Hindawi Publishing Corporation
Advances in Mathematical Physics
Volume 2012, Article ID 505281, 17 pages
doi:10.1155/2012/505281

Review Article

Reduction of Dynamics with Lie Group Analysis

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Received 1 October 2011; Accepted 6 January 2012

Academic Editor: Mariano Torrisi

This paper is mainly a review concerning singular perturbation methods by means of Lie group analysis which has been presented by the author. We make use of a particular type of approximate Lie symmetries in those methods in order to construct reduced systems which describe the long-time behavior of the original dynamical system. Those methods can be used in analyzing not only ordinary differential equations but also difference equations. Although this method has been mainly used in order to derive asymptotic behavior, when we can find exact Lie symmetries, we succeed in construction of exact solutions.

1. Introduction

While the Lie group analysis has played an important role in construction of particular solutions of differential equations in terms of their symmetries [1, 2], it has been shown that the Lie group analysis is also an effective approach for obtaining the asymptotic behavior of systems. About a few decades ago, a method which provides asymptotic behavior of solutions of nonlinear parabolic partial differential equations in terms of the self-similarity of the system was presented [3]. The method was generalized in terms of the Lie symmetry group, which is referred to as the renormalization group symmetry; and a systematic manner was constructed for finding asymptotic solutions [4]. On the other hand, a practical method which introduced an idea of asymptotic Lie symmetry was presented and has succeeded in deriving asymptotic behavior of reaction diffusion equations and some other nonlinear differential equations [5, 6].

It has been shown that the Lie group analysis can also be applied in order to obtain asymptotic behavior in perturbation problems [7–9]. In this paper, we summarize and review the systematic procedure to derive an asymptotic behavior of perturbed dynamical systems by means of the Lie group analysis. This method is especially useful to a particular type of singular perturbation problems in which the naive expansion with respect to the perturbation parameter includes diverging terms, conventionally called secular terms, which give rise to inconsistency with long-time behavior of the solution. One of the differences of the method from the ordinary Lie group analysis is the point that we consider Lie groups which act not only on the variables but also on the parameters which are constants in terms of dynamics. Because we know solutions of the unperturbed system in perturbation problems in general, if we can find such a Lie symmetry group admitted by a perturbed system that acts also on the perturbation parameter, we succeed in the construction of solutions of the perturbed system by generating from the solution of the unperturbed system with the group. Although it is often impossible to find an exact Lie symmetry group, by finding a Lie symmetry group which keeps the system approximately invariant, we can derive dynamical systems which describe asymptotic behavior, what is called the reduced equation in the field of singular perturbation theory.

This paper is organized as follows. In Section 2, we shortly review the concepts concerning Lie symmetry group and the procedure to construct solutions of differential equations. In Section 3, we see the method to derive asymptotic behavior of a particular kind of singular perturbation problems described by ordinary differential equations. In Section 4, the method is extended in order to apply to singular perturbation problems described by difference systems. In Section 5, we see a few simple examples in which we can find an exact Lie symmetry; therefore we can construct an exact solution of the system. In the last section, we summarize this paper and give some discussion about the validity of the method from the viewpoint of approximation of the solution.

2. Lie Symmetry Group of Dynamical Systems

We briefly summarize the Lie group analysis for ordinary differential equations. For detail, see the reference [1, 2], for example. Let us consider an nth order ordinary differential equation such as

$$F(t, z, z_1, \dots, z_n) = 0, \tag{2.1}$$

where $t \in \mathbb{R}$ is the independent variable, $z \in \mathbb{C}^n$ are dependent variables, and $z_k := \mathrm{d}^k z/\mathrm{d}t^k$. Let

$$X = \tau(t, z)\partial_t + \phi(t, z)\partial_z \tag{2.2}$$

be an infinitesimal generator of a Lie group symmetry which leaves the system (2.1) invariant. Then the infinitesimal generator satisfies the infinitesimal criterion for invariance of the system, that is,

$$X^*[F(t, z, z_1, \dots, z_n)]|_{F=0} = 0, \tag{2.3}$$

where X^* is the prolongation of the infinitesimal generator defined by

$$X^* := X + \phi^1 \partial_{z_1} + \phi^2 \partial_{z_2} + \cdots,$$
$$\phi^k := D_t \phi^{k-1} - z_k D_t \tau, \tag{2.4}$$
$$D_t := \partial_t + z_1 \partial_z + z_2 \partial_{z_1} + \cdots.$$

The infinitesimal criterion (2.3) is a linear differential equation in terms of τ and ϕ, which is referred to as the determining equation. By solving this equation, we find an infinitesimal generator of a Lie symmetry group admitted by the system. In terms of the infinitesimal generator, a solution of the system, $f(t, z) = 0$, which satisfies the Lie equation,

$$X[f(t, z)]\big|_{F=0} = 0, \tag{2.5}$$

is referred to as an invariant solution.

3. Singular Perturbation in Ordinary Differential Equations

In this section, we consider perturbation problems governed by ordinary differential equations. This section mainly follows reference [10]. We consider ordinary differential equations which are written as

$$\dot{z} = \Lambda z + \varepsilon g(z), \tag{3.1}$$

where $z \in \mathbb{C}^n$ are dependent valuables, $t \in \mathbb{R}$ is the independent valuable, the overdot denotes the first derivative with respect to t, $g : \mathbb{C}^n \to \mathbb{C}^n$, and Λ is an $n \times n$ matrix. ε represents a small real parameter referred to as the perturbation parameter. To make the notation clear, we consider differential equations such that Λ is a diagonal matrix whose components are written as $\Lambda_{ij} =: \lambda_i \delta_{ij}$. Even if Λ cannot be diagonalized, the procedure presented here can be applied [11].

First, let us see that the perturbed system (3.1) may exhibit a singular perturbation problem where the naive expansion with respect to perturbation parameter of the system includes diverging term in short time which is not appropriate to describe the long-time behavior of the solution. Let

$$z(t; \varepsilon) := \sum_{k=0}^{\infty} \varepsilon^k z^{(k)}(t) \tag{3.2}$$

be the naive expansion of the system (3.1). Then, with direct substitution of this expansion into the both sides of (3.1), and equating the same order of ε, we obtain the recursive equations for $z^{(k)}$ as follows:

$$\dot{z}^{(0)} = \Lambda z^{(0)}, \tag{3.3}$$

$$\dot{z}^{(i)} = \Lambda z^{(i)} + g^{(i-1)}\left(z^{(0)}, \ldots, z^{(i-1)}\right), \quad (i = 1, 2, \ldots), \tag{3.4}$$

where we have set

$$g\left(\sum_{k=0}^{\infty}\varepsilon^k z^{(k)}(t)\right) := \sum_{k=0}^{\infty}\varepsilon^k g^{(k)}\left(z^{(0)},\dots,z^{(k-1)}\right). \tag{3.5}$$

We consider the case that each component of $g(z)$ consists of a polynomial, that is to say, $g(z)$ is given by

$$g(z) = \sum_p C_p z^p, \tag{3.6}$$

where $C_p \in \mathbb{C}^n$, $p \in \mathbb{N}^n$ and $z^p := \prod_{k=1}^n z_k^{p_k}$. Because the unperturbed solution $z^{(0)}(t)$ is

$$z^{(0)}(t) = \mathrm{e}^{\Lambda t} z_0, \tag{3.7}$$

then the differential equation for $z^{(1)}$ becomes

$$\begin{aligned}\dot{z}^{(1)} &= \Lambda z^{(1)} + g\left(z^{(0)}\right)\\ &= \Lambda z^{(1)} + \sum_p C_p \mathrm{e}^{(\lambda\cdot p)t} z_0.\end{aligned} \tag{3.8}$$

The first order of the naive expansion becomes

$$z_i^{(1)} = \sum_{\substack{p \text{ s.t.}\\ \lambda\cdot p-\lambda_i=0}} C_{p,i} t \mathrm{e}^{(\lambda\cdot p)t} z_0^p + \sum_{\substack{p \text{ s.t.}\\ \lambda\cdot p-\lambda_i\neq 0}} \frac{C_{p,i}}{\lambda\cdot p-\lambda_i}\mathrm{e}^{(\lambda\cdot p)t} z_0^p, \tag{3.9}$$

where $\lambda\cdot p := \sum_{j=1}^n \lambda_j p_j$. We see that the nonlinear terms which satisfy $\lambda\cdot p-\lambda_i = 0$ lead to the terms which are proportional to t in the first order in the naive expansion. We can see with straightforward calculation in a similar manner that the higher-order terms, $z^{(n)}$, include the terms proportional to t^n. Such terms are called the resonant terms or the secular terms, and the condition which gives rise to such secular terms, $\lambda\cdot p-\lambda_i = 0$, is called the resonance condition. For example, in perturbed harmonic oscillators, where all of the eigenvalues of the linear system are pure imaginary, the secular terms induce the rapid increase of the approximate solution in a short time; thus the behavior is extremely different from that of the exact solution. This kind of problems in which the naive expansion includes the divergent terms which should not be included to describe the asymptotic behavior is known as one of the types of singular perturbation problem.

Lie group theory can be applied to derive reduced systems which govern the long-time dynamics of the original system to those singular perturbation problems. For the perturbed system (3.1), let

$$X := \partial_\varepsilon + \tau(t,z;\varepsilon)\partial_t + \phi(t,z;\varepsilon)\cdot\partial_z \tag{3.10}$$

be an infinitesimal generator of a Lie group which admits the system (3.1). Note that the group we consider here acts not only on the independent and dependent variables but also on the constant perturbation parameter ε. The reason why the coefficient of ∂_ε is assumed to be 1 is shown in reference [9]. The first prolongation of the infinitesimal generator X^* is written as

$$\begin{gathered} X^* = X + \phi^{\dot{z}}(t, z; \varepsilon) \cdot \partial_{\dot{z}}, \\ \phi^{\dot{z}}(t, z, \dot{z}; \varepsilon) := (\partial_t + \dot{z} \cdot \partial_z)\phi(t, z; \varepsilon) - \dot{z}(\partial_t + \dot{z} \cdot \partial_z)\tau(t, z; \varepsilon). \end{gathered} \tag{3.11}$$

Then the infinitesimal criterion for invariance is given by

$$X^*\left[\dot{z} - \Lambda z - \varepsilon g(z)\right]\Big|_{\dot{z}=\Lambda z+\varepsilon g(z)} = 0. \tag{3.12}$$

Let us find those symmetries which satisfy $\tau = 0$ for simplicity. Then (3.12) reads

$$[\partial_t + (\Lambda z) \cdot \partial_z - \Lambda]\phi - g - \varepsilon\left[\phi \cdot \partial_z g - g \cdot \partial_z \phi\right] = 0, \tag{3.13}$$

which is the determining equation for ϕ. Expanding $\phi(t, z; \varepsilon) = \sum_{k=0}^{\infty} \varepsilon^k \phi^{(k)}(t, z)$, we obtain the recursive equations for $\{\phi^{(k)}\}$ as follows:

$$L\phi^{(0)} = g, \tag{3.14}$$

$$L\phi^{(i)} = \phi^{(i-1)} \cdot \partial_z g - g \cdot \partial_z \phi^{(i-1)}, \quad (i = 1, 2, \ldots), \tag{3.15}$$

where $Lf := [\partial_\varepsilon + (\Lambda z) \cdot \partial_z - \Lambda]f$ for arbitrary function f. Solving these recursive equations from the lowest order, we obtain the infinitesimal generator of the Lie symmetry group, X. Then, we construct the solution invariant under this Lie symmetry group. Such a solution $z = z(t; \varepsilon)$ follows what is called the Lie equation:

$$X[z - z(t; \varepsilon)]|_{z=z(t;\varepsilon)} = 0, \tag{3.16}$$

which reads

$$\frac{\partial z}{\partial \varepsilon}(t; \varepsilon) = \phi(t, z(t; \varepsilon); \varepsilon). \tag{3.17}$$

In solving this equation, we use the solution of the unperturbed system as the boundary condition at $\varepsilon = 0$, that is,

$$z(t; 0) = z^{(0)} = \mathrm{e}^{\Lambda t} z_0 \tag{3.18}$$

for a constant z_0 determined from the initial condition of the unperturbed system. Thus, we obtain the solution of the system $z(t; \varepsilon)$. If we solve the determining equation up to finite order, the solution obtained through this procedure is an approximate solution.

To proceed the calculation more, set the nonlinear polynomial terms $g(z)$ as (3.6), that is,

$$g(z) = \sum_p C_p z^p. \tag{3.19}$$

The recursive equation of the lowest order becomes

$$L\phi^{(0)}(t,z) = \sum_p C_p z^p. \tag{3.20}$$

Making use of the fact that z^p is eigenfunction of L for arbitrary p, namely,

$$Lz^p = (\lambda \cdot p - \lambda_i) z^p, \tag{3.21}$$

we immediately find a solution of (3.20). Particularly, for those terms in $g(z)$ which satisfy $\lambda \cdot p - \lambda = 0$, named the resonance condition in the context of singular perturbation theory, the corresponding eigenfunction of L is a zero eigenfunction. Therefore, in the case that $g(z)$ includes such terms, terms proportional to tz^p appear in $\phi^{(0)}$. Thus, the ith component of $\phi^{(0)}$ is obtained as

$$\phi_i^{(0)}(t,z) = \sum_{\substack{p \text{ s.t.} \\ \lambda\cdot p - \lambda_i = 0}} C_{p,i} t z^p + \sum_{\substack{p \text{ s.t.} \\ \lambda\cdot p - \lambda_i \neq 0}} \frac{C_{p,i}}{\lambda \cdot p - \lambda_i} z^p, \quad (i = 1, 2, \ldots, n). \tag{3.22}$$

The group-invariant solution $z = z(t;\varepsilon)$ for this approximate Lie symmetry group follows:

$$\frac{\partial z(t;\varepsilon)}{\partial \varepsilon} = \phi^{(0)} \quad (t, z(t;\varepsilon);\varepsilon). \tag{3.23}$$

Now, because we are interested in the asymptotic behavior of the system, consider the case $t \gg 1$. Then, the Lie equation is reduced to

$$\frac{\partial z_i(t;\varepsilon)}{\partial \varepsilon} = \sum_{\substack{p \text{ s.t.} \\ \lambda\cdot p - \lambda_i = 0}} C_{p,i} t z^p, \tag{3.24}$$

which reads

$$\frac{\partial z_i(\tau)}{\partial \tau} = \sum_{\substack{p \text{ s.t.} \\ \lambda\cdot p - \lambda_i \neq 0}} C_{p,i} z^p. \tag{3.25}$$

Here, we have introduced a slowly changing variable $\tau := \varepsilon t$. The fact that this reduced equation is a differential equation whose independent variable is the slowly changing time indicates that this equation describes asymptotic behavior of the system. As discussed in the last section, it has been shown that the diverging secular terms in the naive expansion are included to the solution of this reduced equation.

To see this fact concretely, let us consider the following differential equation which is a perturbed harmonic oscillator known as the Rayleigh equation:

$$\ddot{u} + u = \varepsilon\left(\dot{u} - \frac{1}{3}\dot{u}^3\right). \tag{3.26}$$

Under a transformation $(u, \dot{u}) \mapsto (z, \overline{z}) = (u + \mathrm{i}\dot{u}, u - \mathrm{i}\dot{u})$, this dynamical system reads

$$\begin{aligned} \dot{z} &= -\mathrm{i}\omega z + \varepsilon\left[\frac{1}{2}(z - \overline{z}) + \frac{1}{24}(z - \overline{z})^3\right], \\ \dot{\overline{z}} &= \mathrm{i}\omega \overline{z} - \varepsilon\left[\frac{1}{2}(z - \overline{z}) + \frac{1}{24}(z - \overline{z})^3\right]. \end{aligned} \tag{3.27}$$

Because the second equation is the complex conjugate of the first one, it is enough to consider only the first equation in the following. In the same manner shown above, we can calculate the infinitesimal generator of the approximate Lie symmetry group and obtain ϕ up to the lowest order with respect to ε as follows:

$$\phi^{(0)}(t, z) = \frac{1}{2}tz - \frac{1}{8}t|z|^2 z + \frac{1}{4}\mathrm{i}\overline{z} + \frac{1}{48}\mathrm{i}z^3 - \frac{1}{16}\mathrm{i}|z|^2\overline{z} - \frac{1}{96}\mathrm{i}\overline{z}^3. \tag{3.28}$$

The solution invariant under this approximate Lie symmetry group satisfies the Lie equation as follows:

$$\frac{\partial z(t;\varepsilon)}{\partial \varepsilon} = \frac{1}{2}tz - \frac{1}{8}t|z|^2 z + \frac{1}{4}\mathrm{i}\overline{z} + \frac{1}{48}\mathrm{i}z^3 - \frac{1}{16}\mathrm{i}|z|^2\overline{z} - \frac{1}{96}\mathrm{i}\overline{z}^3. \tag{3.29}$$

In order to see the long-time behavior, we consider the case of $t \gg 1$. Then, the Lie equation is reduced to

$$\frac{\partial z(t;\varepsilon)}{\partial \varepsilon} = \frac{1}{2}tz - \frac{1}{8}t|z|^2 z. \tag{3.30}$$

Introducing $\tau := \varepsilon t$ and with the transformation $(z, \overline{z}) \mapsto (A, \theta) \in \mathbb{R}^2$ such that

$$A := |z|, \qquad \theta := \frac{\mathrm{i}}{2}\mathrm{Log}\left(\frac{\overline{z}}{z}\right), \tag{3.31}$$

we obtain

$$\begin{aligned} \frac{\mathrm{d}A}{\mathrm{d}\tau} &= \frac{A}{2}\left(1 - \frac{A^2}{4}\right), \\ \frac{\mathrm{d}\theta}{\mathrm{d}\tau} &= 0. \end{aligned} \tag{3.32}$$

This result indicates that the nonlinear perturbation induces the change of the amplitude of the oscillation and makes the system limit-cycle oscillator in which the asymptotic behavior is described by the differential equation for A; and the amplitude converges 2 in the long-time limit as a consequence.

4. Singular Perturbation in Discrete Dynamical Systems

We consider systems governed by difference equations. Lie group analysis has been developed also in discrete systems thus far [12]. On the other hand, here we can avoid any difficulty in discretizing Lie groups in the derivation of the asymptotic behavior of this system as shown in the following. This section mainly follows reference [8].

We consider a general type of 2D symplectic map $(u_n, v_n) \mapsto (u_{n+1}, v_{n+1})$, $u_n, v_n \in \mathbb{R}$ as follows:

$$\begin{aligned} u_{n+1} &= u_n + v_{n+1}, \\ v_{n+1} &= v_n + a u_n + \varepsilon g(u_n), \end{aligned} \tag{4.1}$$

where $a \in \mathbb{R}$ is a constant coefficient, $\varepsilon \in \mathbb{R}$ is a small perturbation parameter, and $g : \mathbb{R} \to \mathbb{R}$ is a nonlinear function of u_n in general. There occurs the same kind of singular perturbation problem in difference equations as what we have observed in the preceding section for ordinary differential equations, which means that secular terms which are proportional to n emerge in the naive expansion with respect to ε.

Remember that the Lie symmetry groups we have found for ordinary differential equations in the preceding section do not include the action on t, which is discretized variable here. Therefore, as long as we consider autonomous systems as this 2D symplectic map, we can directly applied the method for ordinary differential equations to the systems considered here.

The difference equation is written as

$$\begin{pmatrix} u_{n+1} \\ v_{n+1} \end{pmatrix} = \begin{pmatrix} a+1 & 1 \\ a & 1 \end{pmatrix} \begin{pmatrix} u_n \\ v_n \end{pmatrix} + \begin{pmatrix} \varepsilon g(u_n) \\ \varepsilon g(u_n) \end{pmatrix}. \tag{4.2}$$

With the transformation of variables such that the linear part is diagonalized, that is,

$$\begin{aligned} \begin{pmatrix} z_n \\ \overline{z_n} \end{pmatrix} &= \begin{pmatrix} \dfrac{1}{2\cos(\omega/2)} & \mathrm{i}\dfrac{\exp(\mathrm{i}\omega/2)}{2\sin\omega\cos\omega} \\ \dfrac{1}{2\cos(\omega/2)} & -\mathrm{i}\dfrac{\exp(-\mathrm{i}\omega/2)}{2\sin\omega\cos\omega} \end{pmatrix} \begin{pmatrix} u_n \\ v_n \end{pmatrix}, \\ \Longleftrightarrow \begin{pmatrix} u_n \\ v_n \end{pmatrix} &= \begin{pmatrix} \exp\left(-\dfrac{\mathrm{i}\omega}{2}\right) & \exp\left(\dfrac{\mathrm{i}\omega}{2}\right) \\ -2\mathrm{i}\sin\left(\dfrac{\omega}{2}\right) & 2\mathrm{i}\sin\left(\dfrac{\omega}{2}\right) \end{pmatrix} \begin{pmatrix} z_n \\ \overline{z_n} \end{pmatrix}, \end{aligned} \tag{4.3}$$

we can rewrite the difference equation as

$$\begin{pmatrix} z_{n+1} \\ \overline{z_{n+1}} \end{pmatrix} = \begin{pmatrix} e^{-i\omega} & 0 \\ 0 & e^{i\omega} \end{pmatrix} \begin{pmatrix} z_n \\ \overline{z_n} \end{pmatrix} + \varepsilon \frac{g(u_n)}{2\sin\omega} \begin{pmatrix} i\exp\left(-\dfrac{i\omega}{2}\right) \\ -i\exp\left(\dfrac{i\omega}{2}\right) \end{pmatrix}, \tag{4.4}$$

where $a+2 =: 2\cos\omega$. Because we are particularly interested in the case that the dynamical system exhibits periodic behavior corresponding to oscillation in continuous systems now, we assume that the origin is an elliptic point, namely, $-4 < a < 0$. As is the case in ordinary differential equations, since the second component of the above difference equation is the complex conjugate of the first one, it is sufficient to consider only the equation of the first component:

$$z_{n+1} = e^{-i\omega} z_n + \varepsilon \frac{i\exp(-i\omega/2)}{2\sin\omega} g(u_n). \tag{4.5}$$

Let us construct the singular perturbation method by means of the Lie group analysis. Noting that the difference equation is an algebraic equation for z_n, z_{n+1}, $\overline{z_n}$ and $\overline{z_{n+1}}$, let

$$\begin{aligned} X(n, z_n, \overline{z_n}, z_{n+1}, \overline{z_{n+1}}) = {} & \partial_\varepsilon + \eta^z(n, z_n, \overline{z_n})\partial_{z_n} + \eta^{\overline{z}}(n, z_n, \overline{z_n})\partial_{\overline{z_n}} \\ & + \eta^z(n+1, z_{n+1}, \overline{z_{n+1}})\partial_{z_{n+1}} + \eta^{\overline{z}}(n+1, z_{n+1}, \overline{z_{n+1}})\partial_{\overline{z_{n+1}}} \end{aligned} \tag{4.6}$$

be an infinitesimal generator of a Lie symmetry group which admits the difference equation. We can show with straightforward calculation that

$$\begin{aligned} X(n, z_n, \overline{z_n}, z_{n+1}, \overline{z_{n+1}}) = {} & \partial_\varepsilon + \eta(n, z_n, \overline{z_n})\partial_{z_n} + \overline{\eta(n, z_n, \overline{z_n})}\partial_{\overline{z_n}} \\ & + \eta(n+1, z_{n+1}, \overline{z_{n+1}})\partial_{z_{n+1}} + \overline{\eta(n+1, z_{n+1}, \overline{z_{n+1}})}\partial_{\overline{z_{n+1}}}. \end{aligned} \tag{4.7}$$

Here we simply write $\eta := \eta^z$. Then the infinitesimal criterion for invariance is given by

$$X(n, z_n, \overline{z_n}, z_{n+1}, \overline{z_{n+1}}) \left[z_{n+1} - e^{-i\omega} z_n - \varepsilon \frac{i\exp(-i\omega/2)}{2\sin\omega} g(u_n) \right] \Bigg|_{(50)} = 0. \tag{4.8}$$

In the same manner as that to the ordinary differential equations, we find an approximate Lie symmetry group up to the lowest order with respect to ε. That is to say, we find a solution of

$$\begin{aligned} & X(n, z_n, \overline{z_n}, z_{n+1}, \overline{z_{n+1}}) \left[z_{n+1} - e^{-i\omega} z_n - \varepsilon \frac{i\exp(-i\omega/2)}{2\sin\omega} g(u_n) \right] \Bigg|_{z_{n+1}=e^{-i\omega}z_n} = O(\varepsilon), \\ & \Longleftrightarrow \eta\left(n+1, e^{-i\omega} z_n, e^{i\omega}\overline{z_n}\right) - e^{-i\omega}\eta(n, z_n, \overline{z_n}) = \frac{i\exp(-i\omega/2)}{2\sin\omega} g(u_n). \end{aligned} \tag{4.9}$$

By solving this equation, we obtain the infinitesimal generator X of the Lie group admitted by the difference equation. Using that infinitesimal generator, we construct the invariant solution $z_n = z_n(\varepsilon)$, which satisfies the Lie equation,

$$X[z_n - z_n(\varepsilon)]|_{z_n = z_n(\varepsilon)} = 0, \tag{4.10}$$

which reads

$$\frac{\mathrm{d}z_n(\varepsilon)}{\mathrm{d}\varepsilon} = \eta\left(n, z_n(\varepsilon), \overline{z_n(\varepsilon)}\right). \tag{4.11}$$

If we solve this equation under the boundary condition that

$$z_n(\varepsilon = 0) = z_n^{(0)}, \tag{4.12}$$

which denotes a solution of the unperturbed system, we obtain the approximate solution of the system.

In order to proceed the calculation more, we set the nonlinear perturbation terms as

$$g(u_n) := \sum_j A_j u_n^j, \tag{4.13}$$

where A_j is a real constant. Then, the determining equation becomes

$$\eta\left(n+1, e^{-\mathrm{i}\omega} z_n, e^{\mathrm{i}\omega}\overline{z_n}\right) - e^{-\mathrm{i}\omega}\eta(n, z_n, \overline{z_n}) = \frac{\mathrm{i}\,\exp(-\mathrm{i}\omega/2)}{2\sin\omega} \sum_{l,m} B_{lm}(\omega) z_n^l \overline{z_n}^m, \tag{4.14}$$

where

$$B_{lm}(\omega) = A_{l+m}\binom{l+m}{l}\exp\left[\frac{-\mathrm{i}(l-m)\omega}{2}\right]. \tag{4.15}$$

Unlike the case of ordinary differential equations, the solution of this determining equation depends on whether $\omega/2\pi$ is a rational number or irrational number.

In the case that $\omega/2\pi$ is an irrational number, terms proportional to $z_n^{l+1}\overline{z_n}^l$ in the polynomial g give rise to terms proportional to n in $\eta(n, z_n, \overline{z_n})$. We obtain a solution of (4.9),

$$\begin{aligned}\eta(n, z_n, \overline{z}_n) = {} & \sum_{l,m \text{ s.t. } l-m=1} \frac{\mathrm{i}\,\exp(\mathrm{i}\omega/2) B_{lm}(\omega)}{2\sin\omega} n z_n^l \overline{z_n}^m \\ & + \sum_{l,m \text{ s.t. } l-m\neq 1} \frac{\mathrm{i}\,\exp(-\mathrm{i}\omega/2) B_{lm}(\omega)}{2\sin\omega\{\exp[-\mathrm{i}(l-m)\omega] - \exp[-\mathrm{i}\omega]\}} z_n^l \overline{z_n}^m.\end{aligned} \tag{4.16}$$

The invariant solution under this symmetry group satisfies

$$\begin{aligned}\frac{\mathrm{d}z_n}{\mathrm{d}\varepsilon} &= \sum_{l,m \text{ s.t. } l-m=1} \frac{\mathrm{i}\,\exp(\mathrm{i}\omega/2)B_{lm}(\omega)}{2\sin\omega} n z_n^l \overline{z_n}^m \\ &\quad + \sum_{l,m \text{ s.t. } l-m\neq 1} \frac{\mathrm{i}\,\exp(\mathrm{i}\omega/2)B_{lm}(\omega)}{2\sin\omega\{\exp[-\mathrm{i}(l-m)\omega]-\exp[-\mathrm{i}\omega]\}} z_n^l \overline{z_n}^m.\end{aligned} \tag{4.17}$$

In order to derive the long-time behavior of the system, we consider the case of $n \gg 1$. Then, (4.17) is reduced to

$$\begin{aligned}\frac{\mathrm{d}z_n}{\mathrm{d}\tau} &= \frac{\mathrm{i}\,\exp(\mathrm{i}\omega/2)}{2\sin\omega} \sum_{l=0}^{\infty} B_{l+1\,l}(\omega)|z_n|^{2l} z_n \\ &= \frac{\mathrm{i}}{2\sin\omega} \sum_{l=0}^{\infty} A_{2l+1} \binom{2l+1}{l} |z_n|^{2l} z_n.\end{aligned} \tag{4.18}$$

Here $\tau := \varepsilon n$ is introduced. Noting that $\mathrm{d}|z_n|/\mathrm{d}\varepsilon = 0$, we can obtain the solution as

$$z_n(\tau) = z_n(0) \exp\left(\mathrm{i}\frac{1}{2\sin\omega} \sum_{l=0}^{\infty} A_{2l+1} \binom{2l+1}{l} R^{2l} \tau \right), \tag{4.19}$$

where $R := |z_n|$.

On the other hand, in the case that $\omega/2\pi$ is a rational number, more terms in g induce the resonance. They are the terms $z_n^l \overline{z}_n^m$ which satisfy $l - m = 1 + kp$ for an integer k. Noting this fact, we can obtain the asymptotic behavior in a similar manner; as a result, the reduced equation becomes as follows:

$$\begin{aligned}\frac{\mathrm{d}z_n}{\mathrm{d}\tau} = \mathrm{i}\frac{1}{2\sin\omega} \Bigg[&\sum_{l\geq 1, m\geq 0} A_{2l+mp-1} \binom{2l+mp-1}{l+mp} |z_n|^{2(l-1)} z_n^{mp+1} \\ &+ \sum_{l\geq 0, m\geq 1} A_{2l+mp-1} \binom{2l+mp-1}{l} |z_n|^{2l} \overline{z_n}^{mp-1} \Bigg].\end{aligned} \tag{4.20}$$

It should be remarked that this system is a Hamilton system whose Hamiltonian is given by

$$H(z_n, \overline{z_n}) = \sum_{\substack{l\geq 0, m\geq 0 \\ (l,m)\neq(0,0)}} \left[\mathrm{i} A_{2l+mp-1} \frac{(2l+mp-1)!}{(l+mp)!\; l!} |z_n|^{2l} \left(z_n^{mp} + \overline{z_n}^{mp} \right) \right]. \tag{4.21}$$

This implies that the asymptotic behavior derived with this method holds the simplecticity the original system has.

Moreover, if we discuss the case that the $\omega/2\pi$ is close to an irrational number enough, in the same manner, we can give an explanation for the well-known Poincaré-Birkoff bifurcation (see [8] for detail).

One of the characteristics of this method is that, although we originally consider a system of difference equations, the asymptotic behavior is represented by differential equations, namely, the Lie equation. This may be a merit when we analyze discrete systems since, in general, it is more easier to find a solution of a differential equation than a difference equation.

5. Construction of Exact Solutions

In the preceding sections, we have obtained asymptotic behavior, namely, the reduced equations which govern the long-time dynamics, by finding approximate Lie symmetry groups. However, if we can find an exact Lie symmetry group, namely, an exact solution of the determining equation, we succeed in construction of an exact solution. Here, we see two such simple examples.

We consider the following ordinary differential equation:

$$\dot{v} = cv - v^n, \tag{5.1}$$

where c is a real parameter, $v = v(t;c)$ is the real-valued dependent variable, n is an integer which satisfies $2 \leq n$, and the overdot denotes the derivative with respect to the independent variable t. As known well, this system exhibits a saddle-node bifurcation or a pitch-fork bifurcation at $c = 0$ when n is even or odd, respectively.

Under the transformation of variables

$$(a, \tau, u) := \left(c^{-n}, ct, cv\right), \tag{5.2}$$

the system reads

$$\dot{u} = u - au^n, \tag{5.3}$$

where the overdot denotes the derivative with respect to τ. In the same manner shown in the preceding sections, we set the infinitesimal generator of a Lie symmetry group admitted by the system as

$$X(\tau, u; a) = \alpha(\tau, u; a)\partial_a + \phi(\tau, u; a)\partial_u. \tag{5.4}$$

This also include the transformation of a constant parameter a. Using the first prolongation of X,

$$\begin{aligned} X^* &= X + \phi^{\dot{u}}(\tau, u, \dot{u}; a)\partial_{\dot{u}}, \\ \phi^{\dot{u}}(\tau, u, \dot{u}; \varepsilon) &:= (\partial_\tau + \dot{u}\partial_u)\phi(\tau, u; a), \end{aligned} \tag{5.5}$$

the infinitesimal criterion for invariance of the system is written as

$$X^*[\dot{u} - u + au^n]|_{\dot{u}=u-au^n} = 0, \tag{5.6}$$

which reads

$$\left[\partial_t + (u - au^n)\partial_u - 1 + nau^{n-1}\right]\phi + \alpha u^n = 0. \tag{5.7}$$

In terms of homogeneity, we can easily find the solution as

$$\begin{aligned} \phi(\tau, u; a) &= -\frac{1}{n-1}u^n, \\ \alpha(\tau, u; a) &= 1. \end{aligned} \tag{5.8}$$

Thus, we obtain an exact Lie symmetry group admitted by the system,

$$X(\tau, u; a) = \partial_a - \frac{1}{n-1}u^n\partial_u. \tag{5.9}$$

Then, the Lie equation corresponding to this group is given by

$$\begin{aligned} X[u - u(\tau; a)]|_{u=u(\tau;a)} &= 0, \\ u(\tau;\ 0) &= u_0 e^{\tau}, \end{aligned} \tag{5.10}$$

which reads

$$\begin{aligned} \frac{\partial u(\tau; a)}{\partial a} &= -\frac{1}{n-1}u^n(\tau; a), \\ u(\tau; 0) &= u_0 e^{\tau}, \end{aligned} \tag{5.11}$$

where $u_0 := u(0; 0)$ is a constant. Then the solution follows that, for even n,

$$u(\tau; a) = \left(a + u_0^{-(n-1)} e^{-(n-1)\tau}\right)^{-1/(n-1)}, \tag{5.12}$$

and odd n,

$$u(\tau; a) = \pm\left(a + u_0^{-(n-1)} e^{-(n-1)\tau}\right)^{-1/(n-1)}. \tag{5.13}$$

With the inverse transformation of (5.2), and introducing v_0 as

$$u_0 =: c\left(v_0^{-(n-1)} - c^{-1}\right)^{-1/(n-1)}, \tag{5.14}$$

we obtain the solution of the original equation as follows: for even n,

$$v(t;c) = \left[c^{-1} + \left(v_0^{-(n-1)} - c^{-1}\right)\mathrm{e}^{-(n-1)ct}\right]^{-1/(n-1)}, \tag{5.15}$$

and, for odd n,

$$v(t;c) = \begin{cases} \left[c^{-1} + \left(v_0^{-(n-1)} - c^{-1}\right)\mathrm{e}^{-(n-1)ct}\right]^{-1/(n-1)}, & v_0 > 0, \\ -\left[c^{-1} + \left(v_0^{-(n-1)} - c^{-1}\right)\mathrm{e}^{-(n-1)ct}\right]^{-1/(n-1)}, & v_0 < 0. \end{cases} \tag{5.16}$$

With straightforward calculation in the limit of $t \to \infty$, we can easily confirm that this solution exhibits a saddle-node bifurcation and a pitch-fork bifurcation depending on n.

Next, we consider the following partial differential equation:

$$u_t + cu_x - \varepsilon f(u) = 0. \tag{5.17}$$

Here, t and x are real independent variables, $u(t,x)$ is the real-valued dependent variable, ε is a constant parameter, and f is a real-valued function of u. Let

$$\mathrm{X}(t,x,u;\varepsilon) = \partial_\varepsilon + \eta(t,x,u;\varepsilon)\partial_u \tag{5.18}$$

be the infinitesimal generator of a Lie group admitted by the system. Because its first prolongation is

$$\begin{aligned} \mathrm{X}^*(t,x,u,u_x,u_t;\varepsilon) &= \mathrm{X} + \eta^{u_t}\partial_{u_t} + \eta^{u_x}\partial_{u_x}, \\ \eta^{u_x} &:= (\partial_x + u_x\partial_u)\eta, \\ \eta^{u_t} &:= (\partial_t + u_t\partial_u)\eta, \end{aligned} \tag{5.19}$$

the infinitesimal criterion for invariance is given by

$$\mathrm{X}^*\left[u_t + cu_x - \varepsilon f(u)\right]\big|_{u_t+cu_x-\varepsilon f(u)} = 0, \tag{5.20}$$

which reads

$$\begin{aligned} &\left[\eta^{u_t} + c\eta^{u_x} - f(u) - \varepsilon\eta\partial_u f(u)\right]\big|_{u_t+cu_x-\varepsilon f(u)} = 0, \\ &\Longleftrightarrow (\partial_t + c\partial_x)\eta - f(u) + \varepsilon\left[f(u)\partial_u\eta - \eta\partial_u f(u)\right] = 0. \end{aligned} \tag{5.21}$$

We can immediately find a solution, that is,

$$\eta(t,x,u;\varepsilon) = tf(u). \tag{5.22}$$

Thus, we obtain the infinitesimal generator of an exact Lie symmetry group,

$$X = \partial_\varepsilon + tf(u)\partial_u. \tag{5.23}$$

The solution invariant under this group satisfies the Lie equation:

$$X[u - u(t,x)]|_{u=u(t,x)} = 0, \tag{5.24}$$

which reads

$$\frac{\partial u(t,x)}{\partial \varepsilon} = tf(u(t,x)). \tag{5.25}$$

Then, the corresponding particular solution $u = u(t,x)$ satisfies

$$\int_{u_0}^{u(t,x)} \frac{du'}{f(u')} = t\varepsilon. \tag{5.26}$$

Here, u_0 denotes a solution of the differential equation when $\varepsilon = 0$. Let us consider a simple case, namely, $f(u) = u^n$. Then, we obtain a solution

$$u(t,x) = \frac{u_0(x-ct)}{\left[1 - \varepsilon t u_0(x-ct)^{n-1}\right]^{1/(n-1)}}. \tag{5.27}$$

Here, we have used the fact that the original system in the case of $\varepsilon = 0$ has solutions which satisfies $u = u_0(x - ct)$.

Although the above two examples are not perturbation problems, finding Lie groups which act also on constant parameters plays a role in the construction of an exact solution of differential equations. These examples are simple enough to obtain the solution by means of ordinary quadrature. However, it is possible that the manner presented here is useful in finding a solution of more complex systems because the determining equation is always linear differential equation and the Lie equation is always a first-order ordinary equation. We need to make it clear that to what kind of dynamical systems we can effectively apply this method in the future.

6. Summary and Discussion

We have reviewed methods to derive an asymptotic behavior of dynamical systems which exhibit singular perturbation problem by means of Lie symmetry group. Main characteristics of this method are summarized into two points. (1) We consider Lie groups which also act on constant parameters. (2) We try to find an approximate Lie symmetry group and obtain the reduced dynamics as the Lie equation. As a result, the reduced equations which describe the asymptotic behavior of slowly changing quantities are derived. The quantities are constants in the unperturbed system. As we have seen in the procedure, the secular terms are eliminated so that the symmetry of the dynamical system is approximately maintained. This method can be applied to systems of difference equations where the same kind of singular perturbation problems occurs. In particular cases that we can find a Lie group which admits the system exactly, we succeed in construction of exact solutions.

It seems that construction of solutions with this method consists of more complicated procedure than solving original problems because we have to solve differential equations twice: one of them is the differential equation which the infinitesimal generator of the Lie group symmetry satisfies, namely, the determining equation, and the other is the differential equations which the invariant solution satisfies, namely, the Lie equation. However, this method makes the calculation clear since the Lie group analysis always ensure that (1) the determining equation is always linear differential equation and (2) the Lie equation is always first-order differential equation. Moreover, as we have seen, the solution of the determining equation can be easily found in the case that the nonlinear perturbed terms are polynomial because of the homogeneity. In general, when we construct the perturbation solution, we should solve some recursive equation. This method provides more clear recursive equation for finding the Lie symmetry group, (3.15), compared with the recursive equations for the construction of a naive expansion because of the complicated form of the function $g^{(k)}$ in (3.4). This clear procedure is helpful to find an exact Lie symmetry group and, therefore, to find an exact solution of systems.

Numerous kinds of singular perturbation methods have been developed thus far. Examples include the renormalization group method [11, 13, 14], the normal form method [15], center manifold reduction [16], the multiple time-scale method [15] the averaging method [17], the canonical perturbation theory [18], and geometric singular perturbation theory [19]. All of those methods as well as the method presented in this paper result in the same reduced equation, (3.25). With the method presented in this paper, we can prove that the reduced equations surely describe the asymptotic behavior well [10], that is to say, it is proved that the solution of the reduced equation is exactly the sum of the terms which diverge most rapidly among terms appearing in the naive expansion. For systems of difference equations, although the proof has not been summarized yet, we can easily see in the same way that the corresponding fact also holds.

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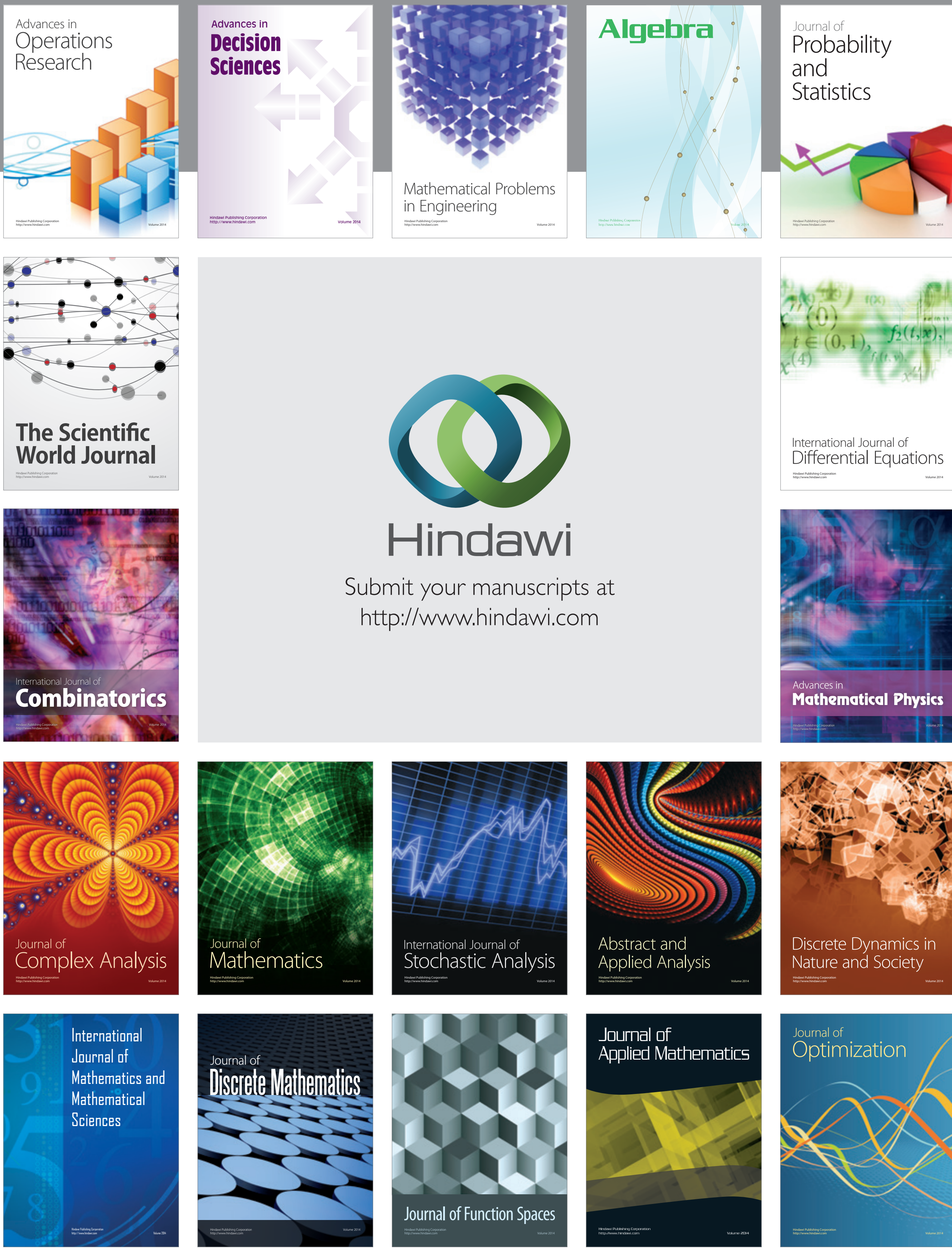
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