(X_b^s)} \right) |||\mathbf{u} - \tilde{\mathbf{u}}|||_{\mathcal{A}_{A_0}(X_b^s)} \\ &\leq \left(12C_0^2 C_5 e^{4A_0} M_0^2 + 4C_0 C_6 e^{4A_0} M_0 \right) T^{\mu} |||\mathbf{u} - \tilde{\mathbf{u}}|||_{\mathcal{A}_{A_0}(X_b^s)} \\ &\leq \frac{1}{2} |||\mathbf{u} - \tilde{\mathbf{u}}|||_{\mathcal{A}_{A_0}(X_b^s)}. \end{aligned} \tag{2.28}$$

Thus, the mapping Φ is contraction from X_{M_0} to itself. We obtain a unique fixed point $u_k \in X_b^s$ satisfying

$$u_k(t) = \psi(t) e^{-t\partial_x^5} \phi_k + \psi(t) \int_0^t e^{-(t-t')\partial_x^5} \psi_T(t') \mathcal{N}_k(u)(t') dt' \tag{2.29}$$

on the time interval $[-T, T]$ and $k = 0, 1, 2, \ldots$. Uniqueness of the solution is also shown by using Bekiranov et al.'s argument in [22]. This completes the proof. □

3. Proof of Lemma 2.4

In this section we prove Lemma 2.4. To prove Lemma 2.4 we prepare the following lemma.

Lemma 3.1 (see [22]). *(1) Let $\alpha, \beta > 0$ and let $\kappa = \min\{\alpha, \beta\}$. If*

$$\alpha + \beta > 1 + \kappa, \tag{3.1}$$

then

$$\left(\int_{-\infty}^{\infty} \frac{dx}{(1+|x-\zeta|)^{\alpha}(1+|x-\eta|)^{\beta}}\right)^{1/2} \leq C_{7,\alpha,\beta}\left(\frac{1}{(1+|\zeta-\eta|)^{\kappa}}\right)^{1/2}, \quad \text{for any } \zeta, \eta \in \mathbb{R}, \tag{3.2}$$

where $C_{7,\alpha,\beta}$ is a constant depending on α and β.

(2) If $\gamma > 1$, then

$$\left(\int_{-\infty}^{\infty} \frac{dx}{(1+|x+\eta|)^{\gamma}}\right)^{1/2} \leq C_{8,\gamma}, \quad \text{for any } \eta \in \mathbb{R}, \tag{3.3}$$

where $C_{8,\gamma}$ is a constant depending on γ.

Proof of Lemma 2.4. By duality, we have

$$\begin{aligned} \|\partial_x((\partial_x u)(\partial_x v))\|_{X^s_{b'-1}} &= \left\| \left\langle \tau - \xi^5 \right\rangle^{(b'-1)} \langle \xi \rangle^s \xi (\mathcal{F}_{t,x}\partial_x u) * (\mathcal{F}_{t,x}\partial_x u) \right\|_{L^2_\tau L^2_\xi} \\ &= \sup_{h \in L^2_\tau L^2_\xi, \|h\|_{L^2_\tau L^2_\xi} \leq 1} \left| \left(\frac{\langle \xi \rangle^s \xi}{\langle \tau - \xi^5 \rangle^{1-b'}} (\mathcal{F}_{t,x}\partial_x u) * (\mathcal{F}_{t,x}\partial_x v), h \right)_{L^2_\tau L^2_\xi} \right|, \end{aligned} \tag{3.4}$$

where $(\cdot,\cdot)_{L^2_\tau L^2_\xi}$ is the inner product in $L^2(\mathbb{R}_\tau \times \mathbb{R}_\xi)$. Setting

$$f(\tau,\xi) = \left\langle \tau - \xi^5 \right\rangle^{b} \langle \xi \rangle^s \mathcal{F}_{t,x} u(\tau,\xi), \qquad g(\tau,\xi) = \left\langle \tau - \xi^5 \right\rangle^{b} \langle \xi \rangle^s \mathcal{F}_{t,x} v(\tau,\xi), \tag{3.5}$$

we have

$$\begin{aligned}
&\left(\frac{\langle\xi\rangle^s\xi}{\langle\tau-\xi^5\rangle^{1-b'}}(\mathcal{F}_{t,x}\partial_x u)*(\mathcal{F}_{t,x}\partial_x v),h\right)_{L^2_\tau L^2_\xi}\\
&=\iint_{\mathbb{R}^2}\frac{\langle\xi\rangle^s\xi}{\langle\tau-\xi^5\rangle^{1-b'}}\\
&\qquad\times\left(\iint_{\mathbb{R}^2}\frac{\xi_1(\xi-\xi_1)f(\tau_1,\xi_1)g(\tau-\tau_1,\xi-\xi_1)}{\langle\xi_1\rangle^s\langle\xi-\xi_1\rangle^s\langle\tau_1-\xi_1^5\rangle^b\left\langle\tau-\tau_1-(\xi-\xi_1)^5\right\rangle^b}d\tau_1d\xi_1\right)h(\tau,\xi)d\tau\, d\xi\\
&=I_{\Omega_{0.0.0}}+I_{\Omega_{0.0.0}{}^c},
\end{aligned}\tag{3.6}$$

where

$$\begin{aligned}
I_{\Omega_{0.0.0}}&=\iiiint_{\Omega_{0.0.0}}\frac{\langle\xi\rangle^s\xi\xi_1(\xi-\xi_1)h(\tau,\xi)f(\tau_1,\xi_1)g(\tau-\tau_1,\xi-\xi_1)}{\langle\tau-\xi^5\rangle^{1-b'}\langle\tau_1-\xi_1^5\rangle^b\left\langle\tau-\tau_1-(\xi-\xi_1)^5\right\rangle^b\langle\xi_1\rangle^s\langle\xi-\xi_1\rangle^s}d\tau_1d\xi_1d\tau\, d\xi,\\
I_{\Omega_{0.0.0}{}^c}&=\iiiint_{\Omega_{0.0.0}{}^c}\frac{\langle\xi\rangle^s\xi\xi_1(\xi-\xi_1)h(\tau,\xi)f(\tau_1,\xi_1)g(\tau-\tau_1,\xi-\xi_1)}{\langle\tau-\xi^5\rangle^{1-b'}\langle\tau_1-\xi_1^5\rangle^b\left\langle\tau-\tau_1-(\xi-\xi_1)^5\right\rangle^b\langle\xi_1\rangle^s\langle\xi-\xi_1\rangle^s}d\tau_1d\xi_1d\tau\, d\xi,
\end{aligned}\tag{3.7}$$

$$\Omega_{0.0.0}=\left\{(\tau,\tau_1,\xi,\xi_1)\in\mathbb{R}^4:|\xi|,|\xi_1|,|\xi-\xi_1|\le 5\right\},\qquad \Omega_{0.0.0}{}^c=\mathbb{R}^4\setminus\Omega_{0.0.0}.\tag{3.8}$$

We split $\Omega_{0.0.0}{}^c$ into three regions, $\Omega_1,\Omega_2,$ and Ω_3:

$$\begin{aligned}
\Omega_1&=\left\{(\tau,\tau_1,\xi,\xi_1)\in\Omega_{0.0.0}{}^c:|\xi|\le\frac{1}{4}|\xi_1|\right\},\\
\Omega_2&=\left\{(\tau,\tau_1,\xi,\xi_1)\in\Omega_{0.0.0}{}^c:\frac{1}{4}|\xi_1|\le|\xi|\le 4|\xi_1|\right\},\\
\Omega_3&=\{(\tau,\tau_1,\xi,\xi_1)\in\Omega_{0.0.0}{}^c:4|\xi_1|\le|\xi|\},
\end{aligned}\tag{3.9}$$

and then, we split Ω_i $(i=1,2,3)$ into three regions:

$$\begin{aligned}
\Omega_{i.1}&=\left\{(\tau,\tau_1,\xi,\xi_1)\in\Omega_i:\left|\tau_1-\xi_1^5\right|,\ \left|\tau-\tau_1-(\xi-\xi_1)^5\right|\le\left|\tau-\xi^5\right|\right\},\\
\Omega_{i.2}&=\left\{(\tau,\tau_1,\xi,\xi_1)\in\Omega_i:\left|\tau-\xi^5\right|,\left|\tau-\tau_1-(\xi-\xi_1)^5\right|\le\left|\tau_1-\xi_1^5\right|\right\},\\
\Omega_{i.3}&=\left\{(\tau,\tau_1,\xi,\xi_1)\in\Omega_i:\left|\tau-\xi^5\right|,\left|\tau_1-\xi_1^5\right|\le\left|\tau-\tau_1-(\xi-\xi_1)^5\right|\right\}.
\end{aligned}\tag{3.10}$$

We further split $\Omega_{1.j}$, $\Omega_{2.j}$, and $\Omega_{3.j}$ $(j = 1, 2, 3)$ into the following regions:

$$\begin{aligned}
&\Omega_{1.j.1} = \left\{(\tau, \tau_1, \xi, \xi_1) \in \Omega_{1.j} : |\xi| \geq 1\right\}, \\
&\Omega_{1.j.2} = \left\{(\tau, \tau_1, \xi, \xi_1) \in \Omega_{1.j} : |\xi| \leq 1, |\xi||\xi_1|^4 \geq 1\right\}, \\
&\Omega_{1.j.3} = \left\{(\tau, \tau_1, \xi, \xi_1) \in \Omega_{1.j} : |\xi||\xi_1|^4 \leq 1\right\}, \\
&\Omega_{2.1.1} = \{(\tau, \tau_1, \xi, \xi_1) \in \Omega_{2.1} : |\xi - \xi_1| \leq 1\}, \\
&\Omega_{2.1.2} = \left\{(\tau, \tau_1, \xi, \xi_1) \in \Omega_{2.1} : |\xi - \xi_1| \geq 1, |\xi - 2\xi_1| \geq |\xi|^{-3/2}\right\}, \\
&\Omega_{2.1.3} = \left\{(\tau, \tau_1, \xi, \xi_1) \in \Omega_{2.1} : |\xi - \xi_1| \geq 1, |\xi - 2\xi_1| \leq |\xi|^{-3/2}\right\}, \\
&\Omega_{2.2.1} = \{(\tau, \tau_1, \xi, \xi_1) \in \Omega_{2.2} : |\xi - \xi_1| \leq 1\}, \\
&\Omega_{2.2.2} = \left\{(\tau, \tau_1, \xi, \xi_1) \in \Omega_{2.2} : |\xi - \xi_1| \geq 1, |2\xi - \xi_1| \geq |\xi_1|^{-3/2}\right\}, \\
&\Omega_{2.2.3} = \left\{(\tau, \tau_1, \xi, \xi_1) \in \Omega_{2.2} : |\xi - \xi_1| \geq 1, |2\xi - \xi_1| \leq |\xi_1|^{-3/2}\right\}, \\
&\Omega_{2.3.1} = \left\{(\tau, \tau_1, \xi, \xi_1) \in \Omega_{2.3} : |\xi - \xi_1| \leq 1, |\xi - \xi_1||\xi_1|^4 \geq 1\right\}, \\
&\Omega_{2.3.2} = \left\{(\tau, \tau_1, \xi, \xi_1) \in \Omega_{2.3} : |\xi - \xi_1||\xi_1|^4 \leq 1\right\}, \\
&\Omega_{2.3.3} = \{(\tau, \tau_1, \xi, \xi_1) \in \Omega_{2.3} : |\xi - \xi_1| \geq 1,\ |\xi + \xi_1| \geq 1\}, \\
&\Omega_{2.3.4} = \left\{(\tau, \tau_1, \xi, \xi_1) \in \Omega_{2.3} : |\xi - \xi_1| \geq 1, |\xi - \xi_1|^{-3/2} \leq |\xi + \xi_1| \leq 1\right\}, \\
&\Omega_{2.3.5} = \left\{(\tau, \tau_1, \xi, \xi_1) \in \Omega_{2.3} : |\xi - \xi_1| \geq 1,\ |\xi + \xi_1| \leq |\xi - \xi_1|^{-3/2}\right\}, \\
&\Omega_{3.j.1} = \left\{(\tau, \tau_1, \xi, \xi_1) \in \Omega_{3.j} : |\xi_1| \geq 1\right\}, \\
&\Omega_{3.j.2} = \left\{(\tau, \tau_1, \xi, \xi_1) \in \Omega_{3.j} : |\xi_1| \leq 1,\ |\xi_1||\xi|^4 \geq 1\right\}, \\
&\Omega_{3.j.3} = \left\{(\tau, \tau_1, \xi, \xi_1) \in \Omega_{3.j} : |\xi_1||\xi|^4 \leq 1\right\},
\end{aligned} \tag{3.11}$$

so that, we have

$$|I_{\Omega_{0.0.0}} + I_{\Omega_{0.0.0}^c}| \leq |I_{\Omega_{0.0.0}}| + \sum_{i,j,k=1}^{3} \left|I_{\Omega_{i.j.k}}\right| + |I_{\Omega_{2.3.4}}| + |I_{\Omega_{2.3.5}}|. \tag{3.12}$$

Now we will estimate $|I_{\Omega_{0.0.0}}|$, $|I_{\Omega_{i.j.k}}|$ $(i, j, k = 1, 2, 3)$, and $|I_{\Omega_{2.3.l}}|$ $(l = 4, 5)$. To estimate these terms, we prepare some estimates. By (3.8), (3.9) we obtain

$$\frac{3}{4}|\xi_1| \leq |\xi - \xi_1| \leq \frac{5}{4}|\xi_1|, \quad \text{in } \Omega_1,$$
$$\frac{3}{4}|\xi| \leq |\xi - \xi_1| \leq \frac{5}{4}|\xi|, \quad \text{in } \Omega_3, \tag{3.13}$$

$$|\xi_1| \geq 4, \quad \text{in } \Omega_1 \cup \Omega_{2.2.3},$$
$$\min\{|\xi_1|, |\xi|\} \geq 1, \quad \text{in } \Omega_2 \setminus \{\Omega_4 \cup \Omega_{2.1.3} \cup \Omega_{2.2.3}\},$$
$$\min\{|\xi_1|, |\xi|\} \geq 3, \quad \text{in } \Omega_4, \tag{3.14}$$
$$|\xi| \geq 4, \quad \text{in } \Omega_3 \cup \Omega_{2.1.3},$$

where $\Omega_4 = \Omega_{2.1.1} \cup \Omega_{2.2.1} \cup \Omega_{2.3.1} \cup \Omega_{2.3.2}$. Since (3.13) and (3.14) yield

$$|\xi - \xi_1| \geq 3, \quad \text{in } \Omega_1, \Omega_3, \tag{3.15}$$

we have by (3.9) and (3.13)–(3.15)

$$\frac{|\xi_1|^2|\xi - \xi_1|^2}{\langle \xi_1 \rangle^{2s} \langle \xi - \xi_1 \rangle^{2s}} \leq C_{9,s}^2 |\xi_1|^{4-4s}, \quad \text{in } \Omega_1,$$
$$\frac{|\xi|^2|\xi_1|^2 \langle \xi \rangle^{2s}}{\langle \xi_1 \rangle^{2s}} \leq C_{9,s}^2 |\xi_1|^4, \quad \text{in } \Omega_2, \tag{3.16}$$
$$\frac{|\xi|^2|\xi - \xi_1|^2 \langle \xi \rangle^{2s}}{\langle \xi - \xi_1 \rangle^{2s}} \leq C_{9,s}^2 |\xi|^4, \quad \text{in } \Omega_3,$$

where $C_{9,s} = 4^{|s|+1}$. Using (3.11), (3.14), and (3.15), we have

$$\langle \xi \rangle^{2s} \leq 2^{2|s|} \max\{1, |\xi|\}^{2s} \quad \text{in } \Omega_1,$$
$$\langle \xi - \xi_1 \rangle^{-2s} \leq 2^{2|s|} \max\{1, |\xi - \xi_1|\}^{-2s} \quad \text{in } \Omega_2, \tag{3.17}$$
$$\langle \xi_1 \rangle^{-2s} \leq 2^{2|s|} \max\{1, |\xi_1|\}^{-2s} \quad \text{in } \Omega_3.$$

In $\Omega_{i_1.j_1.k_1}$ $((i_1.j_1.k_1) = (0.0.0), (1.1.3), (3.2.1), (3.2.2), (i.2.k), (i = 1,2,\ k = 1,2,3))$, we integrate with respect to τ and ξ first, then, we use Schwarz's inequality, Fubini's theorem, and note that $\|h\|_{L^2_\tau L^2_\xi} \le 1$ to have

$$\begin{aligned}
\left|I_{\Omega_{i_1.j_1.k_1}}\right| &\le \|f\|_{L^2_{\tau_1} L^2_{\xi_1}} \\
&\quad\times \left\| \frac{|\xi_1|}{\langle \tau_1 - \xi_1^5\rangle^b \langle \xi_1\rangle^s} \left(\iint_{\mathbb{R}^2} \frac{h(\tau,\xi) g(\tau-\tau_1, \xi-\xi_1) \langle \xi\rangle^s |\xi| |\xi-\xi_1| \chi_{\Omega_{i_1.j_1.k_1}}(\tau,\tau_1,\xi,\xi_1) d\xi\, d\tau}{\langle \tau - \xi^5\rangle^{1-b'} \left\langle \tau - \tau_1 - (\xi-\xi_1)^5 \right\rangle^b \langle \xi - \xi_1\rangle^s} \right) \right\|_{L^2_{\tau_1} L^2_{\xi_1}} \\
&\le \|f\|_{L^2_\tau L^2_\xi} \left\| \left(\iint_{\mathbb{R}^2} |h(\tau,\xi)|^2 \left| g(\tau-\tau_1,\xi-\xi_1)\right|^2 d\xi\, d\tau \right)^{1/2} \right. \\
&\qquad \left. \times \frac{|\xi_1|}{\langle \tau_1 - \xi_1^5\rangle^b \langle \xi_1\rangle^s} \left(\iint_{\mathbb{R}^2} \frac{\langle \xi\rangle^{2s} |\xi|^2 |\xi-\xi_1|^2 \chi_{\Omega_{i_1.j_1.k_1}}(\tau,\tau_1,\xi,\xi_1) d\xi\, d\tau}{\langle \tau - \xi^5\rangle^{2(1-b')} \left\langle \tau - \tau_1 - (\xi-\xi_1)^5 \right\rangle^{2b} \langle \xi - \xi_1\rangle^{2s}} \right)^{1/2} \right\|_{L^2_{\tau_1} L^2_{\xi_1}} \\
&\le \|f\|_{L^2_\tau L^2_\xi} \|g\|_{L^2_\tau L^2_\xi} \\
&\quad\times \left\| \frac{|\xi_1|}{\langle \tau_1 - \xi_1^5\rangle^b \langle \xi_1\rangle^s} \left(\iint_{\mathbb{R}^2} \frac{\langle \xi\rangle^{2s} |\xi|^2 |\xi-\xi_1|^2 \chi_{\Omega_{i_1.j_1.k_1}}(\tau,\tau_1,\xi,\xi_1) d\xi\, d\tau}{\langle \tau - \xi^5\rangle^{2(1-b')} \left\langle \tau - \tau_1 - (\xi-\xi_1)^5 \right\rangle^{2b} \langle \xi - \xi_1\rangle^{2s}} \right)^{1/2} \right\|_{L^\infty_{\tau_1} L^\infty_{\xi_1}},
\end{aligned} \tag{3.18}$$

where

$$\chi_{\Omega_{i_1.j_1.k_1}}(\tau,\tau_1,\xi,\xi_1) \equiv \begin{cases} 1, & \text{if } (\tau,\tau_1,\xi,\xi_1) \in \Omega_{i_1.j_1.k_1}, \\ 0, & \text{if } (\tau,\tau_1,\xi,\xi_1) \notin \Omega_{i_1.j_1.k_1}. \end{cases} \tag{3.19}$$

In $\Omega_{i_2.j_2.k_2}$ $((i_2.j_2.k_2) = (i.1.k), (i = 1,2,3,\ k = 1,2), (2.1.3), (3.1.3), (3.2.3))$, we integrate with respect to τ and ξ first, then, we use the same way as in (3.18) to have

$$\begin{aligned}
\left|I_{\Omega_{i_2.j_2.k_2}}\right| &\le \|f\|_{L^2_\tau L^2_\xi} \|g\|_{L^2_\tau L^2_\xi} \\
&\quad\times \left\| \frac{\langle \xi\rangle^s |\xi|}{\langle \tau - \xi^5\rangle^{(1-b')}} \left(\iint_{\mathbb{R}^2} \frac{|\xi_1|^2 |\xi-\xi_1|^2\ \chi_{\Omega_{i_2.j_2.k_2}}(\tau,\tau_1,\xi,\xi_1) d\xi_1 d\tau_1}{\langle \tau_1 - \xi_1^5\rangle^{2b} \left\langle \tau - \tau_1 - (\xi-\xi_1)^5 \right\rangle^{2b} \langle \xi_1\rangle^{2s} \langle \xi - \xi_1\rangle^{2s}} \right)^{1/2} \right\|_{L^\infty_\tau L^\infty_\xi}.
\end{aligned} \tag{3.20}$$

In $\Omega_{2.3.2}$ we use the change of variables

$$\tau_2 = \tau_1 - \tau, \qquad \xi_2 = \xi_1 - \xi \tag{3.21}$$

to obtain

$$\begin{aligned}
&\iiiint_{\Omega_{2.3.2}} \frac{\langle \xi\rangle^s \xi\xi_1(\xi-\xi_1)h(\tau,\xi)f(\tau_1,\xi_1)g(\tau-\tau_1,\xi-\xi_1)}{\langle \tau-\xi^5\rangle^{1-b'}\langle \tau_1-\xi_1^5\rangle^b\left\langle \tau-\tau_1-(\xi-\xi_1)^5\right\rangle^b\langle\xi_1\rangle^s\langle\xi-\xi_1\rangle^s} d\tau_1 d\xi_1 d\tau\, d\xi \\
&= \iiiint_{\widetilde{\Omega}_{2.3.2}} \frac{\langle \xi_1-\xi_2\rangle^s \xi_1(\xi_1-\xi_2)(-\xi_2)h(\tau_1-\tau_2,\xi_1-\xi_2)f(\tau_1,\xi_1)g(-\tau_2,-\xi_2)}{\left\langle \tau_1-\tau_2-(\xi-\xi_1)^5\right\rangle^{1-b'}\langle \tau_1-\xi_1^5\rangle^b\langle\tau_2-\xi_2^5\rangle^b\langle\xi_1\rangle^s\langle\xi_1\rangle^s\langle\xi_2\rangle^s} d\tau_1 d\xi_1 d\tau_2 d\xi_2 \\
&\equiv J_{\widetilde{\Omega}_{2.3.2}},
\end{aligned} \tag{3.22}$$

where

$$\begin{aligned}
\widetilde{\Omega}_{2.3.2} = \Big\{ &(\tau_1,\tau_2,\xi_1,\xi_2) \in {\Omega_{0.0.0}}^c : \frac{1}{4}|\xi_1-\xi_2| \le |\xi_1| \le 4|\xi_1-\xi_2|, \\
&\left|\tau_1-\tau_2-(\xi_1-\xi_2)^5\right|, \left|\tau_1-\xi_1^5\right| \le \left|\tau_2-\xi_2^5\right|,\ |\xi_2||\xi_1|^4 \le 1 \Big\}.
\end{aligned} \tag{3.23}$$

We integrate with respect to τ_2 and ξ_2 first, then, we use the same way as in (3.18) to have

$$\begin{aligned}
\left|J_{\widetilde{\Omega}_{2.3.2}}\right| \le{}& \|f\|_{L^2_\tau L^2_\xi}\|g\|_{L^2_\tau L^2_\xi} \\
&\times \left\| \frac{|\xi_1|}{\langle\tau_1-\xi_1^5\rangle^b\langle\xi_1\rangle^s}\left(\iint_{\mathbb{R}^2}\frac{\langle\xi_1-\xi_2\rangle^{2s}|\xi_2|^2|\xi_1-\xi_2|^2\chi_{\widetilde{\Omega}_{2.3.2}}(\tau_1,\tau_2,\xi_1,\xi_2)d\xi_2 d\tau_2}{\left\langle\tau_1-\tau_2-(\xi_1-\xi_2)^5\right\rangle^{2(1-b')}\langle\tau_2-\xi_2^5\rangle^{2b}\langle\xi_2\rangle^{2s}}\right)^{1/2}\right\|_{L^\infty_{\tau_1}L^\infty_{\xi_1}}.
\end{aligned} \tag{3.24}$$

In $\Omega_{i_3.3.k_3}$ $((i_3.3.k_3) = (i.3.k), (i = 1,3, k = 1,2,3), (2.3.1), (2.3.3), (2.3.4), (3.3.5))$, we have by a similar argument to (3.22)

$$I_{\Omega_{i_3.3.k_3}} = J_{\widetilde{\Omega}_{i_3.3.k_3}}, \tag{3.25}$$

where $\widetilde{\Omega}_{i_3.3.k_3}$ is the region which is obtained from $\Omega_{i_3.3.k_3}$ by the change of variables $\tau_2 = \tau_1 - \tau$ and $\xi_2 = \xi_1 - \xi$. We integrate with respect to τ_1 and ξ_1 first, then, we use the same way as in (3.18) to have

$$\left|J_{\widetilde{\Omega}_{i_3.3.k_3}}\right| \leq \|f\|_{L^2_\tau L^2_\xi}\|g\|_{L^2_\tau L^2_\xi}$$
$$\times \left\| \frac{|\xi_2|}{\langle \tau_2 - \xi_2^5\rangle^b \langle \xi_2\rangle^s} \left(\iint_{\mathbb{R}^2} \frac{\langle \xi_1 - \xi_2\rangle^{2s} |\xi_1|^2 |\xi_1 - \xi_2|^2 \chi_{\widetilde{\Omega}_{i_3.3.k_3}}(\tau_1,\tau_2,\xi_1,\xi_2)\, d\xi_1 d\tau_1}{\left\langle \tau_1 - \tau_2 - (\xi_1 - \xi_2)^5\right\rangle^{2(1-b')} \langle \tau_1 - \xi_1^5\rangle^{2b} \langle \xi_1\rangle^{2s}} \right)^{1/2} \right\|_{L^\infty_{\tau_2} L^\infty_{\xi_2}}. \tag{3.26}$$

Now we will get bounds for

$$\left\| \frac{|\xi_1|}{\langle \tau_1 - \xi_1^5\rangle^b \langle \xi_1\rangle^s} \left(\iint_{\mathbb{R}^2} \frac{\langle \xi\rangle^{2s} |\xi|^2 |\xi - \xi_1|^2 \chi_{\Omega_{i_1.j_1.k_1}}(\tau,\tau_1,\xi,\xi_1)\, d\xi\, d\tau}{\langle \tau - \xi^5\rangle^{2(1-b')} \left\langle \tau - \tau_1 - (\xi - \xi_1)^5\right\rangle^{2b} \langle \xi - \xi_1\rangle^{2s}} \right)^{1/2} \right\|_{L^\infty_{\tau_1} L^\infty_{\xi_1}},$$
$$\left\| \frac{\langle \xi\rangle^{s} |\xi|}{\langle \tau - \xi^5\rangle^{(1-b')}} \left(\iint_{\mathbb{R}^2} \frac{|\xi_1|^2 |\xi - \xi_1|^2 \chi_{\Omega_{i_2.j_2.k_2}}(\tau,\tau_1,\xi,\xi_1)\, d\xi_1 d\tau_1}{\langle \tau_1 - \xi_1^5\rangle^{2b} \left\langle \tau - \tau_1 - (\xi - \xi_1)^5\right\rangle^{2b} \langle \xi_1\rangle^{2s} \langle \xi - \xi_1\rangle^{2s}} \right)^{1/2} \right\|_{L^\infty_{\tau} L^\infty_{\xi}},$$
$$\left\| \frac{|\xi_1|}{\langle \tau_1 - \xi_1^5\rangle^b \langle \xi_1\rangle^s} \left(\iint_{\mathbb{R}^2} \frac{\langle \xi_1 - \xi_2\rangle^{2s} |\xi_2|^2 |\xi_1 - \xi_2|^2 \chi_{\widetilde{\Omega}_{2.3.2}}\, d\xi_2 d\tau_2}{\left\langle \tau_1 - \tau_2 - (\xi_1 - \xi_2)^5\right\rangle^{2(1-b')} \langle \tau_2 - \xi_2^5\rangle^{2b} \langle \xi_2\rangle^{2s}} \right)^{1/2} \right\|_{L^\infty_{\tau_1} L^\infty_{\xi_1}},$$
$$\left\| \frac{|\xi_2|}{\langle \tau_2 - \xi_2^5\rangle^b \langle \xi_2\rangle^s} \left(\iint_{\mathbb{R}^2} \frac{\langle \xi_1 - \xi_2\rangle^{2s} |\xi_2|^2 |\xi_1 - \xi_2|^2 \chi_{\widetilde{\Omega}_{i_3.3.k_3}}\, d\xi_1 d\tau_1}{\left\langle \tau_1 - \tau_2 - (\xi_1 - \xi_2)^5\right\rangle^{2(1-b')} \langle \tau_1 - \xi_1^5\rangle^{2b} \langle \xi_1\rangle^{2s}} \right)^{1/2} \right\|_{L^\infty_{\tau_2} L^\infty_{\xi_2}}. \tag{3.27}$$

By using the following methods, we estimate (3.27).

The Case of $\Omega_{0.0.0}$

Since

$$\left(\tau_1 + (\xi - \xi_1)^5\right) - \xi^5 = \tau_1 - \xi_1^5 - 5\xi\xi_1(\xi - \xi_1)\left(\xi^2 - \xi\xi_1 + \xi_1^2\right), \tag{3.28}$$

it follows from (3.2) in Lemma 3.1 with $\alpha = 2b$, $\beta = \kappa = 2(1-b')$ that

$$\left\| \frac{|\xi_1|}{\langle \tau_1 - \xi_1^5 \rangle^b \langle \xi_1 \rangle^s} \left(\iint_{\mathbb{R}^2} \frac{\langle \xi \rangle^{2s} |\xi|^2 |\xi - \xi_1|^2 \chi_{\Omega_{0.0.0}}(\tau, \tau_1, \xi, \xi_1) d\xi \, d\tau}{\langle \tau - \xi^5 \rangle^{2(1-b')} \left\langle \tau - \tau_1 - (\xi - \xi_1)^5 \right\rangle^{2b} \langle \xi - \xi_1 \rangle^{2s}} \right)^{1/2} \right\|_{L^\infty_{\tau_1} L^\infty_{\xi_1}}$$

$$\leq C_{7,2(1-b'),2b}$$

$$\times \left\| \frac{|\xi_1| \chi_{\Omega_B}(\xi_1)}{\langle \tau_1 - \xi_1^5 \rangle^b \langle \xi_1 \rangle^s} \left(\int_{\mathbb{R}} \frac{\langle \xi \rangle^{2s} |\xi|^2 |\xi - \xi_1|^2 \chi_{\Omega_{A;\xi_1}}(\xi)}{\langle \tau_1 - \xi_1^5 - 5\xi\xi_1(\xi - \xi_1)(\xi^2 - \xi\xi_1 + \xi_1^2) \rangle^{2(1-b')} \langle \xi - \xi_1 \rangle^{2s}} d\xi \right)^{1/2} \right\|_{L^\infty_{\tau_1} L^\infty_{\xi_1}}, \tag{3.29}$$

where $\Omega_{A;\xi_1} = \{\xi : |\xi|, |\xi - \xi_1| \leq 5\}$ and $\Omega_B = \{\xi_1 : |\xi_1| \leq 5\}$. Since $\langle \xi \rangle^s \leq \max\{1, 6^s\}$, we have

$$C_{7,2(1-b'),2b}$$

$$\times \left\| \frac{|\xi_1| \chi_{\Omega_B}(\xi_1)}{\langle \tau_1 - \xi_1^5 \rangle^b \langle \xi_1 \rangle^s} \left(\int_{\mathbb{R}} \frac{\langle \xi \rangle^{2s} |\xi|^2 |\xi - \xi_1|^2 \chi_{\Omega_{A;\xi_1}}(\xi)}{\langle \tau_1 - \xi_1^5 - 5\xi\xi_1(\xi - \xi_1)(\xi^2 - \xi\xi_1 + \xi_1^2) \rangle^{2(1-b')} \langle \xi - \xi_1 \rangle^{2s}} d\xi \right)^{1/2} \right\|_{L^\infty_{\tau_1} L^\infty_{\xi_1}}$$

$$\leq C_{7,2(1-b'),2b} \max\{1, 6^s\} 5^2 \left(\int_{|\xi| \leq 5} |\xi|^2 d\xi \right)^{1/2}$$

$$\leq M_{1,s,b,b'}, \tag{3.30}$$

where $M_{1,s,b,b'}$ is some constant.

The Case of $\Omega_{1.j.3}$, $\Omega_{3.j.3}$ $(j = 1, 2)$, $\widetilde{\Omega}_{i.3.3}$ $(i = 1, 3)$ *and* $\widetilde{\Omega}_{2.3.2}$

We consider $\widetilde{\Omega}_{2.3.2}$. By (3.2), we have

$$\left\| \frac{|\xi_1|}{\langle \tau_1 - \xi_1^5 \rangle^b \langle \xi_1 \rangle^s} \left(\iint_{\mathbb{R}} \frac{\langle \xi_1 - \xi_2 \rangle^{2s} |\xi_2|^2 |\xi - \xi_1|^2 \chi_{\widetilde{\Omega}_{2.3.2}} d\xi_2 d\tau_2}{\left\langle \tau_1 - \tau_2 - (\xi - \xi_1)^5 \right\rangle^{2(1-b')} \langle \tau_2 - \xi_2^5 \rangle^{2b} \langle \xi_2 \rangle^{2s}} \right)^{1/2} \right\|_{L^\infty_{\tau_1} L^\infty_{\xi_1}}$$

$$\leq C_{7,2(1-b'),2b} C_{9,s} 2^{|s|} \tag{3.31}$$

$$\times \left\| \frac{|\xi_1|^2 \chi_{\widetilde{\Omega}_C}(\xi_1)}{\langle \tau_1 - \xi_1^5 \rangle^b} \left(\int_{\mathbb{R}} \frac{|\xi_2|^2 \chi_{\widetilde{\Omega}_{D;\xi_1}}(\xi_2)}{\langle \tau_1 - \xi_1^5 + 5\xi_1\xi_2(\xi_1 - \xi_2)(\xi_1^2 - \xi_1\xi_2 + \xi_2^2) \rangle^{2(1-b')}} d\xi_2 \right)^{1/2} \right\|_{L^\infty_{\tau_1} L^\infty_{\xi_1}},$$

where $\widetilde{\Omega}_C = \{\xi_1 : |\xi_1| \geq 4\}$ and $\widetilde{\Omega}_{D;\xi_1} = \{\xi_2 : |\xi_2| \leq |\xi_1|^{-4}\}$. Here we have used (3.16) and (3.17) with the change of variables $\xi_2 = \xi_1 - \xi$. Noting

$$2b > 0, \qquad 2(1-b') > 0, \tag{3.32}$$

we have

$$\begin{aligned}
&\left\| \frac{|\xi_1|^2 \chi_{\widetilde{\Omega}_C}(\xi_1)}{\langle \tau_1 - \xi_1^5 \rangle^b} \left(\int_{\mathbb{R}} \frac{|\xi_2|^2 \chi_{\widetilde{\Omega}_{D;\xi_1}}(\xi_2)}{\langle \tau_1 - \xi_1^5 + 5\xi_1\xi_2(\xi_1-\xi_2)(\xi_1^2 - \xi_1\xi_2 + \xi_2^2) \rangle^{2(1-b')}} d\xi_2 \right)^{1/2} \right\|_{L^\infty_{\tau_1} L^\infty_{\xi_1}} \\
&\quad \leq \left\| \left\{ |\xi_1|^4 \chi_{\widetilde{\Omega}_C}(\xi_1) \left(\int_{|\xi_2| \leq |\xi_1|^{-4}} |\xi_2|^2 d\xi_2 \right) \right\}^{1/2} \right\|_{L^\infty_{\xi_1}} \\
&\quad \leq 2^{1/2} \left\| \left(|\xi_1|^{-8} \chi_{\widetilde{\Omega}_C}(\xi_1) \right)^{1/2} \right\|_{L^\infty_{\xi_1}} \\
&\quad \leq 2^{1/2}.
\end{aligned} \tag{3.33}$$

Thus, (3.27) is bounded by

$$M_{2,s,b,b'} = C_{9,s} 2^{|s|+1/2} \max\{C_{7,2(1-b'),2b}, C_{7,2b,2b}\} \tag{3.34}$$

in $\widetilde{\Omega}_{2.3.2}$. In the same manner as (3.31)-(3.33), (3.27) are bounded by $M_{2,s,b,b'}$ in $\Omega_{1.j.3}, \Omega_{3.j.3}$ $(j = 1, 2)$, and $\widetilde{\Omega}_{i.3.3}$ $(i = 1, 3)$.

The Case of $\Omega_{2.1.1}, \Omega_{2.2.1}$

We consider $\Omega_{2.2.1}$. Since

$$\begin{aligned}
&\left| \tau - \xi^5 \right| + \left| \tau_1 - \xi_1^5 \right| + \left| \tau - \tau_1 - (\xi - \xi_1)^5 \right| \\
&\quad \geq \left| \tau - \xi^5 - \left(\tau_1 - \xi_1^5 \right) - \left(\tau - \tau_1 - (\xi - \xi_1)^5 \right) \right| \\
&\quad = 5|\xi||\xi_1||\xi - \xi_1| \left| \xi^2 - \xi\xi_1 + \xi_1^2 \right|,
\end{aligned} \tag{3.35}$$

we obtain

$$\max\left\{ \left| \tau - \xi^5 \right|, \left| \tau_1 - \xi_1^5 \right|, \left| \tau - \tau_1 - (\xi - \xi_1)^5 \right| \right\} \geq \frac{5}{3} |\xi||\xi_1||\xi - \xi_1| \left| \xi^2 - \xi\xi_1 + \xi_1^2 \right|. \tag{3.36}$$

Noting that $-2b < 0$, we have

$$\left\langle \tau_1 - \xi_1^5 \right\rangle^{-2b} \leq 3^{2b} \left\langle 5\xi\xi_1(\xi - \xi_1)\left(\xi^2 - \xi\xi_1 + \xi_1^2 \right) \right\rangle^{-2b} \quad \text{in } \Omega_{2.2.1}. \tag{3.37}$$

Since (3.14) and $|\xi-\xi_1|\le 1$ yield

$$|\xi_1|\ge 3|\xi-\xi_1|, \tag{3.38}$$

we have

$$|2\xi-\xi_1|\ge|\xi_1|-2|\xi-\xi_1|\ge\frac{1}{3}|\xi_1|. \tag{3.39}$$

By (3.2), (3.16), and (3.37), we have

$$\begin{aligned}
&\left\|\frac{|\xi_1|}{\langle\tau_1-\xi_1^5\rangle^b\langle\xi_1\rangle^s}\left(\iint_{\mathbb{R}^2}\frac{|\xi|^2|\xi-\xi_1|^2\langle\xi\rangle^{2s}\chi_{\Omega_{2.2.1}}(\tau,\tau_1,\xi,\xi_1)d\tau\,d\xi}{\langle\tau-\xi^5\rangle^{2(1-b')}\left\langle\tau-\tau_1-(\xi-\xi_1)^5\right\rangle^{2b}\langle\xi-\xi_1\rangle^{2s}}\right)^{1/2}\right\|_{L^\infty_{\tau_1}L^\infty_{\xi_1}}\\
&\quad\le C_{7,2(1-b'),2b}\,C_{9,s}\,2^{|s|}3^b\\
&\qquad\times\left\|\left(\int_{\mathbb{R}}\frac{\langle 5\xi\xi_1(\xi-\xi_1)(\xi^2-\xi\xi_1+\xi_1^2)\rangle^{-2b}|\xi_1|^4|\xi-\xi_1|^2\chi_{\Omega_{E;\xi}}(\xi_1)}{\langle\tau_1-\xi_1^5-5\xi\xi_1(\xi-\xi_1)(\xi^2-\xi\xi_1+\xi_1^2)\rangle^{2(1-b')}}d\xi\right)^{1/2}\right\|_{L^\infty_{\tau_1}L^\infty_{\xi_1}},
\end{aligned}\tag{3.40}$$

where $\Omega_{E;\xi}=\{\xi_1:|\xi-\xi_1|\le 1,\ |\xi_1|\ge 4\}$. Using the change of variable

$$\mu=5\xi\xi_1(\xi-\xi_1)\left(\xi^2-\xi\xi_1+\xi_1^2\right) \tag{3.41}$$

and (3.2), we have

$$\begin{aligned}
&\left\|\left(\int_{\mathbb{R}}\frac{\langle 5\xi\xi_1(\xi-\xi_1)(\xi^2-\xi\xi_1+\xi_1^2)\rangle^{-2b}|\xi_1|^4|\xi-\xi_1|^2\chi_{\Omega_{E;\xi}}(\xi_1)}{\langle\tau_1-\xi_1^5-5\xi\xi_1(\xi-\xi_1)(\xi^2-\xi\xi_1+\xi_1^2)\rangle^{2(1-b')}}d\xi\right)^{1/2}\right\|_{L^\infty_{\tau_1}L^\infty_{\xi_1}}\\
&\quad\le\left\|\left(\int_{|\mu|\le 3|\tau_1-\xi_1^5|}\frac{\langle\mu\rangle^{-2b}|\xi_1|^4d\mu}{\langle\tau_1-\xi_1^5-\mu\rangle^{2(1-b')}5|\xi_1||2\xi-\xi_1|\left|\xi_1^2-2\xi\xi_1+2\xi^2\right|}\right)^{1/2}\right\|_{L^\infty_{\tau_1}L^\infty_{\xi_1}}\\
&\quad\le\left\|\left(\int_{\mathbb{R}}\frac{(5/6)^{-1}d\mu}{\langle\mu\rangle^{2b}\langle\mu-(\tau_1-\xi_1^5)\rangle^{2(1-b')}}\right)^{1/2}\right\|_{L^\infty_{\tau_1}L^\infty_{\xi_1}}\\
&\quad\le C_{7,2(1-b'),2b}\left(\frac{5}{6}\right)^{-1/2},
\end{aligned}\tag{3.42}$$

where we have used

$$5|\xi_1||2\xi-\xi_1|\left|\xi_1^2-2\xi\xi_1+2\xi^2\right| \geq \frac{5}{6}|\xi_1|^4 \quad \text{in } \Omega_{E;\xi}, \tag{3.43}$$

which follows from (3.39) and

$$\left|2\xi^2-2\xi\xi_1+\xi_1^2\right| \geq \left|2\left(\xi-\frac{1}{2}\xi_1\right)^2+\frac{1}{2}\xi_1^2\right| \geq \frac{1}{2}|\xi_1|^2. \tag{3.44}$$

Hence (3.27) is bounded by

$$M_{3,s,b,b'} = \max\left\{C_{7,2(1-b'),2b}^2, C_{7,2(1-b'),2b}C_{7,2b,2b}\right\}C_{9,s}2^{|s|}\left(3^b+3^{1-b'}\right)\left(\frac{5}{6}\right)^{-1/2} \tag{3.45}$$

in $\Omega_{2.2.2}$. By a similar argument to (3.37)–(3.42), (3.27) is also bounded by $M_{3,s,b,b'}$ in $\Omega_{2.2.1}$.

The Case of $\Omega_{2.1.3}$, $\Omega_{2.2.3}$, *and* $\widetilde{\Omega}_{2.3.5}$

We consider $\Omega_{2.2.3}$. Since (3.14) and $|2\xi-\xi_1| \leq |\xi_1|^{-3/2}$ yield

$$|2\xi-\xi_1| \leq \frac{1}{2}|\xi_1|, \tag{3.46}$$

we have

$$\begin{aligned} |\xi-\xi_1| &\geq \frac{1}{2}(|\xi_1|-|2\xi-\xi_1|) \geq \frac{1}{4}|\xi_1|, \\ |\xi-\xi_1| &\leq \frac{1}{2}(|\xi_1|+|2\xi-\xi_1|) \leq \frac{3}{4}|\xi_1|, \end{aligned} \quad \text{in } \Omega_{2.2.3}. \tag{3.47}$$

By (3.9) and (3.47), we have

$$\frac{5}{3}|\xi||\xi_1||\xi-\xi_1|\left|\xi^2-\xi\xi_1+\xi_1^2\right| \geq \frac{5}{48}|\xi_1|^5, \tag{3.48}$$

where we have used

$$\left|\xi^2-\xi\xi_1+\xi_1^2\right| = \left|\left(\xi-\frac{1}{2}\xi_1\right)^2+\frac{3}{4}\xi_1^2\right| \geq \frac{3}{4}|\xi_1|^2. \tag{3.49}$$

Therefore using (3.36), we have

$$\left\langle \tau_1-\xi_1^5\right\rangle^{-2b} \leq \left(\frac{5}{48}\right)^{-2b}|\xi_1|^{-5} \quad \text{in } \Omega_{2.2.3}. \tag{3.50}$$

By (3.2), (3.16), (3.17), (3.47), and (3.50), we have

$$\begin{aligned}
&\left\| \frac{|\xi_1|}{\langle \tau_1 - \xi_1^5 \rangle^b \langle \xi_1 \rangle^s} \left(\iint_{\mathbb{R}^2} \frac{|\xi|^2 |\xi - \xi_1|^2 \langle \xi \rangle^{2s} \chi_{\Omega_{2.2.1}}(\tau, \tau_1, \xi, \xi_1) d\tau\, d\xi}{\langle \tau - \xi^5 \rangle^{2(1-b')} \left\langle \tau - \tau_1 - (\xi - \xi_1)^5 \right\rangle^{2b} \langle \xi - \xi_1 \rangle^{2s}} \right)^{1/2} \right\|_{L^\infty_{\tau_1} L^\infty_{\xi_1}} \\
&\quad \le C_{7,2(1-b'),2b} C_{9,s} 2^{|s|} \max\left\{ \left(\frac{1}{4}\right)^{1-s}, \left(\frac{3}{4}\right)^{1-s} \right\} \left(\frac{5}{48}\right)^{-b} \\
&\qquad \times \left\| \left(\int_{\mathbb{R}} \frac{|\xi_1|^{-5} |\xi_1|^4 |\xi_1|^{2-2s} \chi_{\Omega_G}(\xi_1) \chi_{\Omega_{F;\xi_1}}(\xi) d\xi}{\langle \tau_1 - \xi_1^5 - 5\xi\xi_1(\xi - \xi_1)(\xi^2 - \xi\xi_1 + \xi_1^2) \rangle^{2(1-b')}} \right)^{1/2} \right\|_{L^\infty_{\tau_1} L^\infty_{\xi_1}},
\end{aligned} \tag{3.51}$$

where $\Omega_{F;\xi_1} = \{\xi : |2\xi - \xi_1| \le |\xi_1|^{-3/2}\}$ and $\Omega_G = \{\xi_1 : |\xi_1| \ge 4\}$. Since $|2\xi - \xi_1| \le |\xi_1|^{-3/2}$, we have

$$\left.\begin{aligned}
|\xi| &\ge \frac{1}{2}(|\xi_1| - |2\xi - \xi_1|) \ge \frac{1}{2}\left(|\xi_1| - |\xi_1|^{-3/2}\right), \\
|\xi| &\le \frac{1}{2}(|\xi_1| - |2\xi - \xi_1|) \le \frac{1}{2}\left(|\xi_1| - |\xi_1|^{-3/2}\right),
\end{aligned}\right. \quad \text{in } \Omega_{F;\xi_1}. \tag{3.52}$$

Noting

$$s > -\frac{1}{4}, \qquad 2(1 - b') > 0, \tag{3.53}$$

we have by (3.52)

$$\begin{aligned}
&\left\| \left(\int_{\mathbb{R}} \frac{|\xi_1|^{-5} |\xi_1|^4 |\xi_1|^{2-2s} \chi_{\Omega_G}(\xi_1) \chi_{\Omega_{F;\xi_1}}(\xi) d\xi}{\langle \tau_1 - \xi_1^5 - 5\xi\xi_1(\xi - \xi_1)(\xi^2 - \xi\xi_1 + \xi_1^2) \rangle^{2(1-b')}} \right)^{1/2} \right\|_{L^\infty_{\tau_1} L^\infty_{\xi_1}} \\
&\quad \le \left\| \left\{ |\xi_1|^{1-2s} \chi_{\Omega_G}(\xi_1) \left(\int_{\Omega_{H;\xi_1}} 1\, d\xi \right) \right\}^{1/2} \right\|_{L^\infty_{\tau_1} L^\infty_{\xi_1}} \\
&\quad \le 1,
\end{aligned} \tag{3.54}$$

where

$$\Omega_{H;\xi_1} = \left\{ \xi : \frac{1}{2}\left(|\xi_1| - |\xi_1|^{-3/2}\right) \le |\xi| \le \frac{1}{2}\left(|\xi_1| + |\xi_1|^{-3/2}\right) \right\}. \tag{3.55}$$

Therefore, (3.27) is bounded by

$$M_{4,s,b,b'} = C_{9,s}2^{|s|}\max\{C_{7,2(1-b'),2b}, C_{7,2b,2b}\}\max\left\{\left(\frac{1}{4}\right)^{1-s}, \left(\frac{3}{4}\right)^{1-s}\right\} \tag{3.56}$$

in $\Omega_{2.2.3}$. Using a similar argument to (3.47)–(3.54), we can get bounds of (3.27) in $\Omega_{2.1.3}$ and $\widetilde{\Omega}_{2.3.5}$.

All the Other Cases

We consider $\Omega_{2.1.2}$. By (3.9), we have

$$|\xi - \xi_1| \le |\xi| + |\xi_1| \le 5|\xi|. \tag{3.57}$$

Since $2b' < 1 + 2\sigma$ and $\sigma = \min\{s/5 + 1/20,\ 3/16\}$ yield

$$-\frac{3}{2} + 4(2b' - 1) < -\frac{3}{2} + 8\sigma < 0, \tag{3.58}$$

we have by (3.16), (3.36), and (3.57)

$$\begin{aligned} &\frac{|\xi|^2|\xi_1|^2\langle\xi\rangle^{2s}}{\langle\xi_1\rangle^{2s}} \le C_{9,s}^2\, 5^{3/2}|\xi|^{11/2-8\sigma}|\xi-\xi_1|^{-3/2+8\sigma}, \\ &\left\langle\tau - \xi^5\right\rangle^{2(b'-1)} \le \left(\frac{5}{16}\right)^{2(b'-1)}\left(|\xi-\xi_1||\xi|^4\right)^{-1+2\sigma}, \end{aligned} \quad \text{in } \Omega_{2.1.2}. \tag{3.59}$$

By (3.2), (3.17), and (3.59), we have

$$\begin{aligned} &\left\|\frac{\langle\xi\rangle^{2s}|\xi|^2}{\langle\tau-\xi^5\rangle^{1-b'}}\left(\iint_{\mathbb{R}^2}\frac{|\xi_1|^2|\xi-\xi_1|^2\chi_{\Omega_{2.1.2}}(\tau,\tau_1,\xi,\xi_1)\,d\xi_1 d\tau_1}{\langle\tau_1-\xi_1^5\rangle^{2b}\left\langle\tau-\tau_1-(\xi-\xi_1)^5\right\rangle^{2b}\langle\xi_1\rangle^{2s}\langle\xi-\xi_1\rangle^{2s}}\right)^{1/2}\right\|_{L_\tau^\infty L_\xi^\infty} \\ &\quad\le C_{7,2b,2b}C_{9,s}2^{|s|}5^{3/4}\left(\frac{5}{16}\right)^{(b'-1)} \\ &\qquad\times\left\|\left(\int_{\mathbb{R}}\frac{|\xi|^{(11/2)-8\sigma}|\xi-\xi_1|^{(1/2)-2s+8\sigma}\chi_{\Omega_{I;\xi}}(\xi_1)\,d\xi_1}{\left(|\xi-\xi_1||\xi|^4\right)^{1-2\sigma}\left\langle\tau-\xi^5+5\xi\xi_1(\xi-\xi_1)\left(\xi^2-\xi\xi_1+\xi_1^2\right)\right\rangle^{2b}}\right)^{1/2}\right\|_{L_\tau^\infty L_\xi^\infty}, \end{aligned} \tag{3.60}$$

where $\Omega_{I;\xi} = \{\xi_1 : 1 \le |\xi-\xi_1|, |\xi-2\xi_1| \ge |\xi|^{-3/2}\}$. Noting that

$$2b > 1, \qquad \sigma = \min\left\{\frac{s}{5}+\frac{1}{20}, \frac{3}{16}\right\} \tag{3.61}$$

and using (3.3) and the change of variables

$$\mu = 5\xi\xi_1(\xi-\xi_1)\left(\xi^2-\xi\xi_1+\xi_1^2\right), \tag{3.62}$$

we have

$$\begin{aligned}
&\left\|\left(\int_{\mathbb{R}} \frac{|\xi|^{11/2-8\sigma}|\xi-\xi_1|^{1/2-2s+8\sigma}\chi_{\Omega_{I;\xi}}(\xi_1)}{\left(|\xi-\xi_1||\xi|^4\right)^{1-2\sigma}\left\langle \tau-\xi^5+5\xi\xi_1(\xi-\xi_1)\left(\xi^2-\xi\xi_1+\xi_1^2\right)\right\rangle^{2b}}d\xi_1\right)^{1/2}\right\|_{L_\tau^\infty L_\xi^\infty} \\
&\quad= \left\|\left(\int_{\mathbb{R}} \frac{|\xi|^{3/2}|\xi-\xi_1|^{-1/2-2s+10\sigma}\chi_{\Omega_{I;\xi}}(\xi_1)}{\left\langle \tau-\xi^5+5\xi\xi_1(\xi-\xi_1)\left(\xi^2-\xi\xi_1+\xi_1^2\right)\right\rangle^{2b}}d\xi_1\right)^{1/2}\right\|_{L_\tau^\infty L_\xi^\infty} \\
&\quad\leq \left\|\left(\int_{|\mu|\leq 3|\tau-\xi^5|} \frac{|\xi|^{3/2}}{\left\langle \tau-\xi^5+\mu\right\rangle^{2b}5|\xi||\xi-2\xi_1|\left|\xi^2-2\xi\xi_1+2\xi_1^2\right|}d\mu\right)^{1/2}\right\|_{L_\tau^\infty L_\xi^\infty} \\
&\quad\leq \left(\frac{5}{2}\right)^{-1/2}\left\|\left(\int_{|\mu|\leq 3|\tau-\xi^5|} \frac{d\mu}{\left\langle \tau-\xi^5+\mu\right\rangle^{2b}}\right)^{1/2}\right\|_{L_\tau^\infty L_\xi^\infty} \\
&\quad\leq C_{8,2b}\left(\frac{5}{2}\right)^{-1/2},
\end{aligned} \tag{3.63}$$

where we used the inequality

$$5|\xi||\xi-2\xi_1|\left|\xi^2-2\xi\xi_1+2\xi_1^2\right| \geq \frac{5}{2}|\xi|^{3/4} \quad \text{in } \Omega_{I;\xi}, \tag{3.64}$$

which follows from

$$\left|\xi^2-2\xi\xi_1+2\xi_1^2\right| \geq \frac{1}{2}|\xi|^2, \qquad |\xi-2\xi_1| \geq |\xi|^{-3/2}. \tag{3.65}$$

Thus, (3.27) is bounded by

$$M_{5,s,b,b'} = 5^{3/2}\left(\frac{5}{16}\right)^{(b'-b)-1/2}2^{|s|}C_{9,s}\max\left\{C_{7,2b,2b}C_{8,2b}, C_{7,2(1-b'),2b}3^{b'-1/2}\right\} \tag{3.66}$$

in $\Omega_{2.1.2}$. In the other regions, we can get bounds of (3.27) by a similar argument to (3.57)–(3.63). Therefore we omit the proof.

Now (3.27) are shown to be bounded. Therefore, combining (3.4), (3.6), (3.12), and (3.18)–(3.26) and setting

$$C_{3,s,b,b'} = M_{1,s,b,b'} + 7M_{2,s,b,b'} + 2M_{3,s,b,b'} + 3M_{4,s,b,b'} + 17M_{5,s,b,b'}, \tag{3.67}$$

we have (2.17). This completes the proof of Lemma 2.4. □

Remark 3.2. We briefly state the reason why the term $10u\partial_x^3 u$ is removed from $(1.1)_2$. In order to show the existence of the solution of $(1.1)_2$, we have to prove the following estimate:

$$\left\| u\partial_x^3 v \right\|_{X^s_{b'-1}} \leq C_{3,s,b,b'} \|u\|_{X^s_b} \|v\|_{X^s_b}. \tag{3.68}$$

Unfortunately, we are not able to prove it, because our method can be used to estimate $\partial_x^l(\partial_x^m u \partial_x^n v)$ in the case where $lmn \geq 1$, but not in the case where $lmn = 0$. Therefore, it is necessarily for us to remove the term $10u\partial_x^3 u$ from $(1.1)_2$ unavoidably.

4. Analyticity

In this section we prove the analyticity of the solution $u \equiv u_0$ given in Proposition 2.1. The proof is established by Propositions 4.5–4.9. To prove these propositions we prepare four lemmas (Lemmas 4.1–4.4).

Lemma 4.1 (see [21]). *Let*

$$s > -\frac{7}{4}, \qquad b \in \left(\frac{1}{2}, \frac{1}{2} + \sigma\right), \tag{4.1}$$

where $\sigma = \min\{1/4, (4s+11)/8, (s+6)/5\}$. *Then*

$$\|\partial_x(uv)\|_{X^s_{b-1}} \leq M_{6,s,b} \|u\|_{X^s_b} \|v\|_{X^s_b}, \tag{4.2}$$

where $M_{6,s,b}$ *is a constant depending on* s *and* b.

Lemma 4.2 (see [7]). *Let*

$$f \in H^s(\mathbb{R}^n), \qquad g \in H^r(\mathbb{R}^n). \tag{4.3}$$

Suppose that

$$0 \leq s,\ r \leq \frac{n}{2}, \quad \frac{n}{2} \leq s + r. \tag{4.4}$$

Then, for any $\sigma_1 < s + r - n/2$,

$$\|fg\|_{H^{\sigma_1}(\mathbb{R}^n)} \leq M_7 \|f\|_{H^s(\mathbb{R}^n)} \|g\|_{H^r(\mathbb{R}^n)} \tag{4.5}$$

holds, where $M_7 = M_{7,s,r,\sigma_1,n}$ *is a constant depending on* s, r, σ_1, *and* n.

Lemma 4.3. *Let (t_0, x_0) be an arbitrarily fixed point in $\{(-T,0) \cup (0,T)\} \times \mathbb{R}$.*

(1) *Suppose that $b \in (0,1]$, $r \in (-\infty, 0]$. Then, for a sufficiently small $\varepsilon_1 > 0$ such that*

$$\varepsilon_1 = \begin{cases} \varepsilon^4, & \text{if } -\dfrac{9}{2} < r \le 0, \\ \varepsilon^{\alpha} & \text{with } \alpha > -\dfrac{1}{2} - r, \text{ if } r \le -\dfrac{9}{2}, \end{cases} \tag{4.6}$$

$$\left\| \langle D_{t,x} \rangle^{5b} g \right\|_{L_t^2(\mathbb{R};H_x^r(\mathbb{R}))} \le M_{8,r,b,\varepsilon_1} \left\{ \|g\|_{X_{b-1}^r} + \left\| t\partial_x^5 g \right\|_{X_{b-1}^r} + \left\| P^5 g \right\|_{X_{b-1}^r} \right\}$$

holds for all $g \in X_{b-1}^r$ satisfying

$$\operatorname{supp} g \subset B_{2\varepsilon_1}(t_0, x_0), \quad t\partial_x^5 g, P^5 g \in X_{b-1}^r, \tag{4.7}$$

where $M_{8,r,b,\varepsilon_1} = M_{8,r,b,(t_0,x_0),\varepsilon_1}$ depends on r, b, (t_0, x_0), and ε.

(2) *Let $\mu > 0$. Then, for a sufficiently small $\varepsilon_2 = \varepsilon^4 > 0$,*

$$\| \langle D_{t,x} \rangle^{\mu} g \|_{L_t^2 L_x^2} \le M_{9,\mu,\varepsilon_2} \left\{ \|g\|_{H_{t,x}^{\mu-5}(\mathbb{R}^2)} + \left\| t\partial_x^5 g \right\|_{H_{t,x}^{\mu-5}(\mathbb{R}^2)} + \left\| P^5 g \right\|_{H_{t,x}^{\mu-5}(\mathbb{R})} \right\} \tag{4.8}$$

holds for all $g \in H_{t,x}^{\mu-5}(\mathbb{R}^2)$ satisfying

$$\operatorname{supp} g \subset B_{2\varepsilon_2}(t_0, x_0), \quad t\partial_x^5 g, P^5 g \in H_{t,x}^{\mu-5}\left(\mathbb{R}^2\right), \tag{4.9}$$

where $M_{9,\mu,\varepsilon_2} = M_{9,\mu,(t_0,x_0),\varepsilon_2}$ depends on μ, (t_0, x_0), and ε.

Let $\rho(t,x)$ be a smooth cut-off function around the freezing point (t_0, x_0) such that $\rho \in C_0^\infty(B_{2\varepsilon^4}(t_0, x_0))$.

Lemma 4.4. *Let $s, b \in \mathbb{R}$. Then*

$$\|\rho f\|_{X_b^s} \le M_{10,s,b,\rho,\varepsilon_3} \varepsilon_3^{-|s|-9|b|} \|f\|_{X_b^{s+4|b|}}, \tag{4.10}$$

where $\varepsilon_3 = \varepsilon^4$ and $M_{10,s,b,\rho,\varepsilon_3} = 10^{|b|/2} \| \langle \varepsilon_3^4 \tau - \xi^5 \rangle^{|b|} \langle \xi \rangle^{|s|+4|b|} \mathcal{F}_{t,x}\rho(\tau,\xi) \|_{L_\tau^1 L_\xi^1}$.

Proof. Lemmas 4.3 and 4.4 are proved by the same method as Lemmas 3.2 and 5.2 in [7]. □

Proposition 4.5. *Let $s > 1/8$, and let $b \in (1/2, 23/40)$. Then, for a sufficiently small $\varepsilon^4 > 0$, there exist positive constants $K_{1,\rho}$ and A_1 such that*

$$\left\| \rho P^k u \right\|_{H_{t,x}^{1/3}(\mathbb{R}^2)} + \left\| \rho P^k u \right\|_{L_t^2(\mathbb{R};H_x^1)} \le K_{1,\rho} A_1^k k! \tag{4.11}$$

holds for all $k = 0, 1, 2, \ldots$, where $A_1 = 2A_0^{-1}$ and $K_{1,\rho} = K_{1,s,b,(t_0,x_0),\varepsilon,\rho}$ is a constant depending on s, b, (t_0, x_0), ε and ρ.

Proof. By Plancherel Theorem and Lemma 4.3 with $g = \rho P^k u$, we have

$$\begin{aligned}
&\left\|\rho P^k u\right\|_{H^{1/3}_{t,x}(\mathbb{R}^2)} + \left\|\rho P^k u\right\|_{L^2_t(\mathbb{R};H^1_x)} \\
&\quad \le \frac{2}{(2\pi)^2}\left\|\langle D_{t,x}\rangle^{5b}\rho P^k u\right\|_{L^2_t(\mathbb{R};H^r_x(\mathbb{R}))} \\
&\quad \le \frac{2}{(2\pi)^2} M_{8,r,b,\varepsilon^4}\left\{\left\|\rho P^k u\right\|_{X^r_{b-1}} + \left\|t\partial_x^5\left(\rho P^k u\right)\right\|_{X^r_{b-1}} + \left\|P^5\left(\rho P^k u\right)\right\|_{X^r_{b-1}}\right\},
\end{aligned} \tag{4.12}$$

where

$$\begin{aligned}
r = s-2, &\quad \text{if } \frac{1}{8} < s \le 2, \\
-\frac{15}{8} < r \le 0, &\quad \text{if } s > 2.
\end{aligned} \tag{4.13}$$

We note that $r \le s-2$ holds. Put $K_{2,s,b} = |||\mathbf{u}|||_{\mathcal{A}_{A_0}(X^s_b)}$. Since (2.8) and Remark 2.2 yield

$$\left\|P^k u\right\|_{X^s_b} \le K_{2,s,b}\left(A_0^{-1}\right)^k k!, \quad k = 0,1,2,\ldots, \tag{4.14}$$

it follows from Lemma 4.4 that

$$\begin{aligned}
\left\|\rho P^k u\right\|_{X^r_{b-1}} &\le M_{10,r,b-1,\rho,\varepsilon^4}\varepsilon^{-4|r|-36|b-1|}\left\|P^k u\right\|_{X^s_{b-1}} \\
&\le K_{2,s,b-1}M_{10,r,b-1,\rho,\varepsilon^4}\varepsilon^{-4|r|-36|b-1|}\left(A_0^{-1}\right)^k k!,
\end{aligned} \tag{4.15}$$

$$\begin{aligned}
\left\|P^5\left(\rho P^k u\right)\right\|_{X^r_{b-1}} &\le \sum_{l=0}^{5}\frac{5!}{(5-l)!l!}\left\|P^{5-l}\rho P^{l+k}u\right\|_{X^r_{b-1}} \\
&\le \sum_{l=0}^{5}\frac{5!}{(5-l)!l!}M_{10,r,b-1,\rho_l,\varepsilon^4}\varepsilon^{-4|r|-36|b-1|}\left\|P^{l+k}u\right\|_{X^s_{b-1}} \\
&\le \max_{0\le l\le 5}M_{10,r,b-1,\rho_l,\varepsilon^4}\varepsilon^{-4|r|-36|b-1|}K_{2,s,b-1}\sum_{l=0}^{5}\frac{5!}{(5-l)!l!}\frac{(k+l)!}{2^k k!}\left(A_0^{-1}\right)^l\left(2A_0^{-1}\right)^k k! \\
&\le K_3 A_1^k k!,
\end{aligned} \tag{4.16}$$

where $\rho_l = P^{5-l}\rho$, K_3 is some constant and $A_1 = (2A_0^{-1})$.

Now we estimate $\|t\partial_x^5(\rho P^k u)\|_{X_{b-1}^r}$. By using

$$t\partial_x^5\left(\rho P^k u\right) = t\rho\left(\partial_x^5 P^k u\right) + 5t\partial_x^2\left(\left(\partial_x^2\rho\right)\left(\partial_x P^k u\right)\right) + 5t\partial_x\left(\left(\partial_x\rho\right)\left(\partial_x^3 P^k u\right)\right) + t\left(\partial_x^5\rho\right)P^k u, \tag{4.17}$$

$$\left(\partial_x^5 P^k u\right) = -\frac{1}{5}\left\{P^{k+1}u - x\partial_x P^k u\right\} + t\mathcal{N}_k(u), \tag{4.18}$$

we have

$$\begin{aligned}\left\|t\partial_x^5\left(\rho P^k u\right)\right\|_{X_{b-1}^r} \le{}& \frac{1}{5}\left\{\left\|\rho P^{k+1}u\right\|_{X_{b-1}^r} + \left\|\rho x\partial_x P^k u\right\|_{X_{b-1}^r}\right\} + \|t\rho N_k(u)\|_{X_{b-1}^r} \\ &+ 5\left\|\partial_x^2\left(t\left(\partial_x^2\rho\right)\left(\partial_x P^k u\right)\right)\right\|_{X_{b-1}^r} + 5\left\|\partial_x\left(t(\partial_x\rho)\left(\partial_x^3 P^k u\right)\right)\right\|_{X_{b-1}^r} \\ &+ \left\|t\left(\partial_x^5\rho\right)P^k u\right\|_{X_{b-1}^r}.\end{aligned} \tag{4.19}$$

In the same manner as (4.15), we have

$$\begin{aligned}\left\|\rho P^{k+1}u\right\|_{X_{b-1}^r} &\le M_{10,r,b-1,\rho,\varepsilon^4}\varepsilon^{-4|r|-36|b-1|}\left\|P^{k+1}u\right\|_{X_{b-1}^s} \\ &\le M_{10,r,b-1,\rho,\varepsilon^4}\varepsilon^{-4|r|-36|b-1|}K_{2,s,b-1}\frac{(k+1)!}{2^k k!}\left(A_0^{-1}\right)\left(2A_0^{-1}\right)^k k! \\ &\le K_4 A_1^k k!,\end{aligned} \tag{4.20}$$

where K_4 is some constant. By Lemmas 4.1 and 4.4, we have

$$\begin{aligned}\left\|\rho x\partial_x P^k u\right\|_{X_{b-1}^r} &\le \left\|\partial_x\left(\rho x P^k u\right)\right\|_{X_{b-1}^r} + \left\|(\partial_x(\rho x))P^k u\right\|_{X_{b-1}^r} \\ &\le M_{6,r,b}\|\rho x\|_{X_b^{s-2}}\left\|P^k u\right\|_{X_b^{s-2}} + M_{10,r,b-1,\partial_x(\rho x),\varepsilon^4}\varepsilon^{-4|r|-36|b-1|}\left\|P^k u\right\|_{X_{b-1}^s} \\ &\le \left\{M_{6,r,b}\|\rho x\|_{X_b^{s-2}} + M_{10,r,b-1,\partial_x(\rho x),\varepsilon^4}\varepsilon^{-4|r|-36|b-1|}\right\}K_{2,s,b}\left(A_0^{-1}\right)^k k!.\end{aligned} \tag{4.21}$$

By Lemmas 2.4, 2.5, and 4.4, we have

$$\begin{aligned}
&\left\|t\rho\mathcal{N}_k(u)\right\|_{X^r_{b-1}}\\
&\le M_{10,r,b-1,t\rho,\varepsilon^4}\varepsilon^{-4|r|-36|b-1|}\\
&\quad\times\left\{C_{4,s,b}\sum_{\mathbf{k}}\frac{k!4^{k_4}}{k_1!k_2!k_3!k_4!}\left\|P^{k_1}u\right\|_{X^s_b}\left\|P^{k_2}u\right\|_{X^s_b}\left\|P^{k_3}u\right\|_{X^s_b}\right.\\
&\qquad\left.+C_{3,s,b}\sum_{\mathbf{k}}\frac{k!3^{k_4}}{k_1!k_2!k_3!k_4!}\left\|P^{k_1}u\right\|_{X^s_b}\left\|P^{k_2}u\right\|_{X^s_b}\right\}\\
&\le M_{10,r,b-1,t\rho,\varepsilon^4}\varepsilon^{-4|r|-36|b-1|}\\
&\quad\times\left\{C_{4,s,b}K^3_{2,s,b}\sum_{\mathbf{k}}\frac{4^{k_4}}{k_4!}\left(A_0^{-1}\right)^{k_1+k_2+k_3}+C_{3,s,b}K^2_{2,s,b}\sum_{\mathbf{k}}\frac{3^{k_4}}{k_3!k_4!}\left(A_0^{-1}\right)^{k_1+k_2}\right\}k!\\
&\le M_{10,r,b-1,t\rho,\varepsilon^4}\varepsilon^{-4|r|-36|b-1|}e^{4/A_0^{-1}}\left\{C_{4,s,b}K^3_{2,s,b}\frac{(k+1)}{2^k}+C_{3,s,b}K^2_{2,s,b}\frac{(k+1)(k+2)}{2^k}\right\}\left(2A_0^{-1}\right)^k k!\\
&\le K_5A_1^k k!,
\end{aligned}\tag{4.22}$$

where K_5 is some constant. We also have

$$\begin{aligned}
\left\|\partial_x^2\left(t\left(\partial_x^2\rho\right)\left(\partial_xP^ku\right)\right)\right\|_{X^r_{b-1}}&\le\left\|\partial_x\left(t\left(\partial_x^2\rho\right)\left(\partial_xP^ku\right)\right)\right\|_{X^{s-1}_{b-1}}\\
&\le C_{3,s-1,b}\left\|t\partial_x\rho\right\|_{X^{s-1}_b}\left\|P^ku\right\|_{X^{s-1}_b}\\
&\le C_{3,s-1,b}K_{2,s-1,b}\left\|t\partial_x\rho\right\|_{X^{s-1}_b}\left(A_0^{-1}\right)^k k!,\\
\left\|\sigma_x\left(t(\partial_x\rho)\left(\partial_x^3P^ku\right)\right)\right\|_{X^r_{b-1}}&\le C_{3,s-2,b}\left\|t\rho\right\|_{X^{s-2}_b}\left\|\partial_x^2P^ku\right\|_{X^{s-2}_b}\\
&\le C_{3,s-2,b}\left\|t\rho\right\|_{X^{s-2}_b}\left\|P^ku\right\|_{X^s_b}\\
&\le C_{3,s-2,b}K_{2,s,b}\left\|t\rho\right\|_{X^{s-2}_b}\left(A_0^{-1}\right)^k k!.
\end{aligned}\tag{4.23}$$

In the same manner as (4.15), we have

$$\left\|t\left(\partial_x^5\rho\right)P^ku\right\|_{X^r_{b-1}}\le M_{10,r,b-1,t\partial_x^5\rho,\varepsilon^4}K_{2,s,b-1}\left(A_0^{-1}\right)^k k!.\tag{4.24}$$

Hence

$$\left\|t\partial_x^5\left(\rho P^ku\right)\right\|_{X^r_{b-1}}\le K_6A_1^k k!,\tag{4.25}$$

where K_6 is some constant. Putting

$$K_{1,\rho} = \frac{3}{2\pi^2} M_{8,r,b,\varepsilon^4} \max\left\{ K_{2,s,b-1} M_{10,r,b-1,\rho,\varepsilon^4} \varepsilon^{-4|r|-36|b-1|}, K_3, K_6 \right\}, \tag{4.26}$$

we have (4.11). □

Proposition 4.6. *Under the same assumption as in Proposition 4.5, there exist K_7 and A_2 such that*

$$\left\| \rho_4 P^k u \right\|_{H^{11/2}_{t,x}(\mathbb{R})} \leq K_7 A_2^k k! \tag{4.27}$$

holds for all $k = 0, 1, 2, \ldots$, where ρ_4 is a smooth cut-off function such that

$$\begin{gathered} \rho_4 \leq \min\left\{ \rho, \rho^4 \right\}, \\ \rho_4 \equiv 1 \quad \text{on } \left(t_0 - \varepsilon^4,\ t_0 + \varepsilon^4 \right) \times \left(x_0 - \varepsilon^4, x_0 + \varepsilon^4 \right). \end{gathered} \tag{4.28}$$

Proof. At first, we show that there exists a constant $K_{7,1/2}$ and A_3 such that

$$\left\| \rho_4 P^k u \right\|_{H^{1/2}_{t,x}(\mathbb{R}^2)} \leq K_{7,1/2} A_3^k k!, \quad k = 0, 1, 2, \ldots. \tag{4.29}$$

Applying Lemma 4.3 with $\mu = 1/2$ and $g = \rho_4 P^k u$, we have

$$\begin{aligned} &\left\| \langle D_{t,x} \rangle^{1/2} \rho_4 P^k u \right\|_{L^2_t L^2_x} \\ &\quad \leq M_{9,1/2,\varepsilon^4} \left\{ \left\| \rho_4 P^k u \right\|_{H^{-9/2}_{t,x}(\mathbb{R}^2)} + \left\| t \partial_x^5 \left(\rho_4 P^k u \right) \right\|_{H^{-9/2}_{t,x}(\mathbb{R}^2)} + \left\| P^5 \left(\rho_4 P^k u \right) \right\|_{H^{-9/2}_{t,x}(\mathbb{R}^2)} \right\}. \end{aligned} \tag{4.30}$$

By Proposition 4.5, we have

$$\begin{aligned} \left\| \rho_4 P^k u \right\|_{H^{-9/2}_{t,x}(\mathbb{R}^2)} &\leq K_{1,\rho_4} A_1^k k!, \\ \left\| P^5 \left(\rho_4 P^k u \right) \right\|_{H^{-9/2}_{t,x}(\mathbb{R}^2)} &\leq \sum_{l=0}^{5} \frac{5!}{l!(5-l)!} \left\| P^{5-l} \rho_4 \right\|_{L^\infty_{t,x}(\mathbb{R}^2)} \left\| \rho P^{k+l} u \right\|_{L^2_{t,x}(\mathbb{R}^2)} \\ &\leq K_{1,\rho} \max_{0 \leq l \leq 5} \left\| P^{5-l} \rho_4 \right\|_{L^\infty_{t,x}(\mathbb{R}^2)} \sum_{l=0}^{5} \frac{5!}{(5-l)!l!} \frac{(k+l)!}{2^k k!} \left(A_1^{-1} \right)^l \left(2A_1^{-1} \right)^k k! \\ &\leq K_{1,\rho} K_8 \left(2A_1^{-1} \right)^k k!, \end{aligned} \tag{4.31}$$

where K_8 is some constant.

Now we estimate $\|t\partial_x^5(\rho_4 P^k u)\|_{H_{t,x}^{-9/2}(\mathbb{R}^2)}$. Using (4.18) and

$$\partial_x^5(\rho_4 f) = \rho_4\partial_x^5 f + \sum_{l=1}^{5}(-1)^{l-1}\frac{5!}{l!(5-l)!}\partial_x^{5-l}\left(\left(\partial_x^l\rho_4\right)P^k u\right), \tag{4.32}$$

we obtain

$$\begin{aligned}\left\|t\partial_x^5\left(\rho_4 P^k u\right)\right\|_{H_{t,x}^{-9/2}(\mathbb{R}^2)} &\le \frac{1}{5}\left\{\left\|\rho_4 P^{k+1}u\right\|_{H_{t,x}^{-9/2}(\mathbb{R}^2)} + \left\|\rho_4 x\partial_x P^k u\right\|_{H_{t,x}^{-9/2}(\mathbb{R}^2)}\right\}\\ &\quad + \left\|t\rho_4\mathcal{N}_k(u)\right\|_{H_{t,x}^{-9/2}(\mathbb{R}^2)} + \sum_{l=1}^{5}\frac{5!}{l!(5-l)!}\left\|t\partial_x^{5-l}\left(\left(\partial_x^l\rho_4\right)P^k u\right)\right\|_{H_{t,x}^{-9/2}(\mathbb{R}^2)}.\end{aligned} \tag{4.33}$$

By Proposition 4.5, we have

$$\begin{aligned}\frac{1}{5}\left\|\rho_4 P^{k+1}u\right\|_{H_{t,x}^{-9/2}(\mathbb{R}^2)} &\le \frac{1}{5}\|\rho_4\|_{L_t^\infty L_x^\infty}\left\|\rho P^{k+1}u\right\|_{L_t^2 L_x^2}\\ &\le \frac{1}{5}K_{1,\rho}\|\rho_4\|_{L_t^\infty L_x^\infty}\frac{(k+1)!}{2^k k!}\left(A_1^{-1}\right)\left(2A_1^{-1}\right)^k k!\\ &\le K_{1,\rho}K_9\left(2A_1^{-1}\right)^k k!,\\ \frac{1}{5}\left\|\rho_4 x\partial_x P^k u\right\|_{H_{t,x}^{-9/2}(\mathbb{R}^2)} &\le \frac{1}{5}\left(\|x\rho_4\|_{L_t^\infty L_x^\infty} + \|\partial_x(x\rho_4)\|_{L_t^\infty L_x^\infty}\right)\left\|\rho P^k u\right\|_{L_t^2 L_x^2}\\ &\le \frac{1}{5}\left(\|x\rho_4\|_{L_t^\infty L_x^\infty} + \|\partial_x(x\rho_4)\|_{L_t^\infty L_x^\infty}\right)K_{1,\rho}A_1^k k!,\end{aligned} \tag{4.34}$$

where K_9 is some constant. By Sobolev embedding theorem and Proposition 4.5, we have

$$\begin{aligned}&\left\|t\rho_4\rho^4\partial_x\left(P^{k_2}u\ \ P^{k_3}u\ \ P^{k_4}u\right)\right\|_{H_{t,x}^{-9/2}(\mathbb{R}^2)}\\ &\quad\le \left\|\langle|\tau|+|\xi|\rangle^{-7/2}\right\|_{L_\tau^2 L_\xi^2}\left(\|\partial_x t\rho_4\|_{L_t^\infty L_x^\infty} + \|t\rho_4\|_{L_t^\infty L_x^\infty}\right)\|\rho\|_{L_t^\infty L_x^\infty}\\ &\qquad\times\left\|\rho P^{k_2}u\right\|_{L_t^3 L_x^3}\left\|\rho P^{k_3}u\right\|_{L_t^3 L_x^3}\left\|\rho P^{k_4}u\right\|_{L_t^3 L_x^3}\\ &\quad\le K_{10,-7/2}\|\rho\|_{L_t^\infty L_x^\infty}\left\|\rho P^{k_2}u\right\|_{H_{t,x}^{1/3}(\mathbb{R}^2)}\left\|\rho P^{k_3}u\right\|_{H_{t,x}^{1/3}(\mathbb{R}^2)}\left\|\rho P^{k_4}u\right\|_{H_{t,x}^{1/3}(\mathbb{R}^2)}\\ &\quad\le K_{10,-7/2}\|\rho\|_{L_t^\infty L_x^\infty}K_{1,\rho}^3 A_1^{k_2+k_3+k_4}k_2!k_3!k_4!,\end{aligned}$$

$$\begin{aligned}
&\left\| t\rho_4\rho^4\partial_x((\partial_x P^{k_1}u)\partial_x P^{k_2}u)) \right\|_{H_{t,x}^{-9/2}(\mathbb{R}^2)} \\
&\quad \leq K_{10,-7/2}\left\| \rho^2\partial_x P^{k_1}u \right\|_{L_t^2L_x^2}\left\| \rho^2\partial_x P^{k_2}u \right\|_{L_t^2L_x^2} \\
&\quad \leq K_{10,-7/2}\left(\|\partial_x\rho\|_{L_t^\infty L_x^\infty} + \|\rho\|_{L_t^\infty L_x^\infty} \right)^2 \left\| \rho P^{k_1}u \right\|_{L_t^2(\mathbb{R};H_x^1)}\left\| \rho P^{k_2}u \right\|_{L_t^2(\mathbb{R};H_x^1)} \\
&\quad \leq K_{11,-7/2}K_{1,\rho}^2A_1^{k_1+k_2}k_1!k_2!,
\end{aligned} \tag{4.35}$$

where $K_{10,-7/2}$, $K_{11,-7/2}$ are some constants. Therefore, we have by (4.35)

$$\begin{aligned}
&\| t\rho_4\mathcal{N}_k(u) \|_{H_{t,x}^{-9/2}(\mathbb{R}^2)} \\
&\quad \leq \left\| t\rho_4\rho^4\mathcal{N}_k(u) \right\|_{H_{t,x}^{-9/2}(\mathbb{R}^2)} \\
&\quad \leq \sum_{\mathbf{k}} \frac{k!4^{k_1}}{k_1!k_2!k_3!k_4!}\left\| t\rho_4\rho^4\partial_x(P^{k_2}uP^{k_3}uP^{k_4}u)) \right\|_{H_{t,x}^{-9/2}(\mathbb{R}^2)} \\
&\qquad + \sum_{\mathbf{k}} \frac{k!3^{k_4}}{k_1!k_2!k_3!k_4!}\left\| t\rho_4\rho^4\partial_x\left(\left(\partial_x P^{k_2}u\right)\left(\partial_x P^{k_1}u\right)\right) \right\|_{H_{t,x}^{-9/2}(\mathbb{R}^2)} \\
&\quad \leq K_{10,-7/2}\|\rho\|_{L_t^\infty L_x^\infty}K_{1,\rho}^3k!\sum_{\mathbf{k}}\frac{4^{k_4}}{k_4!}A_1^{k_1+k_2+k_3} + K_{11,-7/2}K_{1,\rho}^2k!\sum_{\mathbf{k}}\frac{3^{k_4}}{k_3!K_4!}A_1^{k_1+k_2} \\
&\quad \leq \left(K_{10,-7/2}\|\rho\|_{L_t^\infty L_x^\infty}K_{1,\rho}^3 + K_{11,-7/2}K_{1,\rho}^2 \right)e^{4/A_1}A_1^kk!.
\end{aligned} \tag{4.36}$$

By Proposition 4.5, we have

$$\begin{aligned}
\sum_{l=1}^{5}\frac{5!}{l!(5-l)!}\left\| \partial_x^{5-l}\left(\left(t\partial_x^l\rho_4\right)P^ku\right) \right\|_{H_{t,x}^{-9/2}(\mathbb{R}^2)} &\leq \sum_{l=1}^{5}\frac{5!}{l!(5-l)!}\left\| t\partial_x^l\rho_4 \right\|_{L_t^\infty L_x^\infty}\left\| \rho P^ku \right\|_{L_t^2L_x^2} \\
&\leq K_{12}K_{1,\rho}A_1^kk!,
\end{aligned} \tag{4.37}$$

where K_{12} is some constant. Hence

$$\left\| t\partial_x^5\left(\rho_4P^ku\right) \right\|_{H_{t,x}^{-9/2}(\mathbb{R}^2)} \leq K_{13,K_{1,\rho},K_{10,-7/2}}\max\left\{ 2A_1^{-1}, A_1 \right\}^k k!, \tag{4.38}$$

where $K_{13,K_{1,\rho},K_{10,-7/2}}$ is some constant. Putting $A_3 = \max\{2A_1^{-1}, A_1\}$,

$$K_{7,1/2} = K_{7,1/2,K_{1,\rho},K_{10,-7/2}} = 3M_{9,1/2,\varepsilon^4}\max\left\{ K_{1,\rho_4}, K_{1,\rho}K_8, K_{14,K_{1,\rho},K_{10,-7/2}} \right\}, \tag{4.39}$$

we have (4.29). Similarly, we can prove by (4.8) with $\mu = 3/2$ and (4.29)

$$\left\|\rho_4 P^k u\right\|_{H^{3/2}_{t,x}(\mathbb{R}^2)} \le K_{7,3/2,K_{7,1/2},K_{10,-5/2}} A_4^k k!, \tag{4.40}$$

where $A_4 = \max\{2A_3^{-1}, A_3\}$. By (4.8) with $\mu = 5/2$ and (4.40), we have

$$\left\|\rho_4 P^k u\right\|_{H^{5/2}_{t,x}(\mathbb{R}^2)} \le K_{7,5/2,K_{7,3/2},K_{10,0}} A_5^k k!, \tag{4.41}$$

where $A_5 = \max\{2A_4^{-1}, A_4\}$ and $K_{10,0} = (\|\partial_x t\rho_4\|_{L_t^\infty L_x^\infty} + \|t\rho_4\|_{L_t^\infty L_x^\infty})\|\rho\|_{L_t^\infty L_x^\infty}$. Repeating the same method as in above, we obtain

$$\begin{aligned}
&\left\|\rho_4 P^k u\right\|_{H^{7/2}_{t,x}(\mathbb{R}^2)} \le K_{7,7/2,K_{8,5/2},M_7} A_6^k k!,\\
&\left\|\rho_4 P^k u\right\|_{H^{9/2}_{t,x}(\mathbb{R}^2)} \le K_{7,9/2,K_{8,7/2},M_7} A_7^k k!,\\
&\left\|\rho_4 P^k u\right\|_{H^{11/2}_{t,x}(\mathbb{R}^2)} \le K_{7,11/2,K_{8,9/2},K_{10,0}} A_2^k k!,
\end{aligned} \tag{4.42}$$

where $A_6 = \max\{2A_5^{-1}, A_5\}$, $A_7 = \max\{2A_6^{-1}, A_6\}$ and $A_2 = \max\{2A_7^{-1}, A_7\}$. Putting $K_7 = K_{7,11/2,K_{8,9/2},K_{10,0}}$, we have (4.27). □

Remark 4.7. When $\mu = 7/2$, $\mu = 9/2$, we can obtain the similar estimates to (4.35) by using Lemma 4.2 and Sobolev embedding theorem.

Proposition 4.8. *Suppose that* (4.27) *holds for all* $k = 0, 1, 2, \ldots$. *Then*

$$\sup_{t\in I_{t_0}} \left\| \left(t^{1/5}\partial_x\right)^l P^k u \right\|_{H^1(I_{x_0})} \le K_7 A_8^{k+l}(k+l)! \tag{4.43}$$

holds for all $k, l = 0, 1, 2, \ldots$, *where*

$$\begin{aligned}
&I_{t_0} = \left(t_0 - \varepsilon^4, t_0 + \varepsilon^4\right), \qquad I_{x_0} = \left(x_0 - \varepsilon^4, x_0 + \varepsilon^4\right),\\
&A_8 \ge \max\left\{ \left(|t_0| + \varepsilon^4\right)^{1/5}, A_2, \left(|x_0| + \varepsilon^4 + 1\right)\left(\left|t_0 - \varepsilon^4\right|\right)^{-1/5} + 1,\right.\\
&\qquad \left. \left(|t_0| + \varepsilon^4\right)^{4/5} K_7 e^{4/A_8} \max\left\{ \left|t_0 - \varepsilon^4\right|^{-2/5},\ K_7 \right\} \right\}.
\end{aligned} \tag{4.44}$$

Proof. We prove (4.43) by induction on l. When $l = 0, 1, 2, 3, 4$, we use the trace theorem and (4.27) to obtain

$$\begin{aligned}\sup_{t\in I_{t_0}}\left\|(t^{1/5}\partial_x)^l P^k u\right\|_{H^1(I_{x_0})} &\le \left(|t_0| + \varepsilon^4\right)^{l/5}\left\|\partial_x^l P^k u\right\|_{H^{3/2}_{t,x}(I_{t_0}\times I_{x_0})}\\ &\le \left(|t_0| + \varepsilon^4\right)^{l/5}\left\|\rho_4 P^k u\right\|_{H^{11/2}_{t,x}(\mathbb{R}^2)}\\ &\le K_7\left(|t_0| + \varepsilon^4\right)^{l/5} A_2^k k!\\ &\le K_7 A_8^{k+l} k!.\end{aligned} \tag{4.45}$$

We assume that (4.43) holds for any $l \ge 5$. Now we will prove

$$\sup_{t\in I_{t_0}}\left\|\left(t^{1/5}\partial_x\right)^{l+1} P^k u\right\|_{H^1(I_{x_0})} \le K_7 A_8^{k+l+1}(k+l+1)!. \tag{4.46}$$

Since

$$\begin{aligned}\left(t^{1/5}\partial_x\right)^l P^k u &= t^{(l-5)/5}\partial_x^{l-5}\left(t\partial_x^5 P^k u\right)\\ &= -\frac{1}{5}t^{(l-5)/5}\partial_x^{l-5}\left\{P^{k+1}u - x\partial_x P^k u\right\} + \left(t^{1/5}\right)^l \partial_x^{l-5}\mathcal{N}_k(u),\end{aligned} \tag{4.47}$$

we have

$$\begin{aligned}\sup_{t\in I_{t_0}}\left\|(t^{1/5}\partial_x)^{l+1} P^k u\right\|_{H^1(I_{x_0})} \le{}& \frac{1}{5}\left\|t^{(l-4)/5}\partial_x^{l-4}P^{k+1}u(t)\right\|_{H^1(I_{x_0})}\\ &+ \frac{1}{5}\left\|t^{(l-4)/5}\partial_x^{l-4}x\partial_x\left(P^k u\right)\right\|_{H^1(I_{x_0})} + \left\|\left(t^{1/5}\right)^{l+1}\partial_x^{l-4}\mathcal{N}_k(u)\right\|_{H^1(I_{x_0})}.\end{aligned} \tag{4.48}$$

By (4.43), we have

$$\frac{1}{5}\left\|t^{(l-4)/5}\partial_x^{l-4}P^{k+1}u(t)\right\|_{H^1(I_{x_0})} \le \frac{1}{5}K_7 A_8^{k+l-3}(k+l-3)!. \tag{4.49}$$

Since

$$t^{(l-4)/5}\partial_x^{l-4}(x\partial_x) = x\partial_x t^{(l-4)/5}\partial_x^{l-4} + (l-4)t^{(l-4)/5}\partial_x^{l-4} \quad (l = 5, 6, 7, \ldots), \tag{4.50}$$

we have by (4.43)

$$\begin{aligned}
&\frac{1}{5}\left\|t^{(l-4)/5}\partial_x^{l-4}x\partial_x\left(P^k u\right)\right\|_{H^1(I_{x_0})}\\
&\quad\le \frac{1}{5}\left\{\left\|x\partial_x t^{(l-4)/5}\partial_x^{l-4}P^k u\right\|_{H^1(I_{x_0})}+(l-4)\left\|t^{(l-4)/5}\partial_x^{l-4}P^k u\right\|_{H^1(I_{x_0})}\right\}\\
&\quad\le \frac{1}{5}\left(|x_0|+\varepsilon^4+1\right)\left|t_0-\varepsilon^4\right|^{-1/5}\left\|t^{(l-3)/5}\partial_x^{l-3}P^k u\right\|_{H^1(I_{x_0})}+\frac{1}{5}(l-4)K_7A_8^{k+l-4}(k+l-4)!\\
&\quad\le \frac{1}{5}\left(|x_0|+\varepsilon^4+1\right)\left|t_0-\varepsilon^4\right|^{-1/5}K_7A_8^{k+l-3}(k+l-3)!+\frac{1}{5}K_7A_8^{k+l-3}(k+l-3)!\\
&\quad\le \frac{1}{5}K_7A_8^{k+l-2}(k+l-3)!.
\end{aligned} \tag{4.51}$$

Now we estimate $\|(t^{1/5})^{l+1}\partial_x^{l-4}\mathcal{N}_k(u)\|_{H^1(I_{x_0})}$. We have

$$\begin{aligned}
\left\|\left(t^{1/5}\right)^{l+1}\partial_x^{l-4}\mathcal{N}_k(u)\right\|_{H^1(I_{x_0})} &\le \left(|t_0|+\varepsilon^4\right)^{4/5}\\
&\quad\times\left\{\sum_{\mathbf{k}}\frac{k!4^{k_4}}{k_1!k_2!k_3!k_4!}\left\|t^{(l-3)/5}\partial_x^{l-3}\left(P^{k_1}u\ \ P^{k_2}u\ \ P^{k_3}u\right)\right\|_{H^1(I_{x_0})}\right.\\
&\quad\left.+\sum_{\mathbf{k}}\frac{k!3^{k_4}}{k_1!k_2!k_3!k_4!}\left\|t^{(l-3)/5}\partial_x^{l-3}\left(\partial_x P^{k_1}u\ \ \partial_x P^{k_2}u\right)\right\|_{H^1(I_{x_0})}\right\}.
\end{aligned} \tag{4.52}$$

Since

$$\sum_{\mathbf{l}}\sum_{\mathbf{k}}\frac{\left(4A_8^{-1}\right)^{k_4}}{k_4!}\frac{(l_1+k_1)!}{k_1!l_1!}\frac{(l_2+k_2)!}{k_2!l_2!}\frac{(l_3+k_3)!}{k_3!l_3!}\frac{(l-3)!k!}{(l+k-3)!}\le e^{4/A_8}(l+k-2), \tag{4.53}$$

we have

$$\begin{aligned}
&\left(|t_0|+\varepsilon^4\right)^{4/5}\sum_{\mathbf{k}}\frac{k!4^{k_4}}{k_1!k_2!k_3!k_4!}\left\|t^{(l-3)/5}\partial_x^{l-3}\left(P^{k_1}u\ \ P^{k_2}u\ \ P^{k_3}u\right)\right\|_{H^1(I_{x_0})}\\
&\quad\le \left(|t_0|+\varepsilon^4\right)^{4/5}\sum_{\mathbf{l}}\sum_{\mathbf{k}}\frac{(l-3)!}{l_1!l_2!l_3!}\frac{k!4^{k_4}}{k_1!k_2!k_3!k_4!}\left\|\left(t^{1/5}\partial_x\right)^{l_1}P^{k_1}u\right\|_{H^1(I_{x_0})}\\
&\quad\times\left\|\left(t^{1/5}\partial_x\right)^{l_2}P^{k_2}u\right\|_{H^1(I_{x_0})}\left\|(t^{1/5}\partial_x)^{l_3}P^{k_3}u\right\|_{H^1(I_{x_0})}
\end{aligned}$$

$$\begin{aligned}
&\leq \left(|t_0| + \varepsilon^4\right)^{4/5} K_7^3 (l+k-3)! A_8^{l+k-3} \\
&\quad \times \sum_{\mathbf{l}} \sum_{\mathbf{k}} \frac{\left(4A_8^{-1}\right)^{k_4}}{k_4!} \frac{(l_1+k_1)!}{k_1! l_1!} \frac{(l_2+k_2)!}{k_2! l_2!} \frac{(l_3+k_3)!}{k_3! l_3!} \frac{(l-3)! k!}{(l+k-3)!} \\
&\leq \left(|t_0| + \varepsilon^4\right)^{4/5} K_7^3 A_8^{l+k-3} e^{4/A_8} (l+k-2)! \\
&\leq \frac{1}{5} K_7 A_8^{k+l} (k+l-2)!.
\end{aligned} \tag{4.54}$$

Similarly,

$$\begin{aligned}
&\left(|t_0| + \varepsilon^4\right)^{4/5} \sum_{\mathbf{k}} \frac{k! 3^{k_4}}{k_1! k_2! k_3! k_4!} \left\| t^{l-3/5} \partial_x^{l-3} \left(\partial_x P^{k_1} u \quad \partial_x P^{k_2} u \right) \right\|_{H^1(I_{x_0})} \\
&\leq \left(|t_0| + \varepsilon^4\right)^{4/5} \sum_{l-3=l_1+l_2} \sum_{\mathbf{k}} \frac{(l-3)!}{l_1! l_2!} \frac{3^{k_3} k!}{k_1! k_2! k_3! k_4!} \\
&\quad \times \left\| (t^{1/5} \partial_x)^{l_1} \partial_x P^{k_1} u \right\|_{H^1(I_{x_0})} \left\| (t^{1/5} \partial_x)^{l_2} \partial_x P^{k_2} u \right\|_{H^1(I_{x_0})} \\
&\leq \left(|t_0| + \varepsilon^4\right)^{4/5} \left| t_0 - \varepsilon^4 \right|^{-2/5} \sum_{l-1=m_1+m_2} \sum_{\mathbf{k}} \frac{(l-3)! m_1 m_2}{m_1! m_2!} \frac{3^{k_3} k!}{k_1! k_2! k_3! k_4!} \\
&\quad \times \left\| (t^{1/5} \partial_x)^{m_1} P^{k_1} u \right\|_{H^1(I_{x_0})} \left\| (t^{1/5} \partial_x)^{m_2} P^{k_2} u \right\|_{H^1(I_{x_0})},
\end{aligned} \tag{4.55}$$

where $m_1 = l_1 + 1$ and $m_2 = l_2 + 1$. By $m_1^2 \geq m_1$ and $m_2^2 \geq m_2$, we have

$$m_1 m_2 = \frac{1}{2} \left\{ (l-1)^2 - \left(m_1^2 + m_2^2 \right) \right\} \leq \left\{ (l-1)^2 - (m_1 + m_2) \right\} \leq (l-2)(l-1). \tag{4.56}$$

Thus, we have by (4.56), (4.43)

$$\begin{aligned}
&\left(|t_0| + \varepsilon^4\right)^{4/5} \left(|t_0| + \varepsilon^4\right)^{-2/5} \sum_{l-1=m_1+m_2} \sum_{\mathbf{k}} \frac{(l-3)!}{m_1! m_2!} \frac{3^{k_3} k!}{k_1! k_2! k_3! k_4!} \\
&\quad \times \left\| (t^{1/5} \partial_x)^{m_1} P^{k_1} u \right\|_{H^1(I_{x_0})} \left\| (t^{1/5} \partial_x)^{m_2} P^{k_2} u \right\|_{H^1(I_{x_0})}
\end{aligned}$$

$$\leq \left(|t_0| + \varepsilon^4\right)^{4/5} \left|t_0 - \varepsilon^4\right|^{-2/5} K_7^2 (l+k-1)! A_g^{l+k-1}$$
$$\times \sum_{l-1=m_1+m_2} \sum_{\mathbf{k}} \frac{\left(3A_8^{-1}\right)^{k_3}!}{k_3!} \frac{A_8^{-k_4}!}{k_4!} \frac{(m_1+k_1)!}{k_1 m_1!} \frac{(m_2+k_2)!}{k_2 m_2!} \frac{(l-1)!k!}{(l+k-1)!}$$
$$\leq \left(|t_0| + \varepsilon^4\right)^{4/5} \left|t_0 - \varepsilon^4\right|^{-2/5} K_7^2 e^{4/A_s} (l+k)! A_8^{l+k-1}$$
$$\leq \frac{1}{5} K_7 A_8^{k+l+1} (K+l)!. \tag{4.57}$$

Combining (4.48)–(4.57), we have (4.46). This completes the proof. □

Proposition 4.9. *Suppose that* (4.43) *holds for all* $k, l = 0, 1, 2, \ldots$. *Then, there exists* $A_9 > 0$ *depending on* A_8, (t_0, x_0), *and* ε *such that*

$$\sup_{t \in I_{t_0}} \left\| \partial_t^m \partial_x^l u \right\|_{H^1(I_{x_0})} \leq K_7 A_9^{m+l} (m+l)! \tag{4.58}$$

holds for all $m, l = 0, 1, 2, \ldots$.

Proof. By induction on m, we prove

$$\left\| (x\partial_x)^m \partial_x^l P^k u \right\|_{H^1(I_{x_0})} \leq K_7 A_{10}^{k+l+m} B_1^m (k+l+m)! \quad k, l, m = 0, 1, 2, \ldots. \tag{4.59}$$

In the case $m = 0$, we have by (4.43)

$$\begin{aligned} \sup_{t \in I_{t_0}} \left\| \partial_x^l P^k u \right\|_{H^1(I_{x_0})} &\leq K_7 A_{10}^l A_8^k (k+l)! \\ &\leq K_7 A_{10}^{k+l} (k+l)!, \quad k, l = 0, 1, 2, \ldots, \end{aligned} \tag{4.60}$$

where $A_{10} = \max\{A_8 |t_0 - \varepsilon^4|^{-1/5}, A_8\}$. We assume (4.59) is valid up to any m. Since

$$\partial_x (x\partial_x)^m = (x\partial_x + 1)^m \partial_x \quad (m = 0, 1, 2, \ldots), \tag{4.61}$$

we have

$$\begin{aligned}
&\left\|(x\partial_x)^{m+1}\partial_x^l P^k u\right\|_{H^1(I_{x_0})}\\
&\le \left(|x_0|+\varepsilon^4+1\right)\left\|(x\partial_x+1)^m\partial_x^{l+1}P^k u\right\|_{H^1(I_{x_0})}\\
&\le \left(|x_0|+\varepsilon^4+1\right)\sum_{m_1=0}^{m}\frac{m!}{m_1!(m-m_1)!}\left\|(x\partial_x)^{m_1}\partial_x^{l+1}P^k u\right\|_{H^1(I_{x_0})}\\
&\le \left(|x_0|+\varepsilon^4+1\right)K_7\mathrm{A}_{10}^{k+m+l+1}B_1^m(k+m+l+1)!\\
&\quad\times\sum_{m_1=0}^{m}\frac{(A_{10}B_1)^{-(m-m_1)}}{(m-m_1)!}\frac{m!}{m_1!}\frac{(k+m_1+l+1)!}{(k+m+l+1)!}\\
&\le \left(|x_0|+\varepsilon^4+1\right)e^{-A_{10}B_1}K_7A_{10}^{k+m+l+1}B_1^m(k+m+l+1)!\\
&\le K_7A_{10}^{k+m+l+1}B_1^{m+1}(k+m+l+1)!,
\end{aligned}\tag{4.62}$$

where $B_1 \ge (|x_0|+\varepsilon^4+1)e^{-A_{10}B_1}$. Since $t\partial_t = (P - x\partial_x)/5$ and

$$P^{n_1}\partial_x^{n_2} = \partial_x^{n_2}(P-n_2)^{n_1} \quad (n_1, n_2 = 0, 1, 2, \ldots),\tag{4.63}$$

it follows from (4.59) that

$$\begin{aligned}
\left\|(t\partial_t)^m\partial_x^l u\right\|_{H^1(I_{x_0})} &\le 5^{-m}\sum_{m=m_1+m_2}\frac{m}{m_1!m_2!}\left\|(x\partial_x)^{m_1}P^{m_2}\partial_x^l u\right\|_{H^1(I_{x_0})}\\
&\le 5^{-m}\sum_{m=m_1+m_2}\frac{m}{m_1!m_2!}\left\|(x\partial_x)^{m_1}\partial_x^l(P-l)^{m_2}u\right\|_{H^1(I_{x_0})}\\
&\le 5^{-m}\sum_{\mathbf{m}}\frac{m!}{m_1!m_2!m_3!}l^{m_3}\left\|(x\partial_x)^{m_1}\partial_x^l P^{m_2}u\right\|_{H^1(I_{x_0})}\\
&\le 5^{-m}K_7(l+m)!\sum_{\mathbf{m}}\frac{m!}{m_1!m_2!m_3!}A_{10}^{l+m-m_3}B_1^{m_1}l^{m_3}\frac{(m_1+m_2+l)!}{(l+m)!}\\
&\le 5^{-m}K_7A_{10}^{l+m}B_1^m(l+m)!\sum_{\mathbf{m}}\frac{m!}{m_1!m_2!m_3!}(A_{11}B_1)^{-m_3}B_1^{-m_2}\\
&\le 5^{-m}K_7\max\{A_{10}B_1, B_1\}^{l+m}(l+m)!\left(1+(A_{10}B_1)^{-1}+B_1^{-1}\right)^m\\
&\le K_7A_{11}^{l+m}(l+m)!,
\end{aligned}\tag{4.64}$$

where $A_{11} = \max\{1, 5^{-1}(1 + (A_{10}B_1)^{-1} + B_1^{-1})\} \cdot \max\{A_{10}B_1, B_1\}$. Thus,

$$\left\| (t\partial_t)^m \partial_x^l u \right\|_{H^1(I_{x_0})} \leq K_7 A_{11}^{l+m}(l+m)! \quad l, m = 0, 1, 2, \ldots. \tag{4.65}$$

By induction on j we prove that (4.65) implies

$$\left\| (t\partial_t)^m \partial_t^j \partial_x^l u \right\|_{H^1(I_{x_0})} \leq K_7 A_{11}^{j+m+l} B_2^j (j+m+l)! \quad j, l, m = 0, 1, 2, \ldots. \tag{4.66}$$

In the case $j = 0$, we have by (4.65)

$$\sup_{t \in I_{t_0}} \left\| (t\partial_t)^m \partial_x^l u \right\|_{H^1(I_{x_0})} \leq K_7 A_{11}^{l+m}(l+m)! \quad k, l = 0, 1, 2, \ldots. \tag{4.67}$$

We assume that (4.66) is valid up to any j. Noting that

$$(t\partial_t)^m \partial_t = \partial_t (t\partial_t - 1)^m \quad (j = 0, 1, 2, \ldots), \tag{4.68}$$

we have

$$\begin{aligned}
\left\| (t\partial_t)^m \partial_t^{j+1} \partial_x^l u \right\|_{H^1(I_{x_0})} &\leq \left\| \partial_t (t\partial_t - 1)^m \partial_t^j \partial_x^l u \right\|_{H^1(I_{x_0})} \\
&\leq \left| t_0 - \varepsilon^4 \right|^{-1} \sum_{m_1=0}^{m} \frac{m!}{m_1!(m-m_1)!} \left\| (t\partial_t)^{m_1+1} \partial_t^j \partial_x^l u \right\|_{H^1(I_{x_0})} \\
&\leq \left| t_0 - \varepsilon^4 \right|^{-1} K_7 A_{11}^{j+l+m+1} B_2^j (j+l+m+1)! \\
&\quad \times \sum_{m_1=0}^{m} \frac{A_{11}^{-(m-m_1)}}{(m-m_1)!} \frac{m!}{m_1!} \frac{(j+l+m_1+1)!}{(j+l+m+1)!} \\
&\leq \left| t_0 - \varepsilon^4 \right|^{-1} e^{-A_{11}} K_7 A_{11}^{j+l+m+1} B_2^j (j+l+m+1)! \\
&\leq K_7 A_{11}^{j+l+m+1} B_2^{j+1} (j+l+m+1)!,
\end{aligned} \tag{4.69}$$

where $B_2 \geq |t_0 - \varepsilon^4|^{-1} e^{-A_{11}}$. Thus (4.66) holds.

Choosing $m = 0$ and $A_9 = \max\{A_{11}B_2, A_{11}\}$ in (4.66), we have (4.58). This completes the proof of Proposition 4.9. □

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