Hindawi Publishing Corporation
Advances in Mathematical Physics
Volume 2013, Article ID 806984, 4 pages
http://dx.doi.org/10.1155/2013/806984

Research Article

Solving Abel's Type Integral Equation with Mikusinski's Operator of Fractional Order

Ming Li[1] and Wei Zhao[2]

[1] *School of Information Science & Technology, East China Normal University, No. 500, Dong-Chuan Road, Shanghai 200241, China*
[2] *Department of Computer and Information Science, University of Macau, Avenida Padre Tomas Pereira, Taipa, Macau*

Correspondence should be addressed to Ming Li; ming_lihk@yahoo.com

Received 21 April 2013; Accepted 10 May 2013

Academic Editor: Carlo Cattani

This paper gives a novel explanation of the integral equation of Abel's type from the point of view of Mikusinski's operational calculus. The concept of the inverse of Mikusinski's operator of fractional order is introduced for constructing a representation of the solution to the integral equation of Abel's type. The proof of the existence of the inverse of the fractional Mikusinski operator is presented, providing an alternative method of treating the integral equation of Abel's type.

1. Introduction

Abel studied a physical problem regarding the relationship between kinetic and potential energies for falling bodies [1]. One of his integrals stated in [1] is expressed in the form

$$f(t) = \int_a^t \frac{g(u)}{\sqrt{t-u}} du, \quad a > 0, \tag{1}$$

where $f(t)$ is known, but $g(t)$ is unknown. The previous expression is in the literature nowadays called Abel's integral equation [2]. In addition to (1), Abel also worked on the integral equation in [1] in the following form:

$$f(t) = \int_a^t \frac{g(u)}{(t-u)^{\lambda}} du, \quad a > 0,\ 0 < \lambda < 1,\ a \le t \le b, \tag{2}$$

which is usually termed the integral equation of Abel's type [3] or the generalized Abel integral equation [4]. The function $(t-u)^{-\lambda}$ may be called Abel's kernel. It is seen that (1) is a special case of (2) for $\lambda = 1/2$. This paper is in the aspect of (2). Without generality losing, for the purpose of facilitating the discussions, we multiply the left side of (1) with the constant $1/\Gamma(\lambda)$, and let $a = 0$. That is, we rewrite (2) by

$$f(t) = \frac{1}{\Gamma(\lambda)} \int_0^t \frac{g(u)}{(t-u)^{\lambda}} du, \quad 0 < \lambda < 1,\ 0 \le t \le b. \tag{3}$$

The integral equation of Abel's type attracts the interests of mathematicians and physicists. In mathematics, for example, for solving the integral equation of Abel's type, [5] discusses a transformation technique, [6] gives a method of orthogonal polynomials, [7] adopts the method of integral operators, [8, 9] utilize the fractional calculus, [10] is with the Bessel functions, [11, 12] study the wavelet methods, [13, 14] describe the methods based on semigroups, [15] uses the almost Bernstein operational matrix, and so forth [16, 17], just to mention a few. Reference [18] represents a nice description of the importance of Abel's integral equations in mathematics as well as engineering, citing [19–23] for the various applications of Abel's integral equations.

The above stands for a sign that the theory of Abel's integral equations is developing. New methods for solving such a type of equations are demanded in this field. This paper presents a new method to describe the integral equation of Abel's type from the point of view of the Mikusinski operator of fractional order. In addition, we will give a solution to the integral equation of Abel's type by using the inverse of the Mikusinski operator of fractional order.

The remainder of this article is organized as follows. In Section 2, we shall express the integral equation of the Abel's type using the Mikusinski operator of fractional order and give the solution to that type of equation in the constructive way based on the inverse of the fractional-order Mikusinski operator. Section 3 consists of two parts. One is the proof of the existence of the inverse of the fractional-order Mikusinski operator. The other is the computation of the solution to Abel's type integral equation. Finally, Section 4 concludes the paper.

2. Constructive Solution Based on Fractional-Order Mikusinski Operator

Denote the operation of Mikusinski's convolution by $\otimes$. Let $\oplus$ be the operation of its inverse. Then, for $a(t), b(t) \in C(0,\infty)$, one has

$$a(t) \otimes b(t) = \int_0^t a(t-\tau)\, b(\tau)\, d\tau = c(t). \tag{4}$$

The inverse of the previous expression is the deconvolution, which is denoted by (see [24–26])

$$c(t) \oplus a(t) = b(t), \qquad c(t) \oplus b(t) = a(t). \tag{5}$$

In (4) and (5), the constraint $a(t), b(t) \in C(0,\infty)$ may be released. More precisely, we assume that $a(t)$ and $b(t)$ may be generalized functions. Therefore, the Diract-δ function in the following is the identity in this convolution system. That is,

$$a(t) \otimes \delta(t) = \delta(t) \otimes a(t) = a(t). \tag{6}$$

Consequently,

$$a(t) \oplus a(t) = \delta(t). \tag{7}$$

Let l be an operator that corresponds to the function $1(t)$ such that

$$la(t) = 1(t) \otimes a(t) = \int_0^t a(\tau)\, d\tau. \tag{8}$$

Therefore, the operator l^2 implies

$$l^2 \Longleftrightarrow 1(t) \otimes 1(t) = \int_0^t d\tau = \frac{t}{1}. \tag{9}$$

For $n = 1, \ldots$, consequently, we have

$$l^n \Longleftrightarrow \frac{t^{n-1}}{(n-1)!}, \tag{10}$$

where $0! = 1$.

The Cauchy integral formula may be expressed by using l^n, so that

$$l^n g(t) = \frac{t^{n-1}}{(n-1)!} \otimes g(t) = \int_0^t \frac{(t-\tau)^{n-1}}{(n-1)!} g(\tau)\, d\tau. \tag{11}$$

Generalizing l^n to l^λ in (12) for $\lambda > 0$ yields the Mikusinski operator of fractional order given by

$$l^\tau \Longleftrightarrow \frac{t^{\lambda-1}}{(\lambda-1)!} = \frac{t^{\lambda-1}}{\Gamma(\lambda)}. \tag{12}$$

Thus, taking into account (12), we may represent the integral equation of Abel's type by

$$l^\lambda g(t) = \frac{t^{\lambda-1}}{\Gamma(\lambda)} \otimes g(t) = \int_0^t \frac{(t-\tau)^{\lambda-1}}{\Gamma(\lambda)} g(\tau)\, d\tau = f(t). \tag{13}$$

Rewrite the above by

$$l^\lambda g(t) = f(t). \tag{14}$$

Then, the solution to Able's type integral equation (3) may be represented by

$$g(t) = l^{-\lambda} f(t), \tag{15}$$

where $l^{-\lambda}$ is the inverse of l^λ.

There are two questions in the constructive solution expressed by (15). One is whether $l^{-\lambda}$ exists. The other is how to represent its computation. We shall discuss the answers next section.

3. Results

3.1. Existence of the Inverse of Mikusinski's Operator of Order λ. Let $\mathbf{G}$ and $\mathbf{F}$ be two normed spaces for $g(t) \in \mathbf{G}$ and $f(t) \in \mathbf{F}$, respectively. Then, the operator l^λ regarding Able's type integral equation (13) may be expressed by

$$l^\lambda : \mathbf{G} \longrightarrow \mathbf{F}. \tag{16}$$

The operator l^λ is obviously linear. Note that (3) is convergent [1]. Thus, one may assume that

$$m \le \int_0^b \left| \frac{(t-\tau)^{\lambda-1}}{\Gamma(\lambda)} g(\tau) \right| d\tau \le M, \tag{17}$$

where

$$m \ge 0, \quad M \ge 0. \tag{18}$$

Define the norm of $f(t)$ by

$$\| f(t) \| = \max_{0<t<b} f(t). \tag{19}$$

Then, we have

$$\left\| l^\lambda g(t) \right\| \le M \| f(t) \|. \tag{20}$$

The above implies that l^λ is bounded. Accordingly, it is continuous [27, 28].

Since

$$\left\| l^\lambda g(t) \right\| \ge m \| f(t) \|, \tag{21}$$

$l^{-\lambda}$ exists. Moreover, the inverse of l^λ is continuous and bounded according to the inverse operator theorem of Banach [27, 28]. This completes the proof of (15).

3.2. Computation Formula. According to the previous analysis, $l^{-\lambda}$ exists. It actually corresponds to the differential of order λ. Thus,

$$g(t) = l^{-\lambda} f(t) = \frac{d^{\lambda} f(t)}{dt^{\lambda}}. \tag{22}$$

In (13), we write $\int_0^t (((t-\tau)^{\lambda-1}/\Gamma(\lambda)) g(\tau)) d\tau = f(t)$ by

$$\int_0^t (t-\tau)^{\lambda-1} g(\tau) \, d\tau = \Gamma(\lambda) f(t). \tag{23}$$

Following [29, p. 13, p. 527], [30], therefore,

$$\begin{aligned} g(t) = l^{-\lambda} f(t) &= \frac{\sin(\pi\lambda)}{\pi} \frac{d}{dt} \int_0^t \frac{\Gamma(\lambda) f(u)}{(t-u)^{1-\lambda}} du \\ &= \frac{\Gamma(\lambda)\sin(\pi\lambda)}{\pi} \left[\frac{f(0)}{t^{1-\lambda}} + \int_0^t \frac{f'(t)\, dt}{(t-u)^{1-\lambda}} \right]. \end{aligned} \tag{24}$$

Since

$$\frac{\sin(\pi\lambda)}{\pi} = \frac{1}{\Gamma(\lambda)\Gamma(1-\lambda)}, \tag{25}$$

we write (24) by

$$g(t) = \frac{1}{\Gamma(1-\lambda)} \left[\frac{f(0)}{t^{1-\lambda}} + \int_0^t \frac{f'(t)\, dt}{(t-u)^{1-\lambda}} \right]. \tag{26}$$

In the solution (26), if $f(0) = 0$, one has

$$g(t) = \frac{1}{\Gamma(1-\lambda)} \int_0^t \frac{f'(t)\, dt}{(t-u)^{1-\lambda}}, \tag{27}$$

which is a result described by Gelfand and Vilenkin in [9, Section 5.5].

Note that Mikusinski's operational calculus is a tool usually used for solving linear differential equations [24–26], but we use it in this research for the integral equation of the Abel's type from a view of fractional calculus. In addition, we suggest that the idea in this paper may be applied to studying other types of equations, for instance, those in [31–50], to make the possible applications of Mikusinski's operational calculus a step further.

4. Conclusions

We have presented the integral equation of Abel's type using the method of the Mikusinski operational calculus. The constructive representation of the solution to Abel's type integral equation has been given with the Mikusinski operator of fractionally negative order, giving a novel interpretation of the solution to Abel's type integral equation.

Acknowledgments

This work was supported in part by the 973 plan under the Project Grant no. 2011CB302800 and by the National Natural Science Foundation of China under the Project Grant nos. 61272402, 61070214, and 60873264.

References

[1] N. H. Abel, "Solution de quelques problèmes à l'aide d'intégrales définies," *Magaziu for Naturvidenskaberue*, Alu-gang I, Bînd 2, Christiania, pp. 11–18, 1823.

[2] R. Gorenflo and S. Vessella, *Abel Integral Equations*, Springer, 1991.

[3] P. P. B. Eggermont, "On Galerkin methods for Abel-type integral equations," *SIAM Journal on Numerical Analysis*, vol. 25, no. 5, pp. 1093–1117, 1988.

[4] A. Chakrabarti and A. J. George, "Diagonalizable generalized Abel integral operators," *SIAM Journal on Applied Mathematics*, vol. 57, no. 2, pp. 568–575, 1997.

[5] J. R. Hatcher, "A nonlinear boundary problem," *Proceedings of the American Mathematical Society*, vol. 95, no. 3, pp. 441–448, 1985.

[6] G. N. Minerbo and M. E. Levy, "Inversion of Abel's integral equation by means of orthogonal polynomials," *SIAM Journal on Numerical Analysis*, vol. 6, no. 4, pp. 598–616, 1969.

[7] J. D. Tamarkin, "On integrable solutions of Abel's integral equation," *The Annals of Mathematics*, vol. 31, no. 2, pp. 219–229, 1930.

[8] D. B. Sumner, "Abel's integral equation as a convolution transform," *Proceedings of the American Mathematical Society*, vol. 7, no. 1, pp. 82–86, 1956.

[9] I. M. Gelfand and K. Vilenkin, *Generalized Functions*, vol. 1, Academic Press, New York. NY, USA, 1964.

[10] C. E. Smith, "A theorem of Abel and its application to the development of a function in terms of Bessel's functions," *Transactions of the American Mathematical Society*, vol. 8, no. 1, pp. 92–106, 1907.

[11] S. Sohrabi, "Comparison Chebyshev wavelets method with BPFs method for solving Abel's integral equation," *Ain Shams Engineering Journal*, vol. 2, no. 3-4, pp. 249–254, 2011.

[12] A. Antoniadis, J. Q. Fan, and I. Gijbels, "A wavelet method for unfolding sphere size distributions," *The Canadian Journal of Statistics*, vol. 29, no. 2, pp. 251–268, 2001.

[13] R. J. Hughes, "Semigroups of unbounded linear operators in Banach space," *Transactions of the American Mathematical Society*, vol. 230, pp. 113–145, 1977.

[14] K. Ito and J. Turi, "Numerical methods for a class of singular integro-differential equations based on semigroup approximation," *SIAM Journal on Numerical Analysis*, vol. 28, no. 6, pp. 1698–1722, 1991.

[15] O. P. Singh, V. K. Singh, and R. K. Pandey, "A stable numerical inversion of Abel's integral equation using almost Bernstein operational matrix," *Journal of Quantitative Spectroscopy and Radiative Transfer*, vol. 111, no. 1, pp. 245–252, 2010.

[16] R. K. Pandey, O. P. Singh, and V. K. Singh, "Efficient algorithms to solve singular integral equations of Abel type," *Computers & Mathematics with Applications*, vol. 57, no. 4, pp. 664–676, 2009.

[17] M. Khan and M. A. Gondal, "A reliable treatment of Abel's second kind singular integral equations," *Applied Mathematics Letters*, vol. 25, no. 11, pp. 1666–1670, 2012.

[18] R. Weiss, "Product integration for the generalized Abel equation," *Mathematics of Computation*, vol. 26, pp. 177–190, 1972.

[19] W. C. Brenke, "An application of Abel's integral equation," *The American Mathematical Monthly*, vol. 29, no. 2, pp. 58–60, 1922.

[20] E. B. Hansen, "On drying of laundry," *SIAM Journal on Applied Mathematics*, vol. 52, no. 5, pp. 1360–1369, 1992.

[21] A. T. Lonseth, "Sources and applications of integral equations," *SIAM Review*, vol. 19, no. 2, pp. 241–278, 1977.

[22] Y. H. Jang, "Distribution of three-dimensional islands from two-dimensional line segment length distribution," *Wear*, vol. 257, no. 1-2, pp. 131–137, 2004.

[23] L. Bougoffa, R. C. Rach, and A. Mennouni, "A convenient technique for solving linear and nonlinear Abel integral equations by the Adomian decomposition method," *Applied Mathematics and Computation*, vol. 218, no. 5, pp. 1785–1793, 2011.

[24] J. Mikusiński, *Operational Calculus*, Pergamon Press, 1959.

[25] T. K. Boehme, "The convolution integral," *SIAM Review*, vol. 10, no. 4, pp. 407–416, 1968.

[26] G. Bengochea and L. Verde-Star, "Linear algebraic foundations of the operational calculi," *Advances in Applied Mathematics*, vol. 47, no. 2, pp. 330–351, 2011.

[27] V. I. Istrăţescu, *Introduction to Linear Operator Theory*, vol. 65, Marcel Dekker, New York, NY, USA, 1981.

[28] M. Li and W. Zhao, *Analysis of Min-Plus Algebra*, Nova Science Publishers, 2011.

[29] A. D. Polyanin and A. V. Manzhirov, *Handbook of Integral Equations*, CRC Press, Boca Raton, Fla, USA, 1998.

[30] http://eqworld.ipmnet.ru/.

[31] C. Cattani and A. Kudreyko, "Harmonic wavelet method towards solution of the Fredholm type integral equations of the second kind," *Applied Mathematics and Computation*, vol. 215, no. 12, pp. 4164–4171, 2010.

[32] C. Cattani, M. Scalia, E. Laserra, I. Bochicchio, and K. K. Nandi, "Correct light deflection in Weyl conformal gravity," *Physical Review D*, vol. 87, no. 4, Article ID 47503, 4 pages, 2013.

[33] C. Cattani, "Fractional calculus and Shannon wavelet," *Mathematical Problems in Engineering*, vol. 2012, Article ID 502812, 26 pages, 2012.

[34] M. Carlini, T. Honorati, and S. Castellucci, "Photovoltaic greenhouses: comparison of optical and thermal behaviour for energy savings," *Mathematical Problems in Engineering*, vol. 2012, Article ID 743764, 10 pages, 2012.

[35] M. Carlini and S. Castellucci, "Modelling the vertical heat exchanger in thermal basin," in *Computational Science and Its Applications*, vol. 6785 of *Lecture Notes in Computer Science*, pp. 277–286, Springer, 2011.

[36] M. Carlini, C. Cattani, and A. Tucci, "Optical modelling of square solar concentrator," in *Computational Science and Its Applications*, vol. 6785 of *Lecture Notes in Computer Science*, pp. 287–295, Springer, 2011.

[37] E. G. Bakhoum and C. Toma, "Mathematical transform of traveling-wave equations and phase aspects of Quantum interaction," *Mathematical Problems in Engineering*, vol. 2010, Article ID 695208, 15 pages, 2010.

[38] E. G. Bakhoum and C. Toma, "Specific mathematical aspects of dynamics generated by coherence functions," *Mathematical Problems in Engineering*, vol. 2011, Article ID 436198, 10 pages, 2011.

[39] C. Toma, "Advanced signal processing and command synthesis for memory-limited complex systems," *Mathematical Problems in Engineering*, vol. 2012, Article ID 927821, 13 pages, 2012.

[40] G. Toma, "Specific differential equations for generating pulse sequences," *Mathematical Problems in Engineering*, vol. 2010, Article ID 324818, 11 pages, 2010.

[41] J. Yang, Y. Chen, and M. Scalia, "Construction of affine invariant functions in spatial domain," *Mathematical Problems in Engineering*, vol. 2012, Article ID 690262, 11 pages, 2012.

[42] J. W. Yang, Z. R. Chen, W.-S. Chen, and Y. J. Chen, "Robust affine invariant descriptors," *Mathematical Problems in Engineering*, vol. 2011, Article ID 185303, 15 pages, 2011.

[43] Z. Jiao, Y.-Q. Chen, and I. Podlubny, "Distributed-Order Dynamic Systems," *Springer*, 2011.

[44] H. Sheng, Y.-Q. Chen, and T.-S. Qiu, *Fractional Processes and Fractional Order Signal Processing*, Springer, 2012.

[45] H. G. Sun, Y.-Q. Chen, and W. Chen, "Random-order fractional differential equation models," *Signal Processing*, vol. 91, no. 3, pp. 525–530, 2011.

[46] S. V. Muniandy, W. X. Chew, and C. S. Wong, "Fractional dynamics in the light scattering intensity fluctuation in dusty plasma," *Physics of Plasmas*, vol. 18, no. 1, Article ID 013701, 8 pages, 2011.

[47] H. Asgari, S. V. Muniandy, and C. S. Wong, "Stochastic dynamics of charge fluctuations in dusty plasma: a non-Markovian approach," *Physics of Plasmas*, vol. 18, no. 8, Article ID 083709, 4 pages, 2011.

[48] C. H. Eab and S. C. Lim, "Accelerating and retarding anomalous diffusion," *Journal of Physics A*, vol. 45, no. 14, Article ID 145001, 17 pages, 2012.

[49] C. H. Eab and S. C. Lim, "Fractional Langevin equations of distributed order," *Physical Review E*, vol. 83, no. 3, Article ID 031136, 10 pages, 2011.

[50] L.-T. Ko, J.-E. Chen, Y.-S. Shieh, H.-C. Hsin, and T.-Y. Sung, "Difference-equation-based digital frequency synthesizer," *Mathematical Problems in Engineering*, vol. 2012, Article ID 784270, 12 pages, 2012.


Advances in Operations Research
Advances in Decision Sciences
Mathematical Problems in Engineering
Algebra
Journal of Probability and Statistics
The Scientific World Journal
International Journal of Differential Equations
Hindawi
Submit your manuscripts at
http://www.hindawi.com
International Journal of Combinatorics
Advances in Mathematical Physics
Journal of Complex Analysis
Journal of Mathematics
International Journal of Stochastic Analysis
Abstract and Applied Analysis
Discrete Dynamics in Nature and Society
International Journal of Mathematics and Mathematical Sciences
Journal of Discrete Mathematics
Journal of Function Spaces
Journal of Applied Mathematics
Journal of Optimization