Hindawi Publishing Corporation
Advances in Mathematical Physics
Volume 2011, Article ID 625978, 16 pages
doi:10.1155/2011/625978

Research Article

Quantum Dynamical Semigroups and Decoherence

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Received 22 June 2011; Accepted 31 August 2011

Academic Editor: Christian Maes

We prove a version of the Jacobs-de Leeuw-Glicksberg splitting theorem for weak* continuous one-parameter semigroups on dual Banach spaces. This result is applied to give sufficient conditions for a quantum dynamical semigroup to display decoherence. The underlying notion of decoherence is that introduced by Blanchard and Olkiewicz (2003). We discuss this notion in some detail.

1. Introduction

The theory of environmental decoherence starts from the question of why macroscopic physical systems obey the laws of classical physics, despite the fact that our most fundamental physical theory—quantum theory—results in contradictions when directly applied to these objects. The infamous Schrödinger cat is a well-known illustration of this problem. This is an embarrassing situation since, from its inception in the 1920s until today, quantum theory has seen a remarkable success and an ever increasing range of applicability. Thus the question of how to reconcile quantum theory with classical physics is a fundamental one, and efforts to find answers to it persisted throughout its history. At present, the most promising and most widely discussed answer is the notion of environmental decoherence. The starting point is the contention that quantum theory is universally valid, in particular in the macroscopic domain, but that one has to take into account the fact that macroscopic objects are strongly interacting with their environment, and that precisely this interaction is the origin of classicality in the physical world. Thus classicality is a dynamically emergent phenomenon due to the essential openness of macroscopic quantum systems, that is, their interaction with other quantum systems surrounding them leads to an effective restriction of the superposition principle and results in a state space with properties different from the pure quantum case.

In order to clarify the status of decoherence and to provide a rigorous definition, Ph. Blanchard and R. Olkiewicz suggested a notion of decoherence formulated in the algebraic framework [1, 2] of quantum physics in [3], drawing on earlier work in [4]. The algebraic framework is especially useful for the discussion of decoherence, since it is able to accommodate classical systems, provides an elegant formulation of superselection rules, and can even describe systems with infinitely many degrees of freedom in a rigorous way. This is why it is becoming increasingly popular in the discussion of foundational and philosophical problems of quantum physics [5, 6].

In the present paper, we assume that the algebra of observables of the system under study is a von Neumann algebra, and due to its openness the time evolution is irreversible and hence given by a family $\{T_t\}_{t\geq 0}$ of normal completely positive and unital linear maps on the von Neumann algebra [7, 8]. In the Markovian approximation, the family $\{T_t\}_{t\geq 0}$ becomes a so-called quantum dynamical semigroup. It is our purpose to discuss the Blanchard-Olkiewicz notion of decoherence for quantum dynamical semigroups. To this end we study a weak* version of the so-called Jacobs-de Leeuw-Glicksberg splitting for one-parameter semigroups on dual Banach spaces. In the Markovian case, the Blanchard-Olkiewicz notion of decoherence relies on the so-called isometric-sweeping splitting, which is similar to the Jacobs-de Leeuw-Glicksberg splitting, and we will be able to prove a new criterion for the appearance of decoherence in the case of uniformly continuous quantum dynamical semigroups by examining the connection between the two asymptotic splittings.

The paper is organized as follows. In Section 2 we establish the Jacobs-de Leeuw-Glicksberg splitting for weak* continuous contractive one-parameter semigroups on dual Banach spaces. We provide a sufficient condition which ensures that the semigroup is weak* stable on the stable subspace of the splitting (Proposition 2.3). In Section 3 we turn to the study of quantum dynamical semigroups on von Neumann algebras. We begin by applying the results of Section 2 in the von Neumann algebra setting (Proposition 3.3). As a complement to Proposition 2.3, we prove Proposition 3.6, which gives another condition for weak* stability on the stable subspace of the splitting. In Section 4, we discuss a notion of decoherence which is very close to that given in [3] and establish some mathematical results related to it. In the final Section 4.2, we use the previous results to give a sufficient condition that a uniformly continuous quantum dynamical semigroup having a faithful normal invariant state displays decoherence.

2. The Jacobs-de Leeuw-Glicksberg Splitting

Suppose that X is a Banach space and assume that it has a predual space denoted by X_*, that is, $(\mathsf{X}_*)^* \cong \mathsf{X}$. If $x \in \mathsf{X}$ and $\varphi \in \mathsf{X}_*$, we will denote the evaluation of x at φ by $\langle x, \varphi\rangle$ and consider this as a dual pairing between X and X_*. The set of all bounded linear operators from X to X, endowed with the operator norm, will be denoted by $\mathrm{L}(\mathsf{X})$, and its unit ball by $\mathrm{L}(\mathsf{X})_1 = \{T \in \mathrm{L}(\mathsf{X}) : \|T\| \leq 1\}$. Operators from $\mathrm{L}(\mathsf{X})_1$ are called contractive. We consider the algebraic tensor product $\mathsf{X}\odot\mathsf{X}_*$ and endow it with the projective cross norm γ; the completion of $\mathsf{X}\odot\mathsf{X}_*$ with respect to γ is a Banach space which will be denoted by $\mathsf{X}\otimes_\gamma\mathsf{X}_*$. Then its dual space $(\mathsf{X}\otimes_\gamma\mathsf{X}_*)^*$ is isometrically isomorphic in a canonical way with $\mathrm{L}(\mathsf{X}, (\mathsf{X}_*)^*) = \mathrm{L}(X)$: If $\psi \in (\mathsf{X}\otimes_\gamma\mathsf{X}_*)^*$, we define $\Phi(\psi) \in \mathrm{L}(\mathsf{X})$ by

$$\langle x \otimes \varphi, \psi\rangle = \langle \Phi(\psi)(x), \varphi\rangle \tag{2.1}$$

for all $x \in \mathsf{X}$ and $\varphi \in \mathsf{X}_*$. It can now be shown that $\psi \mapsto \Phi(\psi)$ extends to an isometric isomorphism, and we can thus write $(\mathsf{X}\otimes_\gamma\mathsf{X}_*)^* \cong \mathrm{L}(\mathsf{X})$.

We now introduce the pointwise weak* topology on $\mathsf{L}(\mathsf{X})$. Let $x \in \mathsf{X}$, $\varphi \in \mathsf{X}_*$, and define the seminorm $\mathsf{L}(\mathsf{X}) \ni T \mapsto p_{x,\varphi}(T) = |\langle T(x), \varphi\rangle|$. The pointwise weak* topology is the locally convex topology on $\mathsf{L}(X)$ induced by the family $\{p_{x,\varphi} : x \in \mathsf{X},\ \varphi \in \mathsf{X}_*\}$ of seminorms. If $T = \Phi(\psi) \in \mathsf{L}(\mathsf{X})$, we see that $p_{x,\varphi}(T) = |\langle T(x), \varphi\rangle| = |\langle x \otimes \varphi, \psi\rangle|$; thus the pointwise weak* topology coalesces with the $\sigma(\mathsf{L}(\mathsf{X}), \mathsf{X}\otimes_\gamma \mathsf{X}_*)$ topology on $\mathsf{L}(\mathsf{X})$, that is, the pointwise weak* topology is a weak* topology as well. Thus we can conclude from Alaoglu's theorem that $\mathsf{L}(\mathsf{X})_1$ is compact in the pointwise weak* topology.

A linear operator $T \in \mathsf{L}(\mathsf{X})$ will be called *normal* provided it is a continuous map from X to X when X is endowed with the weak* topology. We denote the set of all normal operators by $\mathsf{L}_\mathrm{n}(\mathsf{X})$. We can consider the set of all normal contractive operators $\mathsf{L}_\mathrm{n}(\mathsf{X})_1$ as a semigroup under multiplication of operators, that is, if $T_1, T_2 \in \mathsf{L}_\mathrm{n}(\mathsf{X})_1$, then $T_1 \circ T_2$ is normal and contractive; moreover, the multiplication is associative. The semigroup $\mathsf{L}_\mathrm{n}(\mathsf{X})_1$ is semitopological when endowed with the pointwise weak* topology, that is, the multiplication is separately continuous. This means that the maps $T \mapsto T \circ S$ and $S \mapsto T \circ S$ are both continuous with respect to the pointwise weak* topology. Finally we remark that it is important to note that $\mathsf{L}_\mathrm{n}(\mathsf{X})_1$ is not closed in $\mathsf{L}(\mathsf{X})_1$ with respect to the pointwise weak* topology. Moreover, recall that an operator $T \in \mathsf{L}(\mathsf{X})$ is normal if and only if there exists a (unique) predual operator T_* from X_* into X_*, defined by $\langle T(x), \varphi\rangle = \langle x, T_*(\varphi)\rangle$, $x \in \mathsf{X}$, $\varphi \in \mathsf{X}_*$.

In this section, our goal is to study one-parameter semigroups on dual Banach spaces. A contractive one-parameter semigroup [9, 10] is a family $\{T_t\}_{t\ge 0}$ of linear and contractive operators on X, such that $T_s \circ T_t = T_{s+t}$ for all $s, t \ge 0$ and $T_0 = \mathrm{id}_\mathsf{X}$. The semigroup is called weak* continuous provided each T_t is a normal operator and $[0, \infty[\ni t \mapsto T_t(x)$ is weak* continuous for any $x \in \mathsf{X}$. For a weak* continuous semigroup there exists the following concept of a *weak* generator* Z:

$$Zx = \lim_{t\downarrow 0} \frac{T_t(x) - x}{t} \quad \text{in the weak* topology}, \tag{2.2}$$

$$\operatorname{dom} Z = \{x \in \mathsf{X} : \text{the limit in (2.2) exists}\}. \tag{2.3}$$

The predual semigroup $\{T_{t,*}\}_{t\ge 0}$ of a weak* continuous semigroup $\{T_t\}_{t\ge 0}$ is strongly continuous, and the adjoint of its generator Z_* is equal to the weak* generator Z.

Suppose now that $\{T_t\}_{t\ge 0}$ is a weak* continuous contractive semigroup on X, and write $\mathsf{S}_0 = \{T_t : t \ge 0\} \subseteq \mathsf{L}_\mathrm{n}(\mathsf{X})_1$. In the following, we assume that the closure of S_0 in $\mathsf{L}(\mathsf{X})_1$ with respect to the pointwise weak* topology consists of normal operators, that is, we assume that $\mathsf{S} = \overline{\mathsf{S}_0} \subseteq \mathsf{L}_\mathrm{n}(\mathsf{X})_1$, where the bar denotes closure in the pointwise weak* topology. Then S is a compact commutative semitopological subsemigroup of $\mathsf{L}_\mathrm{n}(\mathsf{X})_1$. We now use the fact that every compact commutative semitopological semigroup has a unique minimal ideal $\mathsf{G} \subseteq \mathsf{S}$, the so-called Sushkevich kernel [9], which is given by

$$\mathsf{G} = \bigcap_{R\in \mathsf{S}} R \circ \mathsf{S}, \tag{2.4}$$

and $Q \in \mathsf{G}$ will denote the unit of G. We then have $\mathsf{G} = Q \circ \mathsf{S}$. By compactness of G, it follows that G is, in fact, a commutative topological group. In the following, we will simplify our notation by writing $T_1 T_2$ instead of $T_1 \circ T_2$.

We are now able to prove a weak* version of the Jacobs-de Leeuw-Glicksberg splitting theorem, originally going back to Jacobs [11] and de Leeuw and Glicksberg [12, 13], see also [14]. The present proof mimics the one given in [9] for weakly almost periodic one-parameter semigroups.

Theorem 2.1. *Let $S_0 = \{T_t\}_{t\geq 0}$ be a weak* continuous contractive one-parameter semigroup with generator Z. Assume that $S = \overline{S_0}$ consists of normal operators. Then there exist weak* closed subspaces $\mathsf{X}_{\mathrm{s}}, \mathsf{X}_{\mathrm{r}}$ of X invariant under all operators T_t, $t \geq 0$, such that $\mathsf{X} = \mathsf{X}_{\mathrm{s}} \oplus \mathsf{X}_{\mathrm{r}}$, and*

$$\mathsf{X}_{\mathrm{s}} = \left\{ x \in \mathsf{X} : 0 \in \overline{\{T_t(x) : t \geq 0\}}^{\mathrm{w}^*} \right\}, \tag{2.5}$$

$$\begin{aligned} \mathsf{X}_{\mathrm{r}} &= \overline{\operatorname{lin}\{x \in \operatorname{dom} Z : \exists \alpha \in \mathbb{R} \textit{ such that } Zx = \mathrm{i}\alpha x\}}^{\mathrm{w}^*} \\ &= \overline{\operatorname{lin}\{x \in X : \exists \alpha \in \mathbb{R} \textit{ such that } T_t(x) = \mathrm{e}^{\mathrm{i}\alpha t} x \ \forall t \geq 0\}}^{\mathrm{w}^*}. \end{aligned} \tag{2.6}$$

Proof. Since $Q^2 = Q$, the unit Q is a normal projection such that $[Q, T_t] = 0$ for all $t \geq 0$. The theorem will be established once we prove that $\mathsf{X}_{\mathrm{s}} = \ker Q$ and $\mathsf{X}_{\mathrm{r}} = \operatorname{ran} Q$.

Let $x \in \ker Q$. Since $Q \in \mathsf{S}$, there is a net $\{T_i\}_{i\in I} \subseteq \mathsf{S}_0$ such that $T_i \to Q$ relative to the pointwise weak* topology; hence $T_i(x) \to Qx = 0$, so $0 \in \overline{\{T_t(x) : t \geq 0\}}^{\mathrm{w}^*}$. Conversely, assume $0 \in \overline{\{T_t(x) : t \geq 0\}}^{\mathrm{w}^*}$ for some x. Then there is a net $\{T_i\}_{i\in I} \subseteq \mathsf{S}_0$ such that $T_i(x) \to 0$ relative to the weak* topology. By compactness of S, there is a subnet $\{T_j\}_{j\in J} \subseteq \{T_i\}_{i\in I}$ with $T_j \to R$ relative to the pointwise weak* topology for some $R \in \mathsf{S}$, and it follows that $Rx = 0$. Hence $R'QRx = 0$ for all $R' \in \mathsf{S}$. Choosing R' to be the inverse of QR in G, we get $Qx = 0$, hence $x \in \ker Q$. We have thus proved that $\mathsf{X}_{\mathrm{s}} = \ker Q$.

Let $\widehat{\mathsf{G}}$ be the character group of G. For each $\chi \in \widehat{\mathsf{G}}$ define the operator

$$\mathsf{X} \ni x \longmapsto P_\chi x = \int_{\mathsf{G}} \overline{\chi(S)} S x \, \mathrm{d}\mu(S), \tag{2.7}$$

where μ is the normalized Haar measure of G. The integral is to be understood as a weak* integral, thus P_χ is a well-defined bounded operator in $\mathsf{L}(\mathsf{X})$ with $\|P_\chi\| \leq 1$. Then for all $R \in \mathsf{G}$ we get

$$\begin{aligned} RP_\chi x &= \int_{\mathsf{G}} \overline{\chi(S)} RSx \, \mathrm{d}\mu(S) = \chi(R) \int_{\mathsf{G}} \overline{\chi(RS)} RSx \, \mathrm{d}\mu(S) \\ &= \chi(R) \int_{\mathsf{G}} \overline{\chi(S)} Sx \, \mathrm{d}\mu(S) = \chi(R) P_\chi x, \end{aligned} \tag{2.8}$$

in particular $QP_\chi = P_\chi$; therefore, $T_t P_\chi = T_t Q P_\chi = \chi(T_t Q) P_\chi$ for all $t \geq 0$. Since $t \mapsto \chi(T_t Q)$ is continuous and satisfies the functional equation

$$\chi(T_t Q) \cdot \chi(T_s Q) = \chi(T_{t+s} Q) \in \{z \in \mathbb{C} : |z| = 1\} \tag{2.9}$$

for all $t, s \geq 0$, we have $\chi(T_t Q) = e^{i\alpha t}$ for some $\alpha \in \mathbb{R}$. Thus $T_t P_\chi = e^{i\alpha t} P_\chi$, hence $P_\chi \mathrm{X} \subseteq \operatorname{dom} Z$ and $Z P_\chi = i\alpha P_\chi$ for all $\chi \in \widehat{\mathrm{G}}$. We next define the subspace

$$M = \overline{\operatorname{lin}\left\{\bigcup_{\chi\in\widehat{\mathrm{G}}} P_\chi \mathrm{X}\right\}}^{w^*} \subseteq \mathrm{X}_{\mathrm{r}}. \tag{2.10}$$

We prove that $\operatorname{ran} Q \subseteq M \subseteq \mathrm{X}_{\mathrm{r}}$. Let $\varphi \in M^\perp = \{\varphi \in \mathrm{X}_* : \langle x, \varphi\rangle = 0 \ \forall x \in M\}$. Then $\langle P_\chi x, \varphi\rangle = 0$ for all $x \in \mathrm{X}$, $\chi \in \widehat{\mathrm{G}}$, that is,

$$\int_{\widehat{\mathrm{G}}} \overline{\chi(S)} \langle Sx, \varphi\rangle \mathrm{d}\mu(S) = 0 \tag{2.11}$$

for all $x \in \mathrm{X}$, $\chi \in \widehat{\mathrm{G}}$. Since the character group $\widehat{\mathrm{G}}$ is total in $L^2(G,\mu)$ by the Stone-Weierstraß theorem and since $S \mapsto \langle Sx, \varphi\rangle$ is continuous it follows that $\langle Sx, \varphi\rangle = 0$ for all $S \in \mathrm{G}$ and $x \in \mathrm{X}$. Take $S = Q$, then we obtain $\varphi \in \operatorname{ran} Q^\perp$ and thus $M^\perp \subseteq \operatorname{ran} Q^\perp$. By the bipolar theorem we obtain $\operatorname{ran} Q \subseteq \overline{\mathrm{co}}\ M = M$, since $\operatorname{ran} Q$ is a weak* closed subspace. Conversely, let $x \in \operatorname{dom} Z$ with $Zx = i\alpha x$ for some $\alpha \in \mathbb{R}$. It follows that $T_t(x) = e^{i\alpha t} x$ for all $t \geq 0$ and consequently $Rx = e^{i\alpha} x$ for $R \in \mathrm{S}$. Thus there exists $\beta \in \mathbb{R}$ such that $Qx = e^{i\beta} x = Q^2 x$. Consequently, we must have $\beta = 0$ which implies $Qx = x \in \operatorname{ran} Q$, hence $\mathrm{X}_{\mathrm{r}} \subseteq \operatorname{ran} Q$, and the proof is finished. □

Corollary 2.2. *Under the hypothesis of Theorem 2.1, there exists a weak* continuous one-parameter group* $\{\alpha_t\}_{t\in\mathbb{R}}$ *of isometries on* X_{r} *such that* $\alpha_t = T_t \restriction \mathrm{X}_{\mathrm{r}}$ *for* $t \geq 0$*.*

Proof. Let $T \in \mathrm{S}$, then $QT \in \mathrm{G}$, and let R be the inverse of QT in G, that is, $R(QT) = Q$. Then for all $x \in \mathrm{X}_{\mathrm{r}}$, we have $RTx = RQTx = Qx = x$. Now write $\alpha_t = QT_t$ for $t \geq 0$ and let α_{-t} be the inverse of α_t in G. The foregoing calculation shows that $\{\alpha_t\}_{t\in\mathbb{R}}$ is a one-parameter group on X_{r}. Moreover, it is clear that it is weak* continuous and contractive. Now assume that there is $x \in \mathrm{X}_{\mathrm{r}}$ and $t \geq 0$ such that $\|\alpha_t(x)\| < \|x\|$. Then it follows that $\|\alpha_{-t}\| > 1$, contradiction; thus $\{\alpha_t\}_{t\in\mathbb{R}}$ is isometric. □

The subspace X_{r} is called the *reversible subspace* and X_{s} is called the *stable subspace;* its elements are sometimes called *flight vectors.*

In applications it is sometimes desirable to have a stronger characterization of the subspace X_{s}, namely, we are interested in a stronger stability property of the elements in X_{s}. In particular, this is relevant in the applications to decoherence we discuss in Section 4. The next result provides a sufficient condition for weak* stability to hold on X_{s} based on the boundary spectrum $\operatorname{spec} Z \cap i\mathbb{R}$ of the generator Z.

Proposition 2.3. *Assume that the hypothesis of Theorem 2.1 is satisfied and additionally that* $\operatorname{spec} Z \cap i\mathbb{R}$ *is at most countable. Then the stable subspace* (2.5) *is given by*

$$\mathrm{X}_{\mathrm{s}} = \left\{x \in \mathrm{X} : \lim_{t\to\infty} T_t(x) = 0 \text{ relative to the weak* topology}\right\}. \tag{2.12}$$

Moreover, the convergence in (2.12) *is uniform for* x *in* $\mathrm{X}_{\mathrm{s}} \cap \mathrm{X}_1$*.*

Proof. Consider the predual semigroup $\{T_{t,*}\}_{t\geq 0}$ with generator Z_*; as already remarked, $(Z_*)^*$ is Z. The predual Q_* of Q is a projection and induces a splitting $\mathsf{X}_* = \mathsf{X}_{r,*} \oplus \mathsf{X}_{s,*}$ by way of $\mathsf{X}_{r,*} = \operatorname{ran} Q_*$ and $\mathsf{X}_{s,*} = \ker Q_*$. Let $\{T^s_{t,*}\}_{t\geq 0}$ be the restriction of $\{T_{t,*}\}_{t\geq 0}$ to $\mathsf{X}_{s,*}$. Since $\mathsf{X}_{s,*}$ is closed; the generator Z^s_* of $\{T^s_{t,*}\}_{t\geq 0}$ is given by the restriction $Z^s_* = Z_* \restriction \mathsf{X}_{s,*}$, $\operatorname{dom} Z^s_* = \operatorname{dom} Z_* \cap \mathsf{X}_{s,*}$. A similar construction applies to the reversible subspace X_r. We check that $\operatorname{spec} Z^s_* \subseteq \operatorname{spec} Z_*$. Let $\lambda \in \rho(Z_*)$, that is, the map $(\lambda\mathbb{1} - Z_*) : \operatorname{dom} Z_* \to \mathsf{X}_*$ is bijective. Then clearly the map $(\lambda\mathbb{1} - Z^s_*) = (\lambda\mathbb{1} - Z_*) \restriction \mathsf{X}_{s,*} : \operatorname{dom} Z_* \cap \mathsf{X}_{s,*} \to \mathsf{X}_{s,*}$ is injective. It is also surjective: let $\varphi \in \mathsf{X}_{s,*}$, then there is $\psi = \psi_s \oplus \psi_r \in \operatorname{dom} Z_*$ such that $(\lambda\mathbb{1} - Z_*)\psi = \varphi$. Now

$$(\lambda\mathbb{1} - Z_*)\psi = (\lambda\mathbb{1} - Z_*)\left(\psi_s \oplus \psi_r\right) = \varphi \oplus 0, \tag{2.13}$$

so $(\lambda\mathbb{1} - Z_*)\psi_r = 0$ and $\psi_r = 0$ by injectivity. Thus $(\lambda\mathbb{1} - Z^s_*)$ is bijective and $\lambda \in \rho(Z^s_*)$. In particular, using $\operatorname{spec} Z = \operatorname{spec} Z_*$ we find that

$$\operatorname{spec} Z^s_* \cap i\mathbb{R} \quad \text{is countable.} \tag{2.14}$$

We now see that

$$\operatorname{spec}_p Z_s \cap i\mathbb{R} = \emptyset, \tag{2.15}$$

for if $i\lambda \in \operatorname{spec}_p Z_s$, $\lambda \in \mathbb{R}$, then the corresponding eigenvector $x \in \operatorname{dom} Z_s \subseteq \mathsf{X}_s$ satisfying $Zx = i\lambda x$ must lie in X_r by (2.6), hence $x = 0$, contradiction.

From (2.14) and (2.15), it follows by the Arendt-Batty-Lyubich-V$\tilde{u}$ theorem [15, 16], see also [9], that the semigroup $\{T^s_{t,*}\}_{t\geq 0}$ is strongly stable, that is, for all $x_* \in \mathsf{X}_{s,*}$ we have $\lim_{t\to\infty}\|T^s_{t,*}(x_*)\| = 0$. Thus if $x \in \mathsf{X}_s$ and $x_* \in \mathsf{X}_*$ it follows that

$$\begin{aligned} |\langle T_t(x), x_*\rangle| &= \left|\left\langle Q^\perp(T_t(x)), x_*\right\rangle\right| = \left|\left\langle x, T_{t,*}\left(Q^\perp_*(x_*)\right)\right\rangle\right| \\ &\leq \|x\| \cdot \|T^s_{t,*}\left(Q^\perp_*(x_*)\right)\| \longrightarrow 0 \end{aligned} \tag{2.16}$$

as $t \to \infty$ uniformly for $x \in \mathsf{X}_s \cap \mathsf{X}_1$, where $Q^\perp = \mathrm{id}_{\mathsf{X}} - Q$ denotes the projection onto X_s. □

3. Semigroups on von Neumann Algebras

The results of the previous section apply to the case of Neumann algebras. Let $\mathscr{H}$ be a Hilbert space. A *von Neumann algebra* is a *-subalgebra of the Banach-*-algebra $\mathrm{L}(\mathscr{H})$ of all bounded linear operators acting on $\mathscr{H}$, which is additionally closed in the weak (or equivalently strong) operator topology. The identity operator will be denoted by $\mathbb{1}$, and we will always assume that $\mathbb{1} \in \mathfrak{M}$. The *ultraweak topology* on $\mathfrak{M}$ is defined by the seminorms $p_\rho(x) = |\operatorname{tr}(\rho x)|$, where ρ runs through the trace class operators on $\mathscr{H}$, it agrees with the weak operator topology on bounded portions of $\mathfrak{M}$. The set of all ultraweakly continuous linear functionals on $\mathfrak{M}$ forms a Banach space, and this Banach space is the unique (up to isomorphism) predual space of $\mathfrak{M}$, for this reason we denote it by $\mathfrak{M}_*$. The ultraweak topology on $\mathfrak{M}$ can be shown to be equivalent to the $\sigma(\mathfrak{M}, \mathfrak{M}_*)$ (i.e., weak*) topology. Hence the setup of the previous section applies to this case. The set of all positive operators of $\mathfrak{M}$ will be denoted by $\mathfrak{M}^+$. A functional

φ in $\mathfrak{M}_*$ which is positive (i.e., $\varphi(x) \geq 0$ provided $x \in \mathfrak{M}^+$) and normalized (i.e., $\|\varphi\| = 1$, equivalently $\varphi(\mathbb{1}) = 1$) will be called a *normal state.* A state is called faithful if $x \in \mathfrak{M}^+$ and $\omega(x) = 0$ implies $x = 0$. For proofs of these results, we refer to [10, 17].

Let $T \in \mathrm{L}(\mathfrak{M})$. Then T is called positive if $T(\mathfrak{M}^+) \subseteq \mathfrak{M}^+$. A positive operator is normal (i.e., weak* continuous) if and only if for every uniformly bounded increasing net $\{x_i\}_{i\in I} \subseteq \mathfrak{M}^+$ we have $\sup_i T(x_i) = T(\sup_i x_i)$. Furthermore, T is called *strongly positive* whenever it satisfies Kadison's inequality, that is, $\|T(\mathbb{1})\| T(x^*x) \geq T(x)^* T(x)$ for any $x \in \mathfrak{M}$. Clearly strong positivity implies positivity. An even stronger notion of positivity is complete positivity: $T \in \mathrm{L}(\mathfrak{M})$ is called *completely positive* whenever $\sum_{i,j=1}^{n} y_i^* T(x_i^* x_j) y_j \geq 0$ for all $n \in \mathbb{N}$ and all $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ from $\mathfrak{M}$. The map T is called unital if $T(\mathbb{1}) = \mathbb{1}$; a positive unital map is automatically contractive, that is, $\|T(x)\| \leq \|x\|$ for all $x \in \mathfrak{M}$.

The following result has been established in [18].

Proposition 3.1. *Suppose that $S \subseteq \mathrm{L_n}(\mathfrak{M})_1$ is a subset of normal contractive linear operators. Then the following assertions are equivalent.*

(1) *The set $\{T_*(\varphi) : T \in S\} \subseteq \mathfrak{M}_*$ is relatively weakly compact for every $\varphi \in \mathfrak{M}_*$.*

(2) *The set S is equicontinuous when $\mathfrak{M}$ is endowed with the Mackey topology (i.e., the $\tau(\mathfrak{M}, \mathfrak{M}_*)$ topology).*

(3) *The pointwise weak* closure of S consists of normal operators: $\overline{S} \subseteq L_n(\mathfrak{M})_1$.*

Moreover, these conditions are satisfied whenever there is a faithful normal state ω on $\mathfrak{M}$ such that

$$\omega(T(x^*)T(x)) \leq \omega(x^*x) \quad \text{for any } T \in S,\ x \in \mathfrak{M}. \tag{3.1}$$

In particular, if each element in S is strongly positive we conclude that (3.1) *can be rewritten as $\omega(T(x)) \leq \omega(x)$ for all $x \in \mathfrak{M}^+$, $T \in S$, or briefly $\omega \circ T \leq \omega$, for all $T \in S$.*

If $S \subseteq \mathrm{L_n}(\mathfrak{M})_1$ is a subset of normal contractive linear operators and ω a normal state, we call ω an *invariant state* under S provided $\omega(T(x)) = \omega(x)$ for all $x \in \mathfrak{M}$ and $T \in S$. We now apply the results of Section 2 to weak* continuous semigroups on von Neumann algebras. This gives us the following result.

Corollary 3.2. *Suppose that $\{T_t\}_{t\geq 0}$ is a weak* continuous contractive strongly positive one-parameter semigroup on a von Neumann algebra $\mathfrak{M}$ with ultraweak generator Z, and suppose that there exists a faithful normal invariant state ω. Then there exist weak* closed and T_t-invariant subspaces $\mathfrak{M}_{\mathrm{r}}$ and $\mathfrak{M}_{\mathrm{s}}$ of $\mathfrak{M}$, given by* (2.5) *and* (2.6)*, such that $\mathfrak{M} = \mathfrak{M}_{\mathrm{r}} \oplus \mathfrak{M}_{\mathrm{s}}$.*

Proof. By Kadison's inequality, (3.1) holds; thus Proposition 3.1 implies that the pointwise weak* closure $\mathrm{S} = \overline{\mathrm{S}_0}$, with $\mathrm{S}_0 = \{T_t\}_{t\geq 0}$, consists of normal operators. Hence Theorem 2.1 applies. □

It is worth pointing out that a similar result was recently established in [19] for general semigroups acting on a W*-algebra and possessing a faithful family of subinvariant states.

We now prove that $\mathfrak{M}_{\mathrm{r}}$ is actually a von Neumann subalgebra. Recall that a conditional expectation Q from a C*-algebra $\mathfrak{A}$ onto a C*-subalgebra $\mathfrak{B} \subseteq \mathfrak{A}$ is a completely positive contraction with $Q(x) = x$ for $x \in \mathfrak{B}$ and $Q(xyx) = xQ(y)x$ for $x \in \mathfrak{B}$, $y \in \mathfrak{A}$.

Proposition 3.3. *Let $\{T_t\}_{t\geq 0}$ be a weak* continuous semigroup of strongly positive unital operators and suppose there exists a faithful normal invariant state ω. Then $\mathfrak{M}_r$ is a von Neumann subalgebra of $\mathfrak{M}$ and there exists a group of *-automorphisms $\{\alpha_t\}_{t\in\mathbb{R}}$ on $\mathfrak{M}_r$ such that $T_t \upharpoonright \mathfrak{M}_{\mathrm{r}} = \alpha_t$ for all $t \geq 0$. Moreover, there exists a normal conditional expectation Q from $\mathfrak{M}$ onto $\mathfrak{M}_{\mathrm{r}}$ such that $\omega \circ Q = \omega$. Finally, $\mathfrak{M}_{\mathrm{s}}$ is *-invariant.*

Proof. Since each T_t is a contraction; Corollary 3.2 applies. Let $M_0 = \{x \in \mathfrak{M} : \exists\alpha \in \mathbb{R}$ such that $T_t(x) = \mathrm{e}^{\mathrm{i}\alpha t}x \ \forall t \geq 0\}$, that is, we have $\overline{\operatorname{lin} M_0}^{\mathrm{w}^*} = \mathfrak{M}_{\mathrm{r}}$. As in [20] we define the sesquilinear map $D : \mathfrak{M}\times\mathfrak{M} \to \mathfrak{M}$ by $D(x,y) = T_t(x^*y) - T_t(x)^*T_t(y)$ for some fixed $t \geq 0$. By Kadison's inequality, the sesquilinear form $\varphi \circ D$ is positive-definite for any $\varphi \in \mathfrak{M}_*^+$, so by the Cauchy-Schwarz inequality, $D(x,x) = 0$ if and only if $D(x,y) = 0$ for all $y \in \mathfrak{M}$. Now let $x \in M_0$, then $T_t(x^*x) \geq T_t(x)^*T_t(x) = \mathrm{e}^{-\mathrm{i}\alpha t}\mathrm{e}^{+\mathrm{i}\alpha t}x^*x = x^*x$. Thus $0 \leq \omega(T_t(x^*x) - x^*x) \leq \omega(x^*x - x^*x) = 0$, and by faithfulness $T_t(x^*x) = x^*x$, hence $D(x,x) = 0$ for all $x \in M_0$. So $D(x,y) = 0$ for all $x, y \in M_0$, that is, $T_t(x^*y) = T_t(x)^*T_t(y) = \mathrm{e}^{\mathrm{i}(\alpha_2-\alpha_1)t}x^*y$, and we conclude that $xy \in M_0$ whenever $x, y \in M_0$. It follows that $\operatorname{lin} M_0$ is a *-subalgebra of $\mathfrak{M}$ (containing $\mathbb{1}$) and consequently $\mathfrak{M}_{\mathrm{r}}$ is a von Neumann subalgebra, and $T_t(x^*y) = T_t(x^*)T_t(y)$ for all $x, y \in \mathfrak{M}_{\mathrm{r}}$. By Corollary 2.2, the restriction of T_t to $\mathfrak{M}_{\mathrm{r}}$ extends to a one-parameter group $\{\alpha_t\}_{t\in\mathbb{R}}$ of isometries and the above argument shows that α_t must be a *-homomorphism. Let Q be the Sushkevich kernel of the semigroup $S \subseteq \mathrm{L}_{\mathrm{n}}(\mathfrak{M})_1$. Since Q is a projection and $\|Q\| = 1$ it follows from Tomiyama's theorem [21] that Q is a conditional expectation; since $Q \in \mathrm{S}$; it is also clear that $\omega \circ Q = \omega$. The last assertion is clear as well. □

In the following, we will be interested in the stronger characterization of $\mathfrak{M}_{\mathrm{s}}$ by a stability property as in (2.12). We start by quoting the following result.

Lemma 3.4. *Suppose that $\{T_t\}_{t\geq 0}$ is a weak* continuous one-parameter semigroup of strongly positive unital operators on the von Neumann algebra $\mathfrak{M}$ with a faithful normal invariant state ω. Introduce the subsets*

$$\begin{gathered} M = \{x \in \mathfrak{M} : T_t(x^*x) = T_t(x)^*T_t(x) \ \forall t \geq 0\}, \\ M^* = \{x \in \mathfrak{M} : T_t(xx^*) = T_t(x)T_t(x)^* \ \forall t \geq 0\}, \\ \mathfrak{M}_1 = M \cap M^*. \end{gathered} \tag{3.2}$$

Then $\mathfrak{M}_1$ is a T_t-invariant von Neumann subalgebra of $\mathfrak{M}$, and there exists a group of *-automorphisms $\{\alpha_t\}_{t\in\mathbb{R}}$ on $\mathfrak{M}_1$ such that $T_t \upharpoonright \mathfrak{M}_1 = \alpha_t$ for $t \geq 0$. Moreover, $\mathfrak{M}_1$ is a maximal (in the sense of not being properly contained in a larger von Neumann subalgebra) von Neumann subalgebra on which the restriction of $\{T_t\}_{t\geq 0}$ is given by a group of *-automorphisms.

A proof can be found in [22] (see the proof of Proposition 2). It is easy to see that we always have $\mathfrak{M}_{\mathrm{r}} \subseteq \mathfrak{M}_1$.

Lemma 3.5. *Under the assumptions of Lemma 3.4, for every $x \in \mathfrak{M}$ the weak* limit points of the net $\{T_t(x)\}_{t\in\mathbb{R}^+}$ lie in $\mathfrak{M}_1$.*

A proof of this statement is contained in the proof of Theorem 3.1 of [23].

We can now establish the following result.

Proposition 3.6. *Let $\{T_t\}_{t\geq 0}$ be a weak* continuous semigroup of strongly positive unital operators on the von Neumann algebra $\mathfrak{M}$ with a faithful normal invariant state ω. If $\mathfrak{M}_{\mathrm{r}} = \mathfrak{M}_1$, it follows that*

$$\mathfrak{M}_{\mathrm{s}} = \left\{ x \in \mathfrak{M} : \lim_{t\to\infty} T_t(x) = 0 \textit{ in the weak}^* \textit{ topology} \right\}. \tag{3.3}$$

Proof. Let $x \in \mathfrak{M}_{\mathrm{s}}$ and assume without loss of generality that $\|x\| \leq 1$. By Alaoglu's theorem the net $\{T_t(x)\}_{t\in\mathbb{R}^+}$ contained in the unit ball of $\mathfrak{M}$ has a limit point x_0 for $t \to \infty$. Then using Lemma 3.5, we find that $x_0 \in \mathfrak{M}_1 = \mathfrak{M}_r$. But since $x \in \mathfrak{M}_{\mathrm{s}}$, it follows that also $x_0 \in \mathfrak{M}_{\mathrm{s}}$, that is, $x_0 \in \mathfrak{M}_{\mathrm{s}} \cap \mathfrak{M}_{\mathrm{r}} = \{0\}$; hence $x_0 = 0$. This proves that any limit point of the net $\{T_t(x)\}_{t\in\mathbb{R}^+}$ is equal to 0; therefore we conclude that $\lim_{t\to\infty} T_t(x) = 0$ in the weak* topology for all $x \in \mathfrak{M}_{\mathrm{s}}$. □

Moreover, let us remark the following: suppose that $\{T_t\}_{t\geq 0}$ is a weak* continuous semigroup of strongly positive unital operators with generator Z having a faithful normal invariant state ω, and assume that the peripheral spectrum $\operatorname{spec} Z \cap \mathrm{i}\mathbb{R}$ is at most countable. Then by using Proposition 2.3 the conclusion of Proposition 3.6 holds. These results will be used in the next section when we discuss the notion of decoherence for uniformly continuous quantum dynamical semigroups.

4. Applications to Decoherence

4.1. The Notion of Decoherence in the Algebraic Framework

Consider a closed quantum system whose algebra of observables is a von Neumann algebra $\mathfrak{N}$, and its reversible time evolution is given by a weak* continuous group of *-automorphisms $\{\beta_t\}_{t\in\mathbb{R}}$ on $\mathfrak{N}$. A subsystem can be described by a von Neumann subalgebra $\mathfrak{M} \subseteq \mathfrak{N}$ containing the observables belonging to the subsystem. We will assume that there exists a normal conditional expectation E from $\mathfrak{N}$ onto $\mathfrak{M}$. In this situation, we can define the *reduced dynamics* as follows:

$$T_t(x) = E \circ \beta_t(x), \quad x \in \mathfrak{M},\ t \geq 0. \tag{4.1}$$

This is the Heisenberg picture time evolution an observer whose experimental capabilities are limited to the system described by $\mathfrak{M}$ would witness. Since it is the time evolution of an open system it is, in general, irreversible. From (4.1) we can isolate some mathematical properties of the reduced dynamics.

(1) $\{T_t\}_{t\geq 0}$ is a family of completely positive and normal linear operators on $\mathfrak{M}$.

(2) $T_t(\mathbb{1}) = \mathbb{1}$ for all $t \geq 0$, in particular; each T_t is contractive.

(3) $t \mapsto T_t(x)$ is weak* continuous for any $x \in \mathfrak{M}$.

In general the reduced dynamics $\{T_t\}_{t\geq 0}$ is not Markovian, that is, memory-free, and hence the operators $\{T_t\}_{t\geq 0}$ do not form a one-parameter semigroup. However, in many physically relevant situations it is a good approximation to describe the reduced dynamics by a semigroup satisfying the above properties (1)–(3), that is, a weak* continuous semigroup of completely positive unital maps on the von Neumann algebra $\mathfrak{M}$. Such a semigroup is called a *quantum dynamical semigroup.* We remark that in many physically relevant models we have the following structure: $\mathfrak{N} = \mathfrak{M}\overline{\otimes}\mathfrak{M}_0$, acting on a tensor product $\mathscr{H} \otimes \mathscr{H}_0$ of two Hilbert spaces, where $\mathfrak{M}_0$ describes the environment of the system (e.g., a heat bath). The time evolution of the system and environment is Hamiltonian, that is, $\beta_t(x) = \mathrm{e}^{\mathrm{i}tH} x \mathrm{e}^{-\mathrm{i}tH}$ with $H = H_1 \otimes \mathbb{1} + \mathbb{1} \otimes H_2 + H_{\mathrm{int}}$, where H_1 and H_2 are the Hamiltonians belonging to the system and its environment, and H_{int} is an interaction term. Moreover, the conditional expectation E_ω is given with respect to a reference state ω of the environment, that is, $\varphi \otimes \omega(x) = \varphi(E_\omega(x))$ for all $x \in \mathfrak{N}$ and $\varphi \in \mathfrak{M}_*$. In this situation, the predual time evolution is given by the familiar formula

$$T_{t,*}(\varphi) = \mathrm{tr}_2\left[\mathrm{e}^{-\mathrm{i}tH}(\varphi \otimes \omega)\mathrm{e}^{\mathrm{i}tH}\right], \tag{4.2}$$

where φ is a normal state on $\mathfrak{M}$ and tr_2 denotes the partial trace with respect to the degrees of freedom of the environment.

Since the reduced dynamics is in general not reversible, new phenomena like the approach to equilibrium can appear. In this paper, we are particularly interested in an effect called decoherence. The following general and mathematically rigorous characterization of decoherence in the algebraic framework was introduced by Blanchard and Olkiewicz [3, 4]. Its present form is taken from [24]; the relation of this form and that given in [3] is discussed in [25].

Definition 4.1. We say that the reduced dynamics $\{T_t\}_{t\geq 0}$ displays decoherence if the following assertions are satisfied: there exists a T_t-invariant von Neumann subalgebra $\mathfrak{M}_1$ of $\mathfrak{M}$ and an weak* continuous group $\{\alpha_t\}_{t\in\mathbb{R}}$ of *-automorphisms on $\mathfrak{M}_1$ such that $T_t \restriction \mathfrak{M}_1 = \alpha_t$ for $t \geq 0$, and a T_t-invariant and *-invariant weak* closed subspace $\mathfrak{M}_2$ of $\mathfrak{M}$ such that

$$\mathfrak{M} = \mathfrak{M}_1 \oplus \mathfrak{M}_2, \tag{4.3}$$

$$\lim_{t\to\infty} T_t(x) = 0 \quad \text{in the weak}^* \text{ topology for any } x \in \mathfrak{M}_2. \tag{4.4}$$

Moreover, we require that $\mathfrak{M}_1$ is a maximal von Neumann subalgebra of $\mathfrak{M}$ (in the sense of not being properly contained in any larger von Neumann subalgebra) on which $\{T_t\}_{t\geq 0}$ extends to a group of *-automorphisms. We call $\mathfrak{M}_1$ the algebra of effective observables.

The physical interpretation of this definition is rather clear: if decoherence takes place there is a (maximal) von Neumann subalgebra on which the reduced dynamics is reversible, that is, given by an automorphism group, and a complementary subspace on which the expectation values with respect to any normal state of all its elements tend to zero in time. Thus any observable $x \in \mathfrak{M}$ has a decomposition $x = x_1 + x_2$, where $x_i \in \mathfrak{M}_i$, such that $\varphi(T_t(x_2)) \to 0$ as $t \to \infty$ for any normal state φ on $\mathfrak{M}$. Hence after a sufficiently long time the system behaves effectively like a closed system described by $\mathfrak{M}_1$ and $\{\alpha_t\}_{t\in\mathbb{R}}$. By analyzing the structure of the algebra of effective observables $\mathfrak{M}_1$ and the reversible dynamics $\{\alpha_t\}_{t\in\mathbb{R}}$ various physically relevant and well-known phenomena of decoherence can be identified, like the appearance of pointer states, environment-induced superselection rules, and classical dynamical systems. In this way it is possible to obtain an exhaustive classification of possible decoherence scenarios, see [3] for a thorough discussion of this point. Particularly interesting is the case when $\mathfrak{M}_1$ is a factor, that is, after decoherence we still have a system of pure quantum character. This is of interest in the context of quantum computation since in this way one may obtain a system which retains its quantum character despite decoherence.

According to Definition 4.1, if $\{T_t\}_{t\geq 0}$ is a group of automorphisms, decoherence takes place and the splitting (4.3) is trivial with $\mathfrak{M}_2 = \{0\}$. However, we will keep this slightly unfortunate terminology since it simplifies the statements of theorems, keeping in mind that physically decoherence corresponds to the case when $\mathfrak{M}_2 \neq \{0\}$. This can only happen if $\{T_t\}_{t\geq 0}$ is irreversible.

We remark that the algebra $\mathfrak{M}_1$ has been studied in [26] and explicit representations of $\mathfrak{M}_1$ are obtained for quantum dynamical semigroups with unbounded generators. Moreover, in [27] a different notion of decoherence for quantum dynamical semigroups is introduced in a mathematically rigorous way, and its connection to the Blanchard-Olkiewicz notion is briefly discussed in [28]. In [29] an asymptotic property similar to (4.3) and (4.4) is discussed under the designation "limited relaxation."

In connection with Definition 4.1, the following question arises: if there exists a maximal von Neumann subalgebra of $\mathfrak{M}$ on which $\{T_t\}_{t\geq 0}$ extends to a group of automorphisms, is this subalgebra necessarily unique? The following theorem answers this question.

Theorem 4.2. *Let $\{T_t\}_{t\geq 0}$ be a quantum dynamical semigroup and suppose it has a faithful normal invariant state ω. Then there exists a unique maximal von Neumann subalgebra on which $\{T_t\}_{t\geq 0}$ extends to a group of automorphisms.*

Proof. Let $\mathfrak{M}_i$, where $i \in I$ is some index set, be a collection of von Neumann subalgebras on each of which $\{T_t\}_{t\geq 0}$ extends to a group of automorphisms. Then we put

$$\mathfrak{B} = \bigvee_{i\in I} \mathfrak{M}_i = \overline{\text{lin}}\{x_{i_1} \cdots x_{i_k} : i_1, \ldots, i_k \in I,\ k \in \mathbb{N},\ x_{i_\ell} \in \mathfrak{M}_{i_\ell}\}, \tag{4.5}$$

where the closure is taken in the ultraweak topology. We proceed as in the proof of Proposition 3.3 and introduce $D(x,y) = T_t(x^*y) - T_t(x)^*T_t(y)$ for a fixed $t \geq 0$ and $x, y \in \mathfrak{M}$. Then $D(x,x) = 0$ if and only if $D(x,y) = 0$ for all $y \in \mathfrak{M}$. Now if $x \in \mathfrak{M}_i$, we get $D(x,x) = 0$, thus $D(x,y) = 0$ for $y \in \mathfrak{M}$, where $i \in I$, that is, $T_t(xy) = T_t(x)T_t(y)$. Proceeding inductively we have

$$T_t(x_{i_1} \cdots x_{i_k}) = T_t(x_{i_1}) \cdots T_t(x_{i_k}), \tag{4.6}$$

for an arbitrary monomial $x_{i_1} \cdots x_{i_k}$, where $x_{i_\ell} \in \mathfrak{M}_{i_\ell}$, $i_1, \ldots, i_k \in I$. Therefore, if $B_0 = \text{lin}\{x_{i_1} \cdots x_{i_k} : i_1, \ldots, i_k \in I,\ k \in \mathbb{N}\}$, then $T_t : B_0 \to B_0$ is a *-homomorphism of the *-subalgebra B_0, and $T_t : \mathfrak{B} \to \mathfrak{B}$ is a *-homomorphism as well.

Next we note that $T_t \upharpoonright \mathfrak{B}$ is injective. Namely, if $x \in \ker(T_t \upharpoonright \mathfrak{B})$, we get $\omega(x^*x) = \omega(T_t(x^*x)) = \omega(T_t(x)^*T_t(x)) = 0$, thus $x = 0$ by faithfulness of ω and so $\ker(T_t \upharpoonright \mathfrak{B}) = \{0\}$.

We now establish that $T_t : \mathfrak{B} \to \mathfrak{B}$ is surjective. Since T_t is an injective *-homomorphism on the C*-algebra $\mathfrak{B}$, it follows that $\|T_t(x)\| = \|x\|$ for all $x \in \mathfrak{B}$. Moreover, notice that $T_t : B_0 \to B_0$ is invertible since each restriction $T_t \upharpoonright \mathfrak{M}_i$ is invertible. This implies $\|T_t^{-1}(x)\| = \|T_t(T_t^{-1}(x))\| = \|x\|$ for any $x \in B_0$. Now for the proof of surjectivity choose $y \in \mathfrak{B}$. By the Kaplansky density theorem, there exists a net $\{y_j\}_{j\in J} \subseteq B_0$ such that $\|y_j\| \leq C$ for all $j \in J$, and such that $\lim y_j = y$. Put $x_j = T_t^{-1}(y_j)$. Then $\|x_j\| = \|y_j\| \leq C$, and by Alaoglu's theorem the net $\{x_j\}_{j\in J}$ has an ultraweak limit point x_0. Thus there is a subnet $\{x_{j_e}\} \subseteq \{x_j\}_{j\in J}$ such that $\lim x_{j_e} = x_0$, hence

$$y_{j_e} = T_t\left(x_{j_e}\right) \longrightarrow y = T_t(x_0). \tag{4.7}$$

We conclude by injectivity that any ultraweak limit point of the net $\{x_j\}_{j\in J}$ is equal to x_0, hence this net is convergent to x_0, and $T_t(x_0) = y$, establishing surjectivity. We have thus proved that $T_t \restriction \mathfrak{B}$ is a *-automorphism.

To finish the proof, we have to choose the collection $\{\mathfrak{M}_i : i \in I\}$ in (4.5) to consist of all von Neumann subalgebras on which $\{T_t\}_{t\geq 0}$ extends to a group of automorphisms. □

We next prove that in certain cases the splitting (4.3) is always given by a conditional expectation.

Proposition 4.3. *Suppose that the (not necessarily Markovian) reduced dynamics $\{T_t\}_{t\geq 0}$ on the von Neumann algebra $\mathfrak{M}$ displays decoherence. If $T_t \restriction \mathfrak{M}_1 = \mathrm{id}_{\mathfrak{M}_1}$, then there exists a normal conditional expectation E from $\mathfrak{M}$ onto $\mathfrak{M}_1$.*

Proof. Let $x \in \mathfrak{M}$ and write $x = x_1 + x_2$ with $x_i \in \mathfrak{M}_i$, $i = 1, 2$, and define $E : \mathfrak{M} \to \mathfrak{M}_1$ by $E(x) = x_1$. Then $E(\mathfrak{M}) = \mathfrak{M}_1$ and $E^2 = E$. Since $\mathfrak{M}_2$ is *-invariant, $E(x^*) = E(x_1^* + x_2^*) = x_1^* = E(x)^*$, hence $E(\mathfrak{M}^{sa}) \subseteq \mathfrak{M}^{sa}$. Now let $x \in \mathfrak{M}^+$, consider the decomposition $x = x_1 + x_2$, $x_i \in \mathfrak{M}_i$, $i = 1, 2$, and suppose that $x_1 \in \mathfrak{M}^{sa}$ is not positive. Then there exists $\varphi \in \mathfrak{M}_*^+$ such that $\varphi(x_1) < 0$. This implies

$$0 \leq \varphi(T_t(x)) = \varphi(T_t(x_1)) + \varphi(T_t(x_2)) = \varphi(x_1) + \varphi(T_t(x_2)), \tag{4.8}$$

and letting $t \to \infty$ yields $\varphi(T_t(x_2)) \to 0$, a contradiction. Thus $E(\mathfrak{M}^+) \subseteq \mathfrak{M}^+$, and since $E(\mathbb{1}) = \mathbb{1}$ it follows that $\|E\| \leq 1$. From $E^2 = E$, we get $\|E\| = 1$, so E is a projection of norm 1 and hence by Tomiyama's theorem a conditional expectation. Since $\ker E = \{x \in \mathfrak{M} : E(x) = x_1 = 0\} = \mathfrak{M}_2$ is ultraweakly closed, we obtain by a theorem of Tomiyama [30] that E is normal. □

Given a reduced dynamics, the question arises of under what conditions decoherence will occur. In the Markovian case, sufficient conditions for the appearance of decoherence in the sense of Definition 4.1 have been formulated in [31]. In fact, we have the following theorem.

Theorem 4.4. *Let $\{T_t\}_{t\geq 0}$ be a weak* continuous one-parameter semigroup with a faithful normal state ω. Assume that the following conditions are satisfied.*

(1) *Each T_t, $t \geq 0$, is strongly positive and unital.*

(2) *Let $\{\sigma_t^\omega\}_{t\in\mathbb{R}}$ denote the modular group (see, e.g., [10]) corresponding to the state ω. Assume that $[T_t, \sigma_s^\omega] = 0$ for all $s \in \mathbb{R}$ and $t \geq 0$.*

Then $\{T_t\}_{t\geq 0}$ displays decoherence and there exists a normal conditional expectation E from $\mathfrak{M}$ onto $\mathfrak{M}_1$ such that $[T_t, E] = 0$ for all $t \geq 0$ and $\omega \circ E = \omega$.

The splitting $\mathfrak{M} = \mathfrak{M}_1 \oplus \mathfrak{M}_2$ of $\{T_t\}_{t\geq 0}$ provided by this theorem is called the *isometric-sweeping splitting*. In [31] the theorem was proved under the more general hypothesis that

ω is only a faithful semifinite normal weight. Then some additional technical assumptions about $\{T_t\}_{t\geq 0}$ are necessary. A simpler proof for the theorem as stated above has been given in [24].

The Jacobs-de Leeuw-Glicksberg splitting for weak* continuous semigroups on von Neumann algebras established in Corollary 3.2 can now be applied to establish decoherence.

Corollary 4.5. *Let $\{T_t\}_{t\geq 0}$ be a weak* continuous one-parameter semigroup of strongly positive unital operators with a faithful normal state ω. Assume that $\mathfrak{M}_{\mathrm{r}} = \mathfrak{M}_1$. Then $\{T_t\}_{t\geq 0}$ displays decoherence.*

Proof. According to Corollary 3.2, the Jacobs-de Leeuw-Glicksberg splitting exists, and by Proposition 3.6 we conclude that the requirements of Definition 4.1 are satisfied. □

Remark 4.6. Whenever a conditional expectation E from a von Neumann algebra $\mathfrak{M}$ onto a von Neumann subalgebra $\mathfrak{M}_1$ satisfies $\omega \circ E = \omega$ for a faithful normal state ω, it is uniquely determined by these conditions [17, Corollary II.6.10]. Since in case of the isometric-sweeping splitting as given by Theorem 4.4 we have $\mathfrak{M}_1 = E\mathfrak{M}$ and $\mathfrak{M}_2 = (\mathbb{1} - E)\mathfrak{M}$, and in the Jacobs-de Leeuw-Glicksberg splitting as given by Corollary 3.2 we have $\mathfrak{M}_{\mathrm{r}} = Q\mathfrak{M}$ and $\mathfrak{M}_{\mathrm{s}} = (\mathbb{1} - Q)\mathfrak{M}$, it follows that the isometric-sweeping and Jacobs-de Leeuw-Glicksberg splittings agree whenever $\mathfrak{M}_{\mathrm{r}} = \mathfrak{M}_1$.

4.2. Uniformly Continuous Semigroups

The purpose of this section is to show how the Jacobs-de Leeuw-Glicksberg splitting can be applied to establish decoherence of quantum dynamical semigroups in the sense of Definition 4.1. To avoid complications arising from unbounded generators, we will concentrate on the case of uniformly continuous quantum dynamical semigroups. We will arrive at a result which avoids assumption (2) in Theorem 4.4.

Let $\mathfrak{M}$ be a von Neumann algebra acting on a separable Hilbert space $\mathscr{H}$, and let $\{T_t\}_{t\geq 0}$ be a quantum dynamical semigroup such that $t \mapsto T_t$ is continuous in the uniform topology. Then by [32] the generator Z of $\{T_t\}_{t\geq 0}$, which is a bounded operator on $\mathfrak{M}$, is given by

$$Zx = G^* x + xG + \Phi(x), \tag{4.9}$$

where $G \in \mathrm{L}(\mathscr{H})$ and $\Phi : \mathfrak{M} \to \mathrm{L}(\mathscr{H})$ is a normal completely positive map. Since we have $T_t(\mathbb{1}) = \mathbb{1}$ for all $t \geq 0$, it follows that $Z\mathbb{1} = 0$ which forces $G^* = -G - \Phi(\mathbb{1})$. Upon introducing the operator $H = \mathrm{i}G + (1/2)\mathrm{i}\Phi(\mathbb{1})$, it is seen that H is a bounded selfadjoint operator on $\mathscr{H}$ and Z may be written as

$$Zx = \mathrm{i}[H, x] - \frac{1}{2}\{\Phi(\mathbb{1}), x\} + \Phi(x), \tag{4.10}$$

where $\{\cdot, \cdot\}$ denotes the anticommutator. Let us suppose now that Φ has a Kraus decomposition

$$\Phi(x) = \sum_{n=1}^{\infty} A_n^* x A_n, \tag{4.11}$$

where $\{A_n\}_{n\in\mathbb{N}}$ is a sequence of bounded linear operators on $\mathscr{H}$ and the series converges in the weak* topology. This is always the case if $\mathfrak{M}$ is injective or equivalently, by the Connes theorem [33] and separability of $\mathscr{H}$, that $\mathfrak{M}$ is hyperfinite (this includes the case $\mathfrak{M} = \mathrm{L}(\mathscr{H})$). The preadjoint operator of Z on $\mathfrak{M}_*$ then has the familiar Lindblad form [34]

$$Z_*\rho = -\mathrm{i}[H,\rho] - \frac{1}{2}\sum_{n=1}^{\infty}(\rho A_n^* A_n + A_n^* A_n \rho) + \sum_{n=1}^{\infty} A_n \rho A_n^*, \tag{4.12}$$

where $\rho \in \mathfrak{M}_*$. We are now able to prove the following theorem.

Theorem 4.7. *Let $\{T_t\}_{t\geq 0}$ be as above. Assume that there is a faithful normal state ω such that $\omega \circ T_t = \omega$ for all $t \geq 0$ and that H in* (4.10) *has pure point spectrum. Then $\{T_t\}_{t\geq 0}$ displays decoherence, and for the effective subalgebra $\mathfrak{M}_1$ we have*

$$\mathfrak{M}_1 \subseteq \{A_n, A_n^* : n \in \mathbb{N}\}' \cap \mathfrak{M}, \tag{4.13}$$

where the prime denotes the commutant. Moreover, there exists a normal conditional expectation Q from $\mathfrak{M}$ onto $\mathfrak{M}_1$ such that $\omega \circ Q = \omega$. If the derivation $x \mapsto \mathrm{i}[H,x]$ leaves the subalgebra $\{A_n, A_n^* : n \in \mathbb{N}\}' \cap \mathfrak{M}$ invariant, equality holds in (4.13).

Proof. First note that the assumptions of Proposition 3.3 are satisfied, that is, $\mathfrak{M}_r$ is a von Neumann subalgebra. Consider the subalgebra $\mathfrak{M}_1$ defined in Lemma 3.4, then $\{T_t\}_{t\geq 0}$ restricted to $\mathfrak{M}_1$ extends to a group of automorphisms. We start by proving (4.13). By a simple calculation as in [20], one obtains

$$\begin{aligned} Z(x^*x) - Z(x^*)x - x^*Z(x) &= x^*\Phi(\mathbb{1})x + \Phi(x^*x) - \Phi(x^*)x - x^*\Phi(x) \\ &= \sum_{n=1}^{\infty}[A_n,x]^*[A_n,x]. \end{aligned} \tag{4.14}$$

The generator Z, when restricted to $\mathfrak{M}_1$, is a *-derivation; thus if $x \in \mathfrak{M}_1$, then

$$0 = Z(x^*x) - x^*Z(x) - Z(x^*)x = \sum_{n=1}^{\infty}[A_n,x]^*[A_n,x], \tag{4.15}$$

that is, $[A_n,x] = 0$ for all $n \in \mathbb{N}$, and, moreover, $[A_n^*,x] = 0$ for all $n \in \mathbb{N}$ since $\mathfrak{M}_1$ is a *-subalgebra. This proves that $\mathfrak{M}_1 \subseteq \{A_n, A_n^* : n \in \mathbb{N}\}' \cap \mathfrak{M}$. Conversely, under the assumption that $\mathrm{i}[H,\cdot]$ leaves the right-hand side of (4.13) invariant, we have $Zx = \mathrm{i}[H,x]$ or $T_t(x) = \mathrm{e}^{\mathrm{i}tH}x\mathrm{e}^{-\mathrm{i}tH}$ on $\{A_n, A_n^* : n \in \mathbb{N}\}' \cap \mathfrak{M}$, which implies equality in (4.13).

Now let $x \in \mathfrak{M}_s \cap \mathfrak{M}_1$, $x \neq 0$. Then there exist eigenvectors $\xi, \eta \in \mathscr{H}$ of H with corresponding eigenvalues E_ξ and E_η such that $\langle \xi, x\eta \rangle \neq 0$, thus

$$\langle \xi, T_t(x)\eta \rangle = \left\langle \mathrm{e}^{-\mathrm{i}tH}\xi, x\mathrm{e}^{-\mathrm{i}tH}\eta \right\rangle = \mathrm{e}^{\mathrm{i}t(E_\xi - E_\eta)}\langle \xi, x\eta \rangle \tag{4.16}$$

is bounded away from 0, so 0 is not a weak* limit point of $\{T_t(x) : t \geq 0\}$. Since $x \in \mathfrak{M}_s$, this is a contradiction in view of (2.5), hence $\mathfrak{M}_s \cap \mathfrak{M}_1 = \{0\}$. Now let $x \in \mathfrak{M}_1$ and write $x = x_s + x_r \in \mathfrak{M}_s \oplus \mathfrak{M}_r$. Since $\mathfrak{M}_r \subseteq \mathfrak{M}_1$, we have $x_r \in \mathfrak{M}_1$, and $x_s = x - x_r \in \mathfrak{M}_s \cap \mathfrak{M}_1 = \{0\}$, thus $x \in \mathfrak{M}_r$.

This proves $\mathfrak{M}_1 = \mathfrak{M}_{\mathrm{r}}$ and it follows by Proposition 3.6 that $\mathfrak{M}_{\mathrm{s}}$ has the property (3.3). So we conclude that $\{T_t\}_{t\geq 0}$ displays decoherence. The last assertion is clear from Proposition 3.3. $\square$

We remark that in [29, equation (34)], a class of generators has been given for which equality in (4.13) always holds. As a corollary, we obtain the following result which is similar to the one proved in [35] and is also contained in [36].

Corollary 4.8. *Let $\{T_t\}_{t\geq 0}$ be a uniformly continuous semigroup on $\mathfrak{M}$ consisting of normal completely positive and contractive operators, and suppose it has a faithful normal invariant state ω. If $\{A_n, A_n^* : n \in \mathbb{N}\}' = \mathbb{C}\mathbb{1}$, then*

$$\lim_{t\to\infty} T_t(x) = \omega(x)\mathbb{1} \quad \textit{in the weak}^* \textit{ topology} \tag{4.17}$$

for any $x \in \mathfrak{M}$.

Thus if $\mathfrak{M}_1$ is trivial the semigroup describes the approach to equilibrium.

We remark that the last theorem can be generalized to certain cases when the semigroup $\{T_t\}_{t\geq 0}$ is not uniformly continuous but has an unbounded generator of the form (4.10).

The existence of a faithful normal invariant state of a quantum dynamical semigroup as required by Theorems 4.7 and 4.4 has been discussed in the literature. It is particularly simple in the case of a finite-dimensional von Neumann algebra, that is, a matrix algebra. Suppose $\mathfrak{M} = M_{\mathbb{C}}(d)$ is the $d \times d$-matrix algebra and consider a quantum dynamical semigroup $\{T_t\}_{t\geq 0}$ on $\mathfrak{M}$, then its generator is given by (4.10) and its preadjoint by (4.12), thus if the A_n are normal, $\rho_0 = (1/d)\mathbb{1}$ is a faithful normal invariant state for $\{T_t\}_{t\geq 0}$. Such generators arise, for example, in the singular coupling limit of N-level systems, see [8].

Acknowledgments

Thanks are due to Professor Ph. Blanchard (Bielefeld) for various discussions and to the anonymous referees for a number of remarks which helped to improve the paper and for pointing out an error in it.

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