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Research Article

Conservation of Total Vorticity for a 2D Stochastic Navier Stokes Equation

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We consider N point vortices whose positions satisfy a stochastic ordinary differential equation on $\mathbb{R}^{2N}$ perturbed by spatially correlated Brownian noise. The associated signed point measure-valued empirical process turns out to be a weak solution to a stochastic Navier-Stokes equation (SNSE) with a state-dependent stochastic term. As the number of vortices tends to infinity, we obtain a smooth solution to the SNSE, and we prove the conservation of total vorticity in this continuum limit.

1. Introduction

Our aim is to show that for a two-dimensional incompressible fluid, the total vorticity of the fluid is a conserved quantity (where the vorticity for a rigid body is twice the angular velocity). Following Kotelenez [1] and Marchioro and Pulvirenti [2] (cf. also Amirdjanova, [3, 4], Amirdjanova and Xiong [5]), the distribution of the vorticity satisfies the following:

$$\begin{aligned} \frac{\partial}{\partial t} X(r,t) &= \nu \Delta X(r,t) - \nabla \bullet (U(r,t)X(r,t)), \\ X(r,t) &= \operatorname{curl} U(r,t) = \frac{\partial U_2}{\partial r_1} - \frac{\partial U_1}{\partial r_2}, \quad \nabla \cdot U \equiv 0, \end{aligned} \tag{1.1}$$

where $U(r,t)$ is the velocity field, $r \in \mathbb{R}^2$, $\nu \geq 0$ is the kinematic viscosity, Δ is the Laplacian, ∇ is the gradient, and $\bullet$ denotes the scalar product on $\mathbb{R}^2$. If $\nu > 0$, we obtain the Navier-Stokes equation for the vorticity. If the fluid is inviscid (or ideal), that is, $\nu = 0$, we obtain the Euler equation. Note that by the incompressibility condition $\nabla \bullet U \equiv 0$, we obtain

$$U(r,t) = \int \left(\nabla^{\perp} g\right)(r-q) X(q,t) dq, \tag{1.2}$$

where $g(r) := (1/2\pi)\ln(|r|)$ with $|r|^2 = r_1^2 + r_2^2$ and $\nabla^{\perp} = (-(\partial/\partial r_2), (\partial/\partial r_1))^T$ with T denoting the transpose; $\int () dq$ denotes integration over $\mathbb{R}^2$ with respect to the Lebesgue measure. Consequently, we can obtain the velocity field, U, from the vorticity distribution.

Let $0 < \delta \leq 1$. Let $g_\delta(s)$ be at least twice continuously differentiable with bounded derivatives up to order 2 with $|g'_\delta(s)| \leq |g'(s)|$ and $|g''_\delta(s)| \leq |g''(s)|$, for $s > 0$ such that $g_\delta(r) \equiv g(r)$, for $\delta \leq |r| \leq 1/\delta$. Set

$$K_\delta(r) := \nabla^{\perp} g_\delta(|r|). \tag{1.3}$$

We may assume without loss of generality that $g'_\delta(0) = 0$, which implies $K_\delta(0) = 0$. Thus, we have the smoothed Navier-Stokes equation (NSE)

$$\begin{gathered} \frac{\partial}{\partial t} X(r,t) = \nu \Delta X(r,t) - \nabla \bullet (U_\delta(r,t) X(r,t)), \\ U_\delta(r,t) := \int K_\delta(r-q) X(q,t) dq. \end{gathered} \tag{1.4}$$

Consider N point vortices with intensities $a_i \in \mathbb{R}$, and let r^i be the position of the ith vortex in $\mathbb{R}^2$. Abbreviate $r_N := (r^1, \ldots, r^N) \in \mathbb{R}^{2N}$. Assume that the positions satisfy the stochastic ordinary differential equation (SODE)

$$dr^i(t) = \sum_{j=1}^{N} a_j K_\delta\left(r^i - r^j\right) dt + \sqrt{2\nu}\, dm^i(r_N, t), \quad i = 1, \ldots, N. \tag{1.5}$$

The $m^i(r_N, t)$ are $\mathbb{R}^2$-valued square-integrable continuous martingales ($i = 1, \ldots, N$), which may depend on the positions of the vortices. Let us for the moment assume that for suitably adapted square-integrable initial conditions, (1.5) has a unique (Itô) solution $r_N(t) = (r^1(t), \ldots, r^N(t))$. Set

$$\mathcal{X}_N(t) := \sum_{i=1}^{N} a_i \delta_{r^i(t)}, \tag{1.6}$$

where $r^i(t)$ are the solutions of (1.5) and δ_r is the point measure concentrated in r. We will call $\mathcal{X}_N(t)$ the empirical process associated with the SODE (1.5). Let $L_p(\mathbb{R}^2, dr)$ be the standard L_p-spaces of real-valued functions on $\mathbb{R}^2$ with $p \in [1, \infty]$, where dr is the Lebesgue measure. Set $\mathbf{H}_0 := L_2(\mathbb{R}^2, dr)$ and denote by $\langle \cdot, \cdot \rangle_0$ and $\|\cdot\|_0$ the standard scalar product and its associated norm on $\mathbf{H}_0$. Further, let $\langle \cdot, \cdot \rangle$ be the extension of $\langle \cdot, \cdot \rangle_0$ to a duality between distributions and smooth functions. For $m \in \mathbb{N}$, we define $C_0^m(\mathbb{R}^2, \mathbb{R})$ to be the functions from $\mathbb{R}^2$

into $\mathbb{R}$ which are m times continuously differentiable in all variables such that all derivatives vanish at infinity.

(i) If $\nu = 0$, $X(t)$ is a solution to the Euler equation (1.1) and the initial condition satisfies $X(0) \in L_1(\mathbb{R}^2, dr) \cap L_\infty(\mathbb{R}^2, dr)$, then there is a sequence $K_{\delta(N)}(r) \longrightarrow K(r) := \nabla^\perp g(r)$ as $N \longrightarrow \infty$ such that for $\phi \in C_0^2(\mathbb{R}^2, \mathbb{R})$ (cf. Marchioro and Pulvirenti [2]),

$$\langle \mathcal{X}_N(t), \phi \rangle \longrightarrow \langle X(t), \phi \rangle \quad \text{as } N \longrightarrow \infty. \tag{1.7}$$

(ii) Suppose $\nu > 0$ and $X(t)$ is a solution to the Navier-Stokes equation (1.1). Choose $m^i(r_N, t) := \beta^i(t)$, where $\beta^i(t)$ are i.i.d. $\mathbb{R}^2$-valued standard Wiener processes and half of the intensities a_i equal to $(a^+)(2/N)$ for $a^+ > 0$ and the other half to $(-a^-)(2/N)$ with $a^- > 0$. Let $\varphi \in C_0^2(\mathbb{R}^2, \mathbb{R})$. Apply Itô's formula and compute the quadratic variation

$$\begin{aligned} d[\langle \varphi, \mathcal{X}_N(t) \rangle] &= 2\nu \sum_{\pm} \left[\sum_{j=1}^{N/2} \frac{2a^\pm}{N} \left\{ (\nabla\varphi)\left(r^j(t)\right) \bullet \beta^j(dt) \right\} \right] \\ &= 2\nu \sum_{\pm} \sum_{j=1}^{N/2} \frac{(2a^\pm)^2}{N^2} \left\{ \sum_{k=1}^{2} (\partial_k \varphi)^2 \left(r^j(t)\right) \right\} dt, \end{aligned} \tag{1.8}$$

where we used the independence of $\beta^i(\cdot)$. Hence, for $t \le T$,

$$[\langle \varphi, \mathcal{X}_N(t) \rangle] = O\left(\frac{1}{N}, \varphi, T\right). \tag{1.9}$$

In other words, the empirical vorticity distribution $\mathcal{X}_N(t)$ becomes macroscopic, as $N \longrightarrow \infty$. Choosing a sequence $K_{\delta(N)}(r) \longrightarrow K(r)$ and assuming a suitable convergence of the initial conditions $\mathcal{X}_{N,0}$ towards the initial condition in (1.1), we may expect that

$$\langle \varphi, \mathcal{X}_N(t) \rangle \longrightarrow \langle \varphi, X(t) \rangle, \tag{1.10}$$

where $X(t)$ is the solution to (1.1). As $N \longrightarrow \infty$ describes a *continuum limit* (in this limit, the discrete point particle distribution may become a smooth particle distribution with densities, etc.), (1.10) implies that the *macroscopic limit* and the *continuum limit* coincide. Marchioro and Pulvirenti [2] (cf. also Chorin [6] and the references therein) prove a somewhat weaker result: for the case $m^i(r_N, t) := \beta^i(t)$, assuming in addition to the previous conditions, $\langle E\mathcal{X}_N(0), \varphi \rangle \longrightarrow \langle X(0), \varphi \rangle$ as $N \longrightarrow \infty$, they prove that for all $t > 0$,

$$\langle E\mathcal{X}_N(t), \varphi \rangle \longrightarrow \langle X(t), \varphi \rangle \quad \text{as } N \longrightarrow \infty, \tag{1.11}$$

where E denotes the mathematical expectation with respect to the underlying probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$. (All our stochastic processes are assumed to live on Ω and to be $\mathcal{F}_t$-adapted (including all initial conditions in SODE's and SPDE's), where the filtration $\mathcal{F}_t$ is assumed to be right continuous. Moreover, the processes are assumed to be $(dP \otimes dt)$-measurable, where dt is the Lebesgue measure on $[0, \infty)$.)

In order to separate the macroscopic and continuum limits and to derive a mesoscopic vorticity distribution, Kotelenez [1] introduces spatial correlations of the Brownian noise as follows through correlation functionals convolved with space-time Gaussian white noise as follows: for $i = 1, 2$, $w_i(dp, ds)$ are i.i.d. space-time Gaussian white noise fields, (this is the multiparameter generalization of the increments of a real valued Brownian motion $\beta(ds)$. cf. Walsh [7] and Kotelenez [8]) $\varepsilon > 0$ is the spatial correlation length and define a 2×2-matrix-valued correlation kernel by

$$\widehat{\Gamma}_\epsilon = \begin{pmatrix} \widetilde{\Gamma}_{\epsilon,11} & \widetilde{\Gamma}_{\epsilon,12} \\ \widetilde{\Gamma}_{\epsilon,12} & \widetilde{\Gamma}_{\epsilon,22} \end{pmatrix}. \tag{1.12}$$

$\widetilde{\Gamma}_{\varepsilon,ij} : \mathbb{R}^2 \to \mathbb{R}_+$ are symmetric, bounded, Borel-measurable functions such that the following integrability conditions are satisfied, where $i \in \{1, 2\}$ (the integration domain for $\int$ in what follows is always $\mathbb{R}^2$, unless it is specified to be different).

$$\int \widetilde{\Gamma}^2_{\epsilon,ii}(r - p)dp = 1, \tag{1.13}$$

and there is a finite positive constant c such that for any $r, q \in \mathbb{R}^2$ and $i, j \in \{1, 2\}$

$$\int \left(\widetilde{\Gamma}_{\epsilon,ij}(r - p) - \widetilde{\Gamma}_{\epsilon,ij}(q - p)\right)^2 dp \leq c\varrho(r - q), \tag{1.14}$$

where $\varrho(r - q) := |r - q| \wedge 1$ and $\wedge$ denotes the minimum of two numbers. In this case, the following choice for the square-integrable martingales is made:

$$m^i(r_N, t) := \int_0^t \int \widehat{\Gamma}_\epsilon\left(r^i(s) - p\right) w(dp, ds). \tag{1.15}$$

The two-particle and one-particle diffusion matrices are given by (cf. Kotelenez and Kurtz [9])

$$\begin{gathered} \widetilde{D}_{\epsilon,k\ell}\left(r^i - r^j\right) := \int \sum_{m=1}^{2} \widetilde{\Gamma}_{\epsilon,km}\left(r^i - q\right) \widetilde{\Gamma}_{\epsilon,m\ell}\left(r^j - q\right) dq \quad \forall i, j = 1, \ldots, N, \\ D_{\epsilon,k\ell} := \widetilde{D}_{\epsilon,k\ell}(0), \end{gathered} \tag{1.16}$$

where the spatial shift invariance of the two-particle diffusion matrix and the state independence of the one-particle diffusion matrix follows from the shift invariance of the kernels. Let

$r^1(t)$, $r^2(t)$ be continuous $\mathbb{R}^2$-valued processes, which describe the positions of point particles or point vortices. Levy's theorem then implies that the marginal processes

$$\int_0^t \int \widehat{\Gamma}_\epsilon\Big(r^1(s) - p\Big) w(dp, ds), \qquad \int_0^t \int \widehat{\Gamma}_\epsilon\Big(r^2(s) - p\Big) w(dp, ds), \tag{1.17}$$

are $\mathbb{R}^2$-valued Brownian motions, whereas by (1.16), the joint $\mathbb{R}^4$-valued motion is not Brownian. If we now assume in addition that for all $\delta > 0$

$$\lim_{\epsilon \downarrow 0} \sup_{|r^i - r^j| > \delta} \left| \widetilde{D}_{\epsilon,k\ell}(r^i - r^j) \right| = 0 \quad k, \ell \in \{1, 2\}, \tag{1.18}$$

then the joint $\mathbb{R}^4$-valued motion is approximately Brownian, if the separation between the point vortices is sufficiently large (cf. Kotelenez and Kurtz [9, (2.6)]. Further, in colloids it is an empirical fact that at close range Brownian particles are attracted to one another as a result of the depletion phenomenon. cf. Asakura and Oosawa [10] as well as Kotelenez et al. [11], and the references therein).

Example 1.1. Choose $\widetilde{\Gamma}_{\epsilon,kk}(r) := -c_\epsilon(\partial/\partial r_k)(1/2\pi\epsilon)e^{-|r|^2/4\epsilon}$, where $c_\epsilon > 0$, $k = 1, 2$, and for the off-diagonal elements $(k \neq \ell)$ set $\widetilde{\Gamma}_{\epsilon,kl}(r) := 0$.

Another class of examples can be obtained by taking, for example, the square root of a standard normal distribution with variance ϵ as $\Gamma_{\epsilon,ii}$, $i = 1, 2$, and 0 for the off-diagonal elements of $\widetilde{\Gamma}_\epsilon$. The Chapman-Kolmogorov equation yields

$$\left| \widetilde{D}_{\epsilon,kl}\Big(r^i - r^j\Big) \right| \approx O(\sqrt{\epsilon}). \tag{1.19}$$

The method of perturbing the motion of the point vortices has the benefit that the empirical process $\mathcal{X}_N(t)$ associated with (1.5) satisfies a smoothed stochastic Navier-Stokes equation (SNSE) by Ito's formula (cf. (3.8) in Section 3). From this, Kotelenez [1] derives a priori estimates used to generalize the solution to arbitrary adapted initial conditions. However, the metric used in Kotelenez [1] does not define a metric on the signed measures and, therefore, cannot be used to prove the conservation of total vorticity in the continuum limit. Consequently, we proceed as follows.

(i) (Section 2) We introduce metrics on the space of signed measures. In view of the method of correlation functions, we desire a metric that is complete. Instead, we derive a metric that satisfies a useful "partial-completeness" result which aides in the conservation of vorticity argument.

(ii) (Section 3) We analyze the SNSE equation and derive the existence of solutions. Furthermore, using the results of Section 2, we show that the total vorticity is conserved.

(iii) (Section 4) Based on the recent work of Kotelenez and Kurtz [9], we conjecture the macroscopic limit for the stochastic Navier-Stokes equation.

2. Metrics on the Space of Signed Measures

Let $\mathbb{R}^2$ be equipped with the complete, separable metric $\varrho(r-q) = |r-q| \wedge 1$. We define $\mathbf{M}_f$ to be the set of finite, Borel measures on $\mathbb{R}^2$ and $\mathbf{M}_{f,s}$ the set of finite, Borel signed measures on $\mathbb{R}^2$. For a finite, signed Borel measure, μ, we let μ^+, μ^- denote the Hahn-Jordan decomposition of μ. We also let $C_{L,\infty}(\mathbb{R}^2, \mathbb{R})$ be the set of Lipschitz functions from $\mathbb{R}^2$ to $\mathbb{R}$ which are also bounded. We endow $C_{L,\infty}(\mathbb{R}^2, \mathbb{R})$ with the norm $\|\cdot\|_{L,\infty}$, where $\|f\|_{L,\infty} := \|f\|_\infty \vee \|f\|_L$ for

$$\|f\|_\infty := \sup_q \left|f(q)\right|, \qquad \|f\|_L := \sup_{r,q \in \mathbb{R}^2, r \neq q} \left\{ \frac{\left|f(r) - f(q)\right|}{\varrho(r-q)} \right\}, \tag{2.1}$$

and $\vee$ denotes maximum.

Further, we define for $\mu, \widetilde{\mu} \in \mathbf{M}_{f,s}$

$$\gamma_f(\mu, \widetilde{\mu}) := \sup_{\|f\|_{L,\infty} \leq 1} \left|\langle \mu - \widetilde{\mu}, f\rangle\right|. \tag{2.2}$$

By Kotelenez [8] and Dudley [12], it follows that $(\mathbf{M}_{f,s}, \gamma_f)$ is a metric space on which γ_f is actually a norm. Furthermore, restricting γ_f to $\mathbf{M}_f$ implies that $(\mathbf{M}_f, \gamma_f)$ is a complete and separable metric space (where the linear combination of point measures forms a dense set) (cf., e.g., de Acosta [13]). However, by Kotelenez and Seadler [14], $(\mathbf{M}_{f,s}, \gamma_f)$ is not a complete space. Hence, we introduce the product metric $\widehat{\gamma}_f$ on $\widehat{\mathbf{M}}_{f,s} := \mathbf{M}_{f,s} \times \mathbf{M}_{f,s}$. For $\widehat{\mu} = (\mu_1, \mu_2)$, $\widehat{\nu} = (\nu_1, \nu_2) \in \widehat{\mathbf{M}}_{f,s}$, we define

$$\widehat{\gamma}_f(\widehat{\mu}, \widehat{\nu}) := \gamma_f(\mu_1, \nu_1) + \gamma_f(\mu_2, \nu_2). \tag{2.3}$$

It follows that $(\widehat{\mathbf{M}}_{f,s}, \widehat{\gamma}_f)$ is a metric space, restricted to $\widehat{\mathbf{M}}_f := \mathbf{M}_f \times \mathbf{M}_f$, the metric is complete and separable. We can identify a signed measure μ with the measure pair formed by the Hahn-Jordan decomposition, $\widehat{\mu}^\pm = (\mu^+, \mu^-)$. However, under this identification, $(\mathbf{M}_{f,s}, \widehat{\gamma}_f)$ is still not a complete space. We finally introduce the following quotient-type metric on $\mathbf{M}_{f,s}$ (cf. Kotelenez and Seadler [14]):

$$\gamma_{f,s}(\mu, \nu) := \inf_{\widehat{\eta} \in \widehat{D}_+} \left(\widehat{\gamma}_f(\widehat{\mu}^\pm - \widehat{\nu}^\pm + \widehat{\eta}) \vee \widehat{\gamma}_f(\widehat{\mu}^\pm - \widehat{\nu}^\pm - \widehat{\eta})\right), \tag{2.4}$$

where $\mu, \nu \in \mathbf{M}_{f,s}$ and

$$\widehat{D}_+ := \left\{ \widehat{\nu} = (\nu_1, \nu_2) \in \widehat{\mathbf{M}}_f : \nu_1 = \nu_2 \right\}. \tag{2.5}$$

$(\mathbf{M}_{f,s}, \gamma_{f,s})$ is not a complete space, but it satisfies the following useful partial completeness result (cf. Kotelenez and Seadler [14]).

Theorem 2.1. *(i) $\gamma_{f,s}$ is a metric on $\mathbf{M}_{f,s}$.*

(ii) A Cauchy sequence $\{\mu_n\}_{n\geq 1}$ for $\gamma_{f,s}$ converges if and only if there exists a subsequence $\{\mu_{n_k}\}_{k\geq 1}$ such that $\widehat{\mu}^\pm_{n_k} \to \widehat{\mu}^\pm$ in $\widehat{\gamma}_f$ (i.e., the limit is in Hahn-Jordan form).

Proof. (Sketch) For the forward implication, if given a Cauchy sequence for $\{\mu_n\}_{n\geq 1}$ for $\gamma_{f,s}$, it is routine to show that we can extract a subsequence $\{\hat{\mu}^{\pm}_{n_k}\}_{k\geq 1}$ in $\widehat{\mathbf{M}}_f$ that is Cauchy for $\hat{\gamma}_f$. Consequently, as $(\widehat{\mathbf{M}}_f, \hat{\gamma}_f)$ is complete, there is a limit $\hat{\mu} \in \widehat{\mathbf{M}}_f$. One can show by standard estimates that

$$\hat{\mu}^{\pm}_{n_k} \longrightarrow \hat{\mu} = \hat{\mu}^{\pm} + \hat{\phi} \quad \text{where } \hat{\phi} \in \widehat{D}_+. \tag{2.6}$$

Thus, to prove the forward implication, it suffices to establish the following lemma.

Lemma 2.2. *Let* $\psi : \widehat{\mathbf{M}}_f \to \mathbf{M}_{f,s}$ *by* $\psi((\nu_1, \nu_2)) := \nu_1 - \nu_2$, *then*

$$\gamma_{f,s}\left(\psi(\hat{\mu}^{\pm}_{n_k}), \psi\left(\hat{\mu}^{\pm} + \hat{\phi}\right)\right) \longrightarrow 0 \quad \textit{iff } \hat{\phi} = \hat{0}. \tag{2.7}$$

Proof. (i) "$\Leftarrow$" follows from $\gamma_{f,s}(\psi(\hat{\mu}^{\pm}_{n_k}), \psi(\hat{\mu}^{\pm})) \leq \hat{\gamma}_f(\hat{\mu}^{\pm}_{n_k} - \hat{\mu}^{\pm})$.

(ii) Suppose $\hat{\gamma}_f(\hat{\phi}) > 0$. Then,

$$\gamma_{f,s}\left(\psi(\hat{\mu}^{\pm}_{n_k}), \psi\left(\hat{\mu}^{\pm} + \hat{\phi}\right)\right) \geq \inf_{\hat{\eta}\in\widehat{D}_+} \hat{\gamma}_f(\hat{\mu}^{\pm}_{n_k} - \hat{\mu}^{\pm} - \hat{\eta}) \vee \inf_{\hat{\eta}\in\widehat{D}_+} \hat{\gamma}_f(\hat{\mu}^{\pm} - \hat{\mu}^{\pm}_{n_k} - \hat{\eta}). \tag{2.8}$$

Let $\epsilon > 0$. We compute the second term

$$\begin{aligned}\inf_{\hat{\eta}\in\widehat{D}_+} \hat{\gamma}_f(\hat{\mu}^{\pm} - \hat{\mu}^{\pm}_{n_k} - \hat{\eta}) &= \inf_{\hat{\eta}\in\widehat{D}_+} \hat{\gamma}_f\left(\hat{\mu}^{\pm} + \hat{\phi} - \hat{\mu}^{\pm}_{n_k} - \hat{\phi} - \hat{\eta}\right)\\ &\geq \inf_{\hat{\eta}\in\widehat{D}_+} \hat{\gamma}_f\left(-\hat{\phi} - \hat{\eta}\right) - \hat{\gamma}_f\left(\hat{\mu}^{\pm} + \hat{\phi} - \hat{\mu}^{\pm}_{n_k}\right) \geq \inf_{\hat{\eta}\in\widehat{D}_+} \hat{\gamma}_f\left(\hat{\phi} + \hat{\eta}\right) - \epsilon,\end{aligned} \tag{2.9}$$

for sufficiently large n, since, by assumption, $\hat{\gamma}_f(\hat{\mu}^{\pm} + \hat{\phi} - \hat{\mu}^{\pm}_{n_k}) \to 0$. Hence,

$$\inf_{\hat{\eta}\in\widehat{D}_+} \hat{\gamma}_f(\hat{\mu}^{\pm} - \hat{\mu}^{\pm}_{n_k} - \hat{\eta}) \geq \inf_{\hat{\eta}\in\widehat{D}_+} \hat{\gamma}_f\left(\hat{\phi} + \hat{\eta}\right) = \hat{\gamma}_f\left(\hat{\phi}\right) > 0. \tag{2.10}$$

Therefore, we also have

$$\gamma_{f,s}\left(\psi(\hat{\mu}_{n_k}), \psi\left(\hat{\mu} + \hat{\phi}\right)\right) \geq \hat{\gamma}_f\left(\hat{\phi}\right) > 0. \tag{2.11}$$

□

Part (i) of the proof of Lemma 2.2 also implies the reverse implication for Theorem 2.1.

□

Although the above metrics provide an understanding of the difficulty of completeness for the signed measures, we will also need the Wasserstein metrics on the set of signed measures. Further, for nonnegative numbers b^+, b^-, A, an arbitrary Borel set in $\mathbb{R}^2$, and μ^+, μ^-, we write

$$\mu^{\pm}(A) = b^{\pm} \quad \text{iff } \mu^+(A) = b^+, \qquad \mu^-(A) = b^-. \tag{2.12}$$

If $\mu, \tilde{\mu} \in \mathbf{M}_{f,s}$, we will call positive Borel measures $Q^{\pm}$ on $\mathbb{R}^4$ joint representations of $(\mu^+, \tilde{\mu}^+)$ $[(\mu^-, \tilde{\mu}^-)$, resp.] if $Q^{\pm}(A \times \mathbb{R}^2) = \mu^{\pm}(A)\tilde{\mu}^{\pm}(\mathbb{R}^2)$ and $Q^{\pm}(\mathbb{R}^2 \times B) = \tilde{\mu}^{\pm}(B)^{\pm}\mu^{\pm}(\mathbb{R}^2)$ for arbitrary Borel $A, B \subset \mathbb{R}^2$. In this paper, we will assume unless stated otherwise that $\mu, \tilde{\mu} \in \mathbf{M}_{f,s}$ with $\mu^{\pm}(\mathbb{R}^2) = \tilde{\mu}^{\pm}(\mathbb{R}^2) = a^{\pm}$, where $a^+, a^- > 0$. The set of all joint representations of $(\mu^+, \tilde{\mu}^+)$ $[(\mu^-, \tilde{\mu}^-)$, resp.] will be denoted by $C(\mu^+, \tilde{\mu}^+)$ $[C(\mu^-, \tilde{\mu}^-)$, resp.]. For $\mu, \tilde{\mu} \in \mathbf{M}_{f,s}$ and $p = 1, 2$, set

$$\gamma_{W,p}(\mu, \tilde{\mu}) := \left[\inf_{Q^+ \in C(\mu^+, \tilde{\mu}^+)} \iint Q^+(dr, dq)\varrho^p(r, q) + \inf_{Q^- \in C(\mu^-, \tilde{\mu}^-)} \int\int Q^-(dr, dq)\varrho^p(r, q) \right]^{1/p}. \tag{2.13}$$

$\gamma_{W,p}$ is a metric on $\widehat{\mathbf{M}}_f$, and its restriction to $\mathbf{M}_f$ is a metric, but it is not a metric on $\mathbf{M}_{f,s}$. As we will work with the various types of metrics, we note some basic inequalities relating $\gamma_{W,1}$ and $\gamma_{W,2}$. It follows from the Cauchy-Schwarz inequality and the assumption that ϱ is bounded by one that

$$\gamma_{W,1}(\mu, \tilde{\mu}) \geq \gamma_{W,2}^2(\mu, \tilde{\mu}) \geq \frac{1}{2(a^+ \vee a^-)} \gamma_{W,1}^2(\mu, \tilde{\mu}). \tag{2.14}$$

Furthermore, when restricted to probability measures, $\gamma_{W,1}$ and γ_f define the same metric by the Kantorovich-Rubinstein theorem and Kotelenez [8].

3. Existence and Uniqueness of the SNSE and Conservation of Vorticity

Let us return to the SODE (1.5) and note that if we define $m^i(r_N, t)$ by (1.15), then (1.5) becomes

$$dr^i(t) := \sum_{j=1}^{N} a_j K_\delta\left(r^i - r^j\right)dt + \sqrt{2\nu} \int \widehat{\Gamma}_\epsilon\left(r^i - p\right)w(dp, dt). \tag{3.1}$$

For metric spaces $\mathbb{M}_1, \mathbb{M}_2$, and $C(\mathbb{M}_1, \mathbb{M}_2)$ is the space of continuous functions from $\mathbb{M}_1$ into $\mathbb{M}_2$. If μ is a finite (signed) Borel measure on $\mathbb{R}^2$, we set

$$\mu \int \widehat{\Gamma}_\epsilon(\cdot - p)w(dp, t) := \int \widehat{\Gamma}_\epsilon(\cdot - p)w(dp, t)\mu(d\cdot), \tag{3.2}$$

that is, $\int \widehat{\Gamma}_\epsilon(\cdot - p)w(dp, t)$ is treated as a density with respect to μ.

If μ itself has a density with respect to the Lebesgue measure, we will denote this density also by μ (i.e., $\langle \phi, \mu \rangle = \langle \phi, \mu \rangle_0$), and the above expression reduces to pointwise multiplication between μ and the stochastic integral.

Lemma 3.1. *To each $\mathcal{F}_0$-adapted initial condition $q_N(0) \in \mathbb{R}^{2N}$, (3.1) has a unique $\mathcal{F}_t$-adapted solution $q_N(\cdot) \in C([0, \infty); \mathbb{R}^{2N})$ a.s.; which is an $\mathbb{R}^{2N}$-valued Markov process.*

Proof. Compare with Kotelenez [1, 8]. □

For the square-integrable martingales in (1.5), we denote by $\langle\!\langle m_k^i(r_N,t), m_l^j(r_N,t)\rangle\!\rangle$ the mutual quadratic variation process of the one-dimensional components where $k,l \in \{1,2\}$, $i,j \in \{1,\dots,N\}$ (cf. Métivier and Pellaumail [15]). For $\phi \in C_0^2(\mathbb{R}^2;\mathbb{R})$, it follows from Ito's formula that the empirical process associated with (1.5), $\mathcal{X}_N$ (defined by (1.6)) satisfies the following:

$$\begin{aligned} d\langle \mathcal{X}_N(t),\phi\rangle = {} & \langle \mathcal{X}_N(t),(U_{\delta,N}\bullet\nabla)\phi\rangle dt \\ & + \nu\sum_{i=1}^N a_i \sum_{k,l=1}^2 \frac{\partial^2}{\partial r_k\partial r_l}\phi\left(r^i(t)\right) d\langle\!\langle m_k^i(r_N,t), m_l^i(r_N,t)\rangle\!\rangle \\ & + \sqrt{2\nu}\sum_{i=1}^N a_i\left(\nabla\phi\left(r^i(t)\right)\right)\bullet dm^i(r_N,t), \end{aligned} \tag{3.3}$$

where $r=(r_1,r_2)$ and

$$U_{\delta,N}(r,t) := \int K_\delta(r-q)\mathcal{X}_N(dq,t). \tag{3.4}$$

Recalling that the marginals in (1.15) are standard $\mathbb{R}^2$-valued Brownian motions, we obtain

$$\nu\sum_{i=1}^N a_i \sum_{k,l=1}^2 \frac{\partial^2}{\partial r_k\partial r_l}\phi\left(r^i(t)\right) d\left\langle\!\!\left\langle m_k^i(r_N,t), m_l^i(r_N,t)\right\rangle\!\!\right\rangle = \nu\sum_{i=1}^N a_i\Delta\phi\left(r^i(t)\right)dt = \nu\langle(\Delta\phi)(\cdot),\mathcal{X}_N(t)\rangle, \tag{3.5}$$

and (in what follows, we use the duality between $\mathbb{R}^2$-valued functions and $\mathbb{R}^2$-valued generalized functions by first applying the scalar product $\bullet$ and then for each component computing the duality)

$$\begin{aligned} & \sqrt{2\nu}\sum_{i=1}^N a_i\left(\nabla\phi\left(r^i(t)\right)\right)\bullet\int\widehat{\Gamma}_\epsilon\left(r^i(t)-p\right)w(dp,dt) \\ & \quad = \left\langle(\nabla\phi(\cdot), \int\sqrt{2\nu}\widehat{\Gamma}_\epsilon\left(r^i(t)-p\right)w(dp,dt)\mathcal{X}_N(t)\right\rangle. \end{aligned} \tag{3.6}$$

Therefore, (3.3) yields

$$\begin{aligned} d\langle \mathcal{X}_N(t),\phi\rangle = {} & \langle \mathcal{X}_N(t),(U_{\delta,N}\bullet\nabla)\phi\rangle dt \\ & + \nu\langle(\Delta\phi)(\cdot),\mathcal{X}_N(t)\rangle dt + \left\langle(\nabla\phi(\cdot), \int\sqrt{2\nu}\widehat{\Gamma}_\epsilon\left(r^i(t)-p\right)w(dp,dt)\mathcal{X}_N(t)\right\rangle. \end{aligned} \tag{3.7}$$

Integrating in (3.7) by parts in the sense of generalized funtions, we obtain that the empirical process is the weak solution of the following stochastic Navier-Stokes equation (SNSE) on $\mathbf{M}_{f,s}$:

$$d\mathcal{X}(t) = \left[\nu\Delta\mathcal{X} - \nabla \bullet \left(\widetilde{U}_\delta\mathcal{X}\right)\right]dt - \sqrt{2\nu}\nabla \bullet \left(\mathcal{X}\widehat{\Gamma}_\epsilon(\cdot - p)\right)w(dp, dt),$$
$$\widetilde{U}_\delta(r,t) := \int K_\delta(r-q)\mathcal{X}(dq,t). \tag{3.8}$$

Lemma 3.2 (conservation of vorticity for discrete initial conditions). *Consider*

$$\mathcal{X}_N^{\pm}\left(\mathbb{R}^2, t\right) = \mathcal{X}_N^{\pm}\left(\mathbb{R}^2, 0\right) = a^{\pm} \quad a.s. \tag{3.9}$$

Proof. By Kotelenez and Seadler [14], it suffices to verify that the coefficients of (3.1) satisfy Lipschitz conditions. For the stochastic component, we note that if $r, q \in \mathbb{R}^2$, we obtain that if $\{\widetilde{\phi}_n\}_{n\in\mathbb{N}}$ is a complete orthonormal system in $L_2(\mathbb{R}^2, dr)$ and

$$\phi_n := \begin{pmatrix} \widetilde{\phi}_n & 0 \\ 0 & \widetilde{\phi}_n \end{pmatrix}, \tag{3.10}$$

then it follows that

$$\int \widehat{\Gamma}_\epsilon(r-p)w(dp,t) = \sum_{n=1}^{\infty}\int \widehat{\Gamma}_\epsilon(r-p)\phi_n(p)dp\beta^n(t), \tag{3.11}$$

where $\beta^n(t) := \int \phi_n(p)w(dp,t)$ are i.i.d. $\mathbb{R}^2$-valued standard Wiener processes (cf. Kotelenez [8]). It follows from the definition of the correlation function that for $i, j \in \{1, 2\}$

$$\sum_{n=1}^{\infty}\left[\int\left(\widetilde{\Gamma}_{\epsilon,ij}(r-p) - \widetilde{\Gamma}_{\epsilon,ij}(q-p)\right)\widetilde{\phi}_n(p)dp\right]^2 = \int\left(\widetilde{\Gamma}_{\epsilon,ij}(r-p) - \widetilde{\Gamma}_{\epsilon,ij}(q-p)\right)^2 dp$$
$$\leq c\varrho^2(r-q). \tag{3.12}$$

Now, we note that the drift coefficient can be represented by $F(\mathcal{X}_N(t), r) : \mathbf{M}_{f,s} \times \mathbb{R}^2 \to \mathbb{R}^2$, where $F(\mu, r) := K_\delta * \mu(r)$, $*$ denotes convolution, and $\mu \in \mathbf{M}_{f,s}$. For $\mu_1, \mu_2 \in \mathbf{M}_{f,s}, r_1, r_2 \in \mathbb{R}^2$, we have the following (cf. Kotelenez [8, page 81]):

$$\left|F(\mu_1, r_1) - F(\mu_2, r_2)\right| \leq c_{K,\delta}\left(a\varrho(r_1, r_2) + \widehat{\gamma}_f\left(\widehat{\mu}_1^{\pm}, \widehat{\mu}_2^{\pm}\right)\right), \tag{3.13}$$

where $\widehat{\mu}^{\pm} = (\mu^+, \mu^-)$ and μ^+, μ^- is the Hahn-Jordan decomposition of μ and $a := a^+ + a^-$ is the total vorticity. □

We wish to extend the result of Lemma 3.2 to arbitrary adapted initial conditions and not just discrete adapted initial conditions. To accomplish this, we must derive a priori estimates on the empirical distribution. We first introduce the following notation.

If $(\mathbf{Y}, \lambda)$ is some metric space and $p \geq 1$, $L_p(\Omega; \mathbf{Y})$ is the metric space of $\mathbf{Y}$-valued p-integrable random variables with metric $(E\lambda^p(\xi, \eta))^{1/p})$ for $\xi, \eta \in L_p(\Omega; Y)$. Set

$$
\begin{gathered}
\mathbb{M}_{f,s,d} := \left\{\mu \in \mathbf{M}_{f,s} : \mu \text{ is a finite linear combination of point measures on } \mathbb{R}^2\right\}, \\
\widetilde{\mathcal{M}}_0 := L_2(\Omega; \mathbf{M}_{f,s,d}), \\
\mathcal{M}_0 := L_2(\Omega; \mathbf{M}_{f,s}), \\
\mathcal{M}_{[0,T]} := L_2(\Omega; C([0,T]; \mathbf{M}_{f,s})).
\end{gathered} \tag{3.14}
$$

Let $r_N(t) := r_N(t, Z_1(t), \mathcal{X}_{N,0})$ and $q_N(t) := q_N(t, Z_2(t), \mathcal{Y}_{N,0})$ be the solutions of the SODE (3.1) with $\mathcal{F}_0$-measurable initial empirical distributions $\mathcal{X}_{N,0}, \mathcal{Y}_{N,0} \in \mathbf{M}_{f,s}$. Denote the empirical processes associated with $r_N(t, Z_1(t), \mathcal{X}_{N,0})$ and $q_N(t, Z_2(t), \mathcal{Y}_{N,0})$ by $\mathcal{X}_N(t, Z_1, \mathcal{X}_{N,0})$ and $\mathcal{Y}_N(t, Z_2, \mathcal{Y}_{N,0})$, respectively, where $Z_1(\cdot), Z_2(\cdot)$ are also adapted $M_{f,s}$-valued processes, replacing in (3.1) the empirical processes. Set

$$
\varrho_N(r_N, q_N) := \max_{i=1,\ldots,N} \varrho\left(r_N^i, q_N^i\right). \tag{3.15}
$$

Theorem 3.3. *For any $T > 0$,*

$$
E \sup_{0 \leq t \leq T \wedge \tau} \varrho_N^2(r_N(t), q_N(t)) \leq c_{T,\delta,\epsilon}\left(E\varrho_N^2(r_N(0), q_N(0)) + E\int_0^t \hat{\gamma}_f^2(Z_1(s), Z_2(s))ds\right). \tag{3.16}
$$

Proof. This follows from Theorem 2.1 in Kotelenez and Seadler [14]. □

We can now derive the necessary a priori estimates to extend from discrete initial conditions to arbitrary adapted initial conditions.

Lemma 3.4. *For any $T > 0$, there is a $c := c_{\delta,\epsilon,T} > 0$ such that for all $N \in \mathbb{N}$*

$$
E \sup_{0 \leq t \leq T} \hat{\gamma}_f^2\left(\widehat{\mathcal{X}}_N(t), \widehat{\mathcal{Y}}_N(t)\right) \leq cE\hat{\gamma}_f^2\left(\widehat{\mathcal{X}}_N(0), \widehat{\mathcal{Y}}_N(0)\right). \tag{3.17}
$$

Proof. (i) Define $\mathcal{X}_N^+(t, Z_k) := \sum_{a_i \geq 0} a_i \delta_{r^i(t, Z_k, r_0^i)}$ and $\mathcal{X}_N^-(t, Z_k) = \sum_{a_i < 0} a_i \delta_{x^i(t, Z_k, r_0^i)}$. Similarly, we decompose $\mathcal{Y}_N(t, Z_k)$, where $k = 1, 2$.

$$
\begin{aligned}
&E \sup_{0 \leq t \leq T} \hat{\gamma}_f^2\left(\widehat{\mathcal{X}}_N(t, Z_1), \widehat{\mathcal{Y}}_N(t, Z_2)\right) \\
&\quad \leq 4\left(E \sup_{0 \leq t \leq T} \gamma_f^2(\mathcal{X}_N^+(t, Z_1), \mathcal{Y}_N^+(t, Z_2)) + E \sup_{0 \leq t \leq T} \gamma_f^2(\mathcal{X}_N^-(t, Z_1), \mathcal{Y}_N^-(t, Z_2))\right).
\end{aligned} \tag{3.18}
$$

We show the estimate for $E\sup_{0\le t\le T}\gamma_f^2(\mathcal{X}_N^+(t,Z_1),\mathcal{Y}_N^+(t,Z_2))$ as a similar estimate will hold for the second term. Note that

$$
\begin{aligned}
&E\sup_{0\le t\le T}\gamma_f^2(\mathcal{X}_N^+(t,Z_1),\mathcal{Y}_N^+(t,Z_2))\\
&\quad\le 4\left(E\sup_{0\le t\le T}\gamma_f^2(\mathcal{X}_N^+(t,Z_1),\mathcal{X}_N^+(t,Z_2))+E\sup_{0\le t\le T}\gamma_f^2(\mathcal{X}_N^+(t,Z_2),\mathcal{Y}_N^+(t,Z_2))\right).
\end{aligned}
\tag{3.19}
$$

Recall that $\mathcal{X}_{N,0}$ and $\mathcal{Y}_{N,0}$ are the initial measures of $\mathcal{X}(t)$ and $\mathcal{Y}(t)$, respectively. Let $Q_0\in C(\mathcal{X}_N^+(0),\mathcal{Y}_N^+(0))$, then, by Cauchy-Schwarz and Theorem 3.3, the right hand side of the last inequality equals

$$
\begin{aligned}
&4E\sup_{0\le t\le T}\left(\sup_{\|f\|_{L,\infty}\le 1}\int\left(f\left(r^i(t,Z_1,q)\right)-f\left(r^i(t,Z_2,q)\right)\right)\mathcal{X}_{N,0}(dq)\right)^2\\
&\quad+4E\sup_{0\le t\le T}\left(\sup_{\|f\|_{L,\infty}\le 1}\iint\left(f\left(r^i(t,Z_2,q)\right)-f\left(q^i(t,Z_2,\tilde{q})\right)\right)Q_0(dq,d\tilde{q})\right)^2\\
&\le E\sup_{0\le t\le T}c\int\varrho^2\left(r^i(t,Z_1,q)-r^i(t,Z_2,q)\right)\mathcal{X}_{N,0}(dq)\\
&\quad+E\sup_{0\le t\le T}c\iint\varrho^2\left(r^i(t,Z_2,q)-q^i(t,Z_2,\tilde{q})\right)Q_0(dq,d\tilde{q}).\\
&\le\tilde{c}_{T,a}\left(E\gamma_f^2\left(\mathcal{X}_{N,0}^+,\mathcal{Y}_{N,0}^+\right)+\int_0^T E\hat{\gamma}_f^2\left(\hat{Z}_1(s),\hat{Z}_2(s)\right)ds\right),
\end{aligned}
\tag{3.20}
$$

by Theorem 3.3, (2.14), and the Kantorovich-Rubinste in theorem.

Combining the terms for the positive and negative parts, choosing $Z_1\equiv\mathcal{X}_N$ and $Z_2\equiv\mathcal{Y}_N$ and applying Gronwall's Inequality yields the claim. □

The following theorem asserts that the total vorticity of the fluid is conserved.

Theorem 3.5 (conservation of vorticity in the continuum limit). *(1) The map $\mathcal{X}_N(0)\mapsto\mathcal{X}_\epsilon(\cdot,\mathcal{X}_N(0))$ from $\widetilde{\mathcal{M}}_0$ into $\mathcal{M}_{[0,T]}$ extends uniquely to a map $\mathcal{X}_0\mapsto\mathcal{X}_\epsilon(\cdot,\mathcal{X}_0)$ from $\mathcal{M}_0$ into $\mathcal{M}_{[0,T]}$, and this extension is a weak solution of* (3.8). *Moreover, for any $\mathcal{X}_0,\mathcal{Y}_0\in\mathcal{M}_0$, there exists a constant $\tilde{c}:=\tilde{c}_{T,\delta,a}<\infty$*

$$
E\sup_{0\le t\le T}\hat{\gamma}_f^2\left(\hat{\mathcal{X}}_\epsilon(t,\mathcal{X}_0),\hat{\mathcal{X}}_\epsilon(t,\mathcal{Y}_0)\right)\le\tilde{c}E\hat{\gamma}_f^2\left(\hat{\mathcal{X}}_0,\hat{\mathcal{Y}}_0\right),
\tag{3.21}
$$

(2)

$$
\mathcal{X}_\epsilon^\pm\left(\mathbb{R}^2,t,\mathcal{X}_0\right)=\mathcal{X}_\epsilon^\pm\left(\mathbb{R}^2,0,\mathcal{X}_0\right)=a^\pm\quad a.s.
\tag{3.22}
$$

Proof. (i) By (3.17), $\widehat{\mathcal{X}}_N(0) \mapsto \widehat{\mathcal{X}}_\epsilon(\cdot, \widehat{\mathcal{X}}_N(0))$ is uniformly continuous. Hence, we can extend our solutions of (3.8) by continuity to all $\mathcal{X}_0 \in \mathcal{M}_0$ by the density of $\widetilde{\mathcal{M}}_0$ in $\mathcal{M}_0$. We can see that (3.21) follows from (3.17).

Since $\phi \in C_0^2(\mathbb{R}^2;\mathbb{R})$, $\|\Delta\phi\|_{L.\infty} < \infty$ and $\|(\partial/\partial r_l)\phi\|_{L,\infty} < \infty$, $l = 1, 2$. So, the right-hand side of (3.7) is defined if we replace $\mathcal{X}_N(t)$ by $\mathcal{X}_\epsilon(t, \mathcal{X}_0)$.

(ii) Set $f_N(t) := \mathcal{X}_\epsilon(t, \mathcal{X}_0) - \mathcal{X}(t, \mathcal{X}_N(0))$. Then,

$$\begin{aligned} E&\left(\int_0^t \left\langle f_N(s), \int \widehat{\Gamma}_\epsilon(\cdot - p) w(dp, ds) \cdot \nabla\phi \right\rangle\right)^2 \\ &= \sum_{l,k,\tilde{l},\tilde{k}=1}^{2} E\int_0^t \iint f_N(s, dr) f_N(s, dq) \\ &\quad\times \int \widetilde{\Gamma}_{\epsilon,lk}(r, p)\widetilde{\Gamma}_{\epsilon,\tilde{l}\tilde{k}}(q, p)\, dp \frac{\partial}{\partial r_l}\phi(r) \frac{\partial}{\partial q_{\tilde{l}}}\phi(q)\, ds. \end{aligned} \tag{$*$}$$

Since for any q,

$$\left|\int \left[\widetilde{\Gamma}_{\epsilon,lk}(r-p) - \widetilde{\Gamma}_{\epsilon,lk}(\tilde{r}-p)\right]\widetilde{\Gamma}_{\epsilon,\tilde{l}\tilde{k}}(q-p)\, dp\right| \le c\left(\int (\widetilde{\Gamma}_{\epsilon,lk}(r-p) - \widetilde{\Gamma}_{\epsilon,lk}(\tilde{r}-p))^2\right)^{1/2} \le \tilde{c}\varrho(q, \tilde{q}), \tag{3.23}$$

we obtain that $\int \widetilde{\Gamma}_{\epsilon,lk}(r, p)\widetilde{\Gamma}_{\epsilon,\tilde{l}\tilde{k}}(q, p)\, dp(\partial/\partial r_l)\phi(r)$ is a bounded Lipschitz function in r uniformly in q. Similarly, if the roles of r and q are reversed, the right-hand side of $(*)$ tends to zero as $N \to \infty$ as a consequence of Lemma 3.4.

(iii) Because $\sup_p \|K_\delta(\cdot - p)\|_{L,\infty} \le c < \infty$, the analogue to step (ii) also holds for the deterministic integrals on the right-hand side of (3.8).

Next, we must show conservation of vorticity for all $\mathcal{X}_0 \in \mathcal{M}_0$. For $\mathcal{X}_0 \in \mathcal{M}_0$, choose a sequence $\{\mathcal{X}_{N,0}\}_{N\ge1} \subset \widetilde{\mathcal{M}}_0$ such that $\widehat{\mathcal{X}}_{N,0} \to \widehat{\mathcal{X}}_0$ in $\widehat{\gamma}_f$. We first solve (3.8) on the product space by Corollary 3.3 in Kotelenez and Seadler [14]. By Lemma 3.4 and the fact that $\widehat{\gamma}_f$ dominates $\widehat{\gamma}_{f,s}$, we have that $\mathcal{X}_\epsilon(\cdot, \mathcal{X}_{N,0})$ is a converging Cauchy sequence for $\gamma_{f,s}$. By Theorem 2.1, we have that the limit, $\mathcal{X}_\epsilon(\cdot, \mathcal{X}_0)$ must be in Hahn-Jordan form. Since positive and negative vorticities are conserved, we have conservation of total vorticity (3.22). □

4. The Macroscopic Limit

Set

$$\Lambda_N := \left\{ r_N \in \mathbb{R}^{2N} : \exists(i, j),\ 1 \le i < j \le N,\ \text{such that } r^i = r^j \right\}, \tag{4.1}$$

and denote by "⇒" weak convergence. Based on the recent work of Kotelenez and Kurtz [9], we conjecture the following.

Conjecture 4.1. *For each $N \in \mathbb{N}$ suppose $r_N(0) \notin \Lambda_N$ a.s. and suppose that the two-dimensional coordinates of $r_N(0)$ are exchangeable. Let $\phi \in C_0^2(\mathbb{R}^2, \mathbb{R})$ and suppose that $\langle \mathcal{X}_{N,0}, \phi \rangle \Rightarrow \langle X(0), \phi \rangle_0$, as $N \to \infty$. Then, there is a sequence $\delta(N) \to 0$, as $N \to \infty$ such that for any $t > 0$,*

$$\left\langle \mathcal{X}_{\epsilon,\delta(N)}(t), \phi \right\rangle \Longrightarrow \left\langle X(t), \phi \right\rangle_0 \quad \text{as } \epsilon \longrightarrow 0,\ N \longrightarrow \infty, \tag{4.2}$$

where $X(\cdot)$ is the solution of (1.1).

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