

Research Article

A Mathematical Characterization for Patterns of a Keller-Segel Model with a Cubic Source Term

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This paper deals with a Neumann boundary value problem for a Keller-Segel model with a cubic source term in a d -dimensional box ($d = 1, 2, 3$), which describes the movement of cells in response to the presence of a chemical signal substance. It is proved that, given any general perturbation of magnitude δ , its nonlinear evolution is dominated by the corresponding linear dynamics along a finite number of fixed fastest growing modes, over a time period of the order $\ln(1/\delta)$. Each initial perturbation certainly can behave drastically differently from another, which gives rise to the richness of patterns. Our results provide a mathematical description for early pattern formation in the model.

1. Introduction

Keller and Segel in their pioneering work [1] proposed the following model

$$\begin{aligned} U_t &= \nabla \cdot (D_u \nabla U - \chi U \nabla V), \\ V_t &= D_v \nabla^2 V + \alpha U - \beta V, \end{aligned} \quad (1)$$

where $U(x, t)$ is cell density, $V(x, t)$ is chemoattractant concentration, D_u is the amoeboid motility, χ is the chemotactic sensitivity, D_v is the diffusion rate of cyclic adenosine monophosphate (cAMP), α is the rate of cAMP secretion per unit density of amoebae, and β is the rate of degradation of cAMP in environment. Keller and Segel wanted to model the chemotactic movement of the cellular slime mold *Dictyostelium discoideum* during its aggregation phase, where population growth does not occur. Therefore, they considered a population in the absence of “death” and “birth.” For some main results on the Keller-Segel model, please see [2–4] and references therein.

Recently, Guo and Hwang in [5] investigated the nonlinear dynamics near an unstable constant equilibrium of the Keller-Segel model satisfying the Neumann boundary

conditions for $U(x, t)$ and $V(x, t)$ on a d -dimensional box $\mathbb{T}^d = (0, \pi)^d$ ($d = 1, 2, 3$); that is,

$$\frac{\partial U}{\partial x_i} = \frac{\partial V}{\partial x_i} = 0, \quad \text{at } x_i = 0, \pi, \text{ for } 1 \leq i \leq d. \quad (2)$$

Let $[\bar{U}, \bar{V}]$ be the uniform constant solution of the Keller-Segel model, and $u(x, t) = U(x, t) - \bar{U}$, $v(x, t) = V(x, t) - \bar{V}$. Then $[u, v]$ satisfies the equivalent Keller-Segel system below:

$$\begin{aligned} u_t &= D_u \nabla^2 u - \chi \bar{U} \nabla^2 v - \chi \nabla \cdot (u \nabla v), \\ v_t &= D_v \nabla^2 v + \alpha u - \beta v. \end{aligned} \quad (3)$$

Guo and Hwang proved that linear fastest growing modes determine unstable patterns for the above system. Their result can be interpreted as a rigorous mathematical characterization for early pattern formation in the Keller-Segel model.

In recent years, more and more attention has been given to the Keller-Segel model with the reaction terms, that is, the following chemotaxis-diffusion-growth model:

$$\begin{aligned} U_t &= \nabla \cdot (D_u \nabla U - \chi U \nabla V) + f(U), \\ V_t &= D_v \nabla^2 V + \alpha U - \beta V. \end{aligned} \quad (4)$$

For $f(U) = rU(1 - U/K)$, Painter and Hillen [6] demonstrated the capacity of the above model to self-organize into multiple cellular aggregations which, according to its position in parameter space, either form a stationary pattern or undergo a sustained spatiotemporal sequence of merging (two aggregations coalesce) and emerging (a new aggregation appears). This spatiotemporal patterning can be further subdivided into either a time-periodic or time-irregular fashion. Numerical explorations into the latter indicate a positive Lyapunov exponent (sensitive dependence to initial conditions) together with a rich bifurcation structure. In particular, they found stationary patterns that bifurcate onto a path of periodic patterns which, prior to the onset of spatiotemporal irregularity, undergo a “periodic-doubling” sequence. Based on these results and comparisons with other systems, they argued that the spatiotemporal irregularity observed here describes a form of spatiotemporal chaos.

For $f(U) = rU(1 - U)$, Banerjee et al. [7] showed that the dynamics of the chemotaxis-diffusion-growth model may lead to steady states, to divergencies in a finite time, and to the formation of spatiotemporal irregular patterns. The latter, in particular, appears to be chaotic in part of the range of bounded solutions, as demonstrated by the analysis of wavelet power spectra. Steady states are achieved with sufficiently large values of the chemotactic coefficient χ and/or with growth rate r below a critical value r_c . For $r > r_c$, the solutions of the differential equations of the model diverge in a finite time. They also reported on the pattern formation regime, for different value of χ , r and of the diffusion coefficient D_u . For the same $f(U)$, Kuto et al. [8] considered some qualitative behaviors of stationary solutions from global and local (bifurcation) viewpoints. They studied the asymptotic behavior of stationary solutions as the chemotactic intensity grows to infinity and construct local bifurcation branches of stripe and hexagonal stationary solutions in the special case when the habitat domain is a rectangle. For this case, the directions of the branches near the bifurcation points are also obtained. Finally, they exhibited several numerical results for the stationary and oscillating patterns.

In [9], Okuda and Osaki studied the chemotaxis-diffusion-growth model with $f = pU(1 - U) + qU(1 - U)(U - v)$ in a rectangular domain by applying the center manifold theory, where constant $v \in (0, 1)$ and either $p > 0$, $q = 0$, or $p = 0$, $q > 0$. It is observed that the trivial solutions are destabilized due to the chemotaxis term. They obtained the normal form on the center manifold, and it is proved that the locally asymptotically stable hexagonal patterns exist.

Another extended formation of logistic source term is the cubic source term $f = U(b_1 + b_2U - b_3U^2)$, where $b_1 \geq 0$ is the intrinsic growth rate, the sign of b_2 is undetermined, $b_3 > 0$ is a positive constant, and $b_2U - b_3U^2$ is the density restriction term (see [10, 11] for more information and references). Recently, Cao and Fu in [11] studied global existence and convergence of solutions to a cross-diffusion cubic predator-prey system with stage structure for the prey. In this paper, we investigate dynamics of the chemotaxis-diffusion-growth

model with the source term $f = (b_1 + b_2U - b_3U^2)U$; that is,

$$\begin{aligned} U_t &= \nabla(D_u \nabla U - \chi U \nabla V) \\ &\quad + (b_1 + b_2U - b_3U^2)U, \\ V_t &= D_v \nabla^2 V + \alpha U - \beta V, \end{aligned} \quad (5)$$

where D_u , χ , D_v , α , and β are positive constants and $U(x, t)$, $V(x, t)$ satisfies the Neumann boundary conditions. We will prove that given any general perturbation of magnitude δ , its nonlinear evolution is dominated by the corresponding linear dynamics along a fixed finite number of fastest growing modes, over a time period of $\ln(1/\delta)$. Each initial perturbation certainly can behave drastically differently from another, which gives rise to the richness of patterns. Our results provide a mathematical description for early pattern formation in the model (5).

The organization of this paper is as follows: In Section 2, we prove that the positive constant equilibrium solution of (5) without chemotaxis is globally asymptotically stable if $b_1 > 0$. In Section 3, we investigate the growing modes of (5). In Section 4, we present and prove the Bootstrap lemma. In Section 5, given any general perturbation of magnitude δ , we prove that its nonlinear evolution is dominated by the corresponding linear dynamics along a fixed finite number of fastest growing modes, over a time period of $\ln(1/\delta)$.

2. Stability of Positive Equilibrium Point of (5) without Chemotaxis

The corresponding semilinear system of (5) without chemotaxis is as follows:

$$\begin{aligned} U_t &= D_u \nabla^2 U + (b_1 + b_2U - b_3U^2)U, \quad x \in \mathbb{T}^d, \quad t > 0, \\ V_t &= D_v \nabla^2 V + \alpha U - \beta V, \quad x \in \mathbb{T}^d, \quad t > 0, \\ \frac{\partial U}{\partial x_i} &= \frac{\partial V}{\partial x_i} = 0, \quad x_i = 0, \pi, \quad 1 \leq i \leq d, \end{aligned} \quad (6)$$

$$\begin{aligned} U(x, 0) &= U_0(x) \geq 0, \\ V(x, 0) &= V_0(x) \geq 0, \quad x \in \mathbb{T}^d \quad (d = 1, 2, 3). \end{aligned}$$

Obviously, $[\bar{U}, \bar{V}] \equiv [(b_2 + \sqrt{b_2^2 + 4b_1b_3})/2b_3, \alpha(b_2 + \sqrt{b_2^2 + 4b_1b_3})/2\beta b_3]$ is a positive equilibrium point of (6) and only if either of the following two cases happens: (i) $b_1 > 0$, $b_2 \in \mathbb{R}$, (ii) $b_1 = 0$, $b_2 > 0$. In the following we will discuss the stability of $[\bar{U}, \bar{V}]$ in (6).

Let $W = [U, V]$, $\bar{W} = [\bar{U}, \bar{V}]$, and $0 = \mu_1 < \mu_2 < \mu_3 < \dots$ be the eigenvalues of the operator $-\Delta$ on \mathbb{T}^d ($d = 1, 2, 3$) with the homogeneous Neumann boundary condition, and let $E(\mu_i)$ be the eigenspace corresponding to μ_i in $L^2(\mathbb{T}^d)$. Let $X = [L^2(\mathbb{T}^d)]^2$, $\{\phi_{ij} : j = 1, \dots, \dim E(\mu_i)\}$ be an orthonormal basis of $E(\mu_i)$, and $X_{ij} = \{c \cdot \phi_{ij} \mid c \in \mathbb{R}^2\}$. Then

$$X = \Phi_{i=1}^{\infty} X_i, \quad X_i = \Phi_{j=1}^{\dim E(\mu_i)} X_{ij}. \quad (7)$$

Let $\mathfrak{D} = \text{diag}(D_u, D_v)$ and $\mathfrak{L} = \mathfrak{D}\Delta + G_W(\bar{W})$, where

$$G_W(\bar{W}) = \begin{pmatrix} -3b_3\bar{U}^2 + 2b_2\bar{U} + b_1 & 0 \\ \alpha & -\beta \end{pmatrix}. \quad (8)$$

Then the linearization of (6) at $[\bar{U}, \bar{V}]$ is $W_t = \mathfrak{L}(W - \bar{W})$.

For each $i \geq 1$, X_i is invariant under the operator \mathfrak{L} , and λ is an eigenvalue of \mathfrak{L} on X_i if and only if it is an eigenvalue of the matrix

$$-\mu_i \mathfrak{D} + G_W(\bar{W}) = \begin{pmatrix} -\mu_i D_u - 3b_3\bar{U}^2 + 2b_2\bar{U} + b_1 & 0 \\ \alpha & -\mu_i D_v - \beta \end{pmatrix}. \quad (9)$$

Notice that $(b_2 + \sqrt{b_2^2 + 3b_1b_3})/3b_3$ is the positive root of $-3b_3x^2 + 2b_2x + b_1 = 0$ and

$$\bar{U} > \frac{b_2 + \sqrt{b_2^2 + 3b_1b_3}}{3b_3}. \quad (10)$$

Thus, $-\mu_i \mathfrak{D} + G_W(\bar{W})$ has two negative eigenvalues $-\mu_i D_u - 3b_3\bar{U}^2 + 2b_2\bar{U} + b_1$ and $-\mu_i D_v - \beta$. It follows from [12, Theorem 5.1.1] that $[\bar{U}, \bar{V}]$ is locally asymptotically stable.

Let $[U, V]$ be a unique nonnegative global solution of (6). It is not hard to verify by the maximum principle that

$$\begin{aligned} 0 \leq U &\leq \max\{\bar{U}, |U_0(x)|_\infty\}, \\ 0 \leq V &\leq \max\{\bar{V}, |V_0(x)|_\infty\}, \end{aligned} \quad \forall t \geq 0. \quad (11)$$

Moreover, if $U_0, V_0 \geq (\neq) 0$, then $U(x, t) > 0, V(x, t) > 0$, for all $t > 0$.

According to the main result in [13], we have

$$\begin{aligned} \|U(\cdot, t)\|_{C^{2,\alpha}(\mathbb{T}^d)} &\leq C, \\ \|V(\cdot, t)\|_{C^{2,\alpha}(\mathbb{T}^d)} &\leq C, \end{aligned} \quad \forall t \geq 1. \quad (12)$$

(i) If $b_1 > 0, b_2 \in \mathbb{R}$, then $b_3\bar{U} - b_2 > 0$. We define the Lyapunov function

$$E(t) = \int_{\mathbb{T}^d} \left[p \left(U - \bar{U} - \bar{U} \ln \frac{U}{\bar{U}} \right) + (V - \bar{V})^2 \right] dx, \quad (13)$$

where $p = 2\alpha^2/(\beta(b_3\bar{U} - b_2))$.

Calculating the derivative of $E(t)$ along positive solution of (6) by integration by parts and the Cauchy inequality, we have

$$\begin{aligned} \frac{dE(u, v)}{dt} &\leq - \int_{\mathbb{T}^d} \left\{ \frac{pD_u\bar{U}}{U^2} |\nabla U|^2 + 2D_v|\nabla V|^2 \right\} dx \\ &\quad - \int_{\mathbb{T}^d} \left\{ p(b_3U + b_3\bar{U} - b_2)(U - \bar{U})^2 \right. \\ &\quad \left. - 2\alpha(U - \bar{U})(V - \bar{V}) + 2\beta(V - \bar{V})^2 \right\} dx \\ &\leq - \int_{\mathbb{T}^d} \left\{ p(b_3\bar{U} - b_2)(U - \bar{U})^2 \right. \\ &\quad \left. - 2\alpha(U - \bar{U})(V - \bar{V}) + 2\beta(V - \bar{V})^2 \right\} dx \\ &\leq -\frac{\alpha^2}{\beta} \int_{\mathbb{T}^d} (U - \bar{U})^2 dx - \beta \int_{\mathbb{T}^d} (V - \bar{V})^2 dx. \end{aligned} \quad (14)$$

(ii) If $b_1 = 0, b_2 > 0$, then $b_2 = b_3\bar{U}$. We define the Lyapunov function

$$E(t) = \int_{\mathbb{T}^d} \left[\frac{q}{U} (U - \bar{U})^2 (V - \bar{V})^2 \right] dx, \quad (15)$$

where $q = 2\alpha^2/\beta b_3\bar{U}$.

Calculating in the same way as (14), we have

$$\begin{aligned} \frac{dE(u, v)}{dt} &\leq - \int_{\mathbb{T}^d} 2\frac{qD_u\bar{U}^2}{u^3} |\nabla U|^2 + D_v|\nabla V|^2 dx \\ &\quad - \int_{\mathbb{T}^d} q b_3 (U + \bar{U})(U - \bar{U})^2 \\ &\quad \quad - 2\alpha(U - \bar{U})(V - \bar{V}) + 2\beta(V - \bar{V})^2 dx \\ &\leq - \int_{\mathbb{T}^d} q b_3 \bar{U} (U - \bar{U})^2 - 2\alpha(U - \bar{U})(V - \bar{V}) \\ &\quad \quad + 2\beta(V - \bar{V})^2 dx \\ &\leq -\frac{\alpha^2}{\beta} \int_{\mathbb{T}^d} (U - \bar{U})^2 dx - \beta \int_{\mathbb{T}^d} (V - \bar{V})^2 dx. \end{aligned} \quad (16)$$

Combining (12)–(16) and Lemma 3.2 in [11], we conclude that

$$\begin{aligned} \lim_{t \rightarrow \infty} \|U(\cdot, t) - \bar{U}\|_{L^2(\mathbb{T}^d)} &= 0, \\ \lim_{t \rightarrow \infty} \|V(\cdot, t) - \bar{V}\|_{L^2(\mathbb{T}^d)} &= 0. \end{aligned} \quad (17)$$

The global asymptotic stability of $[\bar{U}, \bar{V}]$ follows from (17) and the local stability of $[\bar{U}, \bar{V}]$.

Theorem 1. *The positive equilibrium point $[\bar{U}, \bar{V}]$ of (6) is locally asymptotically stable. If either $b_1 > 0$, $b_2 \in \mathbb{R}$ or $b_1 = 0$, $b_2 > 0$ holds, then $[\bar{U}, \bar{V}]$ is globally asymptotically stable.*

3. Growing Modes in the System (5)

Let $u(\mathbf{x}, t) = U(\mathbf{x}, t) - \bar{U}$, $v(\mathbf{x}, t) = V(\mathbf{x}, t) - \bar{V}$. Then

$$\begin{aligned} u_t &= D_u \nabla^2 u - \chi \bar{U} \nabla^2 v - \left(3b_3^2 \bar{U}^2 - 2b_2 \bar{U} - b_1 \right) u \\ &\quad - \chi \nabla (u \nabla v) + \left(b_2 - 3b_3 \bar{U} \right) u^2 - b_3 u^3, \\ v_t &= D_v \nabla^2 v + \alpha u - \beta v. \end{aligned} \quad (18)$$

The corresponding linearized system takes the form

$$\begin{aligned} u_t &= D_u \nabla^2 u - \chi \bar{U} \nabla^2 v - \left(3b_3^2 \bar{U}^2 - 2b_2 \bar{U} - b_1 \right) u, \\ v_t &= D_v \nabla^2 v + \alpha u - \beta v. \end{aligned} \quad (19)$$

Let $\mathbf{w}(\mathbf{x}, t) \equiv [u(\mathbf{x}, t), v(\mathbf{x}, t)]$, $\mathbf{q} = (q_1, \dots, q_d) \in \Omega = \mathbb{N}^d$, and $e_{\mathbf{q}}(\mathbf{x}) = \prod_{i=1}^d \cos(q_i x_i)$. Then $\{e_{\mathbf{q}}(\mathbf{x})\}_{\mathbf{q} \in \Omega}$ forms a basis of the space of functions in \mathbb{T}^d that satisfy Neumann boundary conditions. We look for a normal mode to the linearized system (19) of the following form

$$\mathbf{w}(\mathbf{x}, t) = \mathbf{r}_{\mathbf{q}} \exp(\lambda_{\mathbf{q}} t) e_{\mathbf{q}}(\mathbf{x}), \quad (20)$$

where $\mathbf{r}_{\mathbf{q}}$ is a vector depending on \mathbf{q} . Plugging (20) into (19) we have the following dispersion formula for $\lambda_{\mathbf{q}}$

$$\begin{aligned} \lambda_{\mathbf{q}}^2 + \left\{ q^2 (D_u + D_v) + \beta + 3b_3 \bar{U}^2 - 2b_2 \bar{U} - b_1 \right\} \lambda_{\mathbf{q}} \\ + q^2 \left\{ D_u D_v q^2 + \beta D_u + \left(3b_3 \bar{U}^2 - 2b_2 \bar{U} - b_1 \right) D_v \right. \\ \left. - \alpha \chi \bar{U} \right\} + \left(3b_3 \bar{U}^2 - 2b_2 \bar{U} - b_1 \right) \beta = 0. \end{aligned} \quad (21)$$

Thus we deduce the following well-known aggregation (i.e., linear instability) criterion by requiring there exists a \mathbf{q} such that

$$\begin{aligned} q^2 \left\{ D_u D_v q^2 + \beta D_u + \left(3b_3 \bar{U}^2 - 2b_2 \bar{U} - b_1 \right) D_v - \alpha \chi \bar{U} \right\} \\ + \left(3b_3 \bar{U}^2 - 2b_2 \bar{U} - b_1 \right) \beta < 0 \end{aligned} \quad (22)$$

to ensure that (21) has at least one positive root $\lambda_{\mathbf{q}}$. This implies that for \mathbf{q} ,

$$\begin{aligned} \left\{ q^2 (D_u + D_v) + \beta + 3b_3 \bar{U}^2 - 2b_2 \bar{U} - b_1 \right\}^2 \\ - 4 \left\{ q^2 \left\{ D_u D_v q^2 + \beta D_u + \left(3b_3 \bar{U}^2 - 2b_2 \bar{U} - b_1 \right) D_v \right. \right. \\ \left. \left. - \alpha \chi \bar{U} \right\} + \left(3b_3 \bar{U}^2 - 2b_2 \bar{U} - b_1 \right) \beta \right\} \\ = \left\{ q^2 (D_u - D_v) - \left(\beta - 3b_3 \bar{U}^2 + 2b_2 \bar{U} + b_1 \right) \right\}^2 \\ + 4\alpha \chi \bar{U} q^2 \geq 0. \end{aligned} \quad (23)$$

If $\mathbf{q} = \mathbf{0}$, then (21) has two negative roots $-\beta$ and $-3b_3 \bar{U}^2 + 2b_2 \bar{U} + b_1$. Therefore, the positive equilibrium point of (18) is locally asymptotically stable.

Now we investigate nonlinear dynamics near the unstable constant equilibrium solution of (18) in the case $\mathbf{q} \neq \mathbf{0}$. If $\mathbf{q} \neq \mathbf{0}$, the right side of (23) is positive. Therefore, there exist two distinct real roots $\lambda_{\mathbf{q}}^{\pm}$ for all $\mathbf{q} \neq \mathbf{0}$ to the quadratic equation (21). We denote the corresponding (linearly independent) eigenvectors by $\mathbf{r}_{-}(\mathbf{q})$ and $\mathbf{r}_{+}(\mathbf{q})$, such that

$$\mathbf{r}_{\pm}(\mathbf{q}) = \left[\frac{\lambda_{\mathbf{q}}^{\pm} + D_v q^2 + \beta}{\alpha}, 1 \right]. \quad (24)$$

Clearly, for q large

$$\begin{aligned} q^2 \left\{ D_u D_v q^2 + \beta D_u + \left(3b_3 \bar{U}^2 - 2b_2 \bar{U} - b_1 \right) D_v - \alpha \chi \bar{U} \right\} \\ + \left(3b_3 \bar{U}^2 - 2b_2 \bar{U} - b_1 \right) \beta > 0. \end{aligned} \quad (25)$$

Hence, there are only finitely many \mathbf{q} such that $\lambda_{\mathbf{q}}^{+} > 0$. We denote the largest eigenvalue by $\lambda_{\max} > 0$ and define $\Omega_{\max} \equiv \{\mathbf{q} \in \Omega \mid \lambda_{\mathbf{q}}^{+} = \lambda_{\max}\}$. It is easy to see that there is one q^2 (possibly two) having $\lambda^{+}(q^2) = \lambda_{\max}$ if we regard $\lambda_{\mathbf{q}}^{+}$ as a function of q^2 . We also denote $\nu > 0$ to be the gap between the λ_{\max} and the rest. Given any initial perturbation $\mathbf{w}(\mathbf{x}, 0)$, we can expand it as

$$\begin{aligned} \mathbf{w}(\mathbf{x}, 0) &= \sum_{\mathbf{q} \in \Omega} \mathbf{w}_{\mathbf{q}} e_{\mathbf{q}}(\mathbf{x}) \\ &= \sum_{\mathbf{q} \in \Omega} \left\{ w_{\mathbf{q}}^{-} \mathbf{r}_{-}(\mathbf{q}) + w_{\mathbf{q}}^{+} \mathbf{r}_{+}(\mathbf{q}) \right\} e_{\mathbf{q}}(\mathbf{x}), \end{aligned} \quad (26)$$

so that

$$\mathbf{w}_{\mathbf{q}} = w_{\mathbf{q}}^{-} \mathbf{r}_{-}(\mathbf{q}) + w_{\mathbf{q}}^{+} \mathbf{r}_{+}(\mathbf{q}). \quad (27)$$

The unique solution $\mathbf{w}(\mathbf{x}, t) = [u(\mathbf{x}, t), v(\mathbf{x}, t)]$ to (19) is given by

$$\begin{aligned} \mathbf{w}(\mathbf{x}, t) &= \sum_{\mathbf{q} \in \Omega} \left\{ w_{\mathbf{q}}^{-} \mathbf{r}_{-}(\mathbf{q}) \exp(\lambda_{\mathbf{q}}^{-} t) \right. \\ &\quad \left. + w_{\mathbf{q}}^{+} \mathbf{r}_{+}(\mathbf{q}) \exp(\lambda_{\mathbf{q}}^{+} t) \right\} e_{\mathbf{q}}(\mathbf{x}) \equiv e^{\mathbf{R}t} \mathbf{w}(\mathbf{x}, 0). \end{aligned} \quad (28)$$

For any $\mathbf{g}(\cdot, t) \in [L^2(\mathbb{T}^d)]^2$, we denote $\|\mathbf{g}(\cdot, t)\| \equiv \|\mathbf{g}(\cdot, t)\|_{L^2}$. Our main result of this section is the following lemma.

Lemma 2. *Suppose that the instability criterion (22) holds. Let $\mathbf{w}(\mathbf{x}, t) \equiv e^{\mathbf{R}t} \mathbf{w}(\mathbf{x}, 0)$ be a solution to the linearized system (19) with initial condition $\mathbf{w}(\mathbf{x}, 0)$. Then there exists a constant $C_1 \geq 1$ depending on $D_u, \chi, \bar{U}, b_1, b_2, b_3, D_v, \alpha$, and β , such that*

$$\|\mathbf{w}(\cdot, t)\| \leq C_1 \exp(\lambda_{\max} t) \|\mathbf{w}(\cdot, 0)\|, \quad \forall t \geq 0. \quad (29)$$

Proof. We first consider the case for $t \geq 1$. By analyzing (21), for q large, we have

$$\lim_{q \rightarrow \infty} \frac{\lambda_{\mathbf{q}}^{\pm}}{q^2} = -D_u, -D_v, \quad (30)$$

respectively. From the quadratic formula of (21), we can see that

$$\frac{\lambda_{\mathbf{q}}^+ - \lambda_{\mathbf{q}}^-}{q^2} \geq \frac{2\sqrt{\alpha\chi\bar{U}}}{q}. \quad (31)$$

It follows from (27) that

$$|w_{\mathbf{q}}^{\pm}| \leq \frac{|\mathbf{r}_{\pm}(\mathbf{q})| \times |\mathbf{w}_{\mathbf{q}}|}{|\det[\mathbf{r}_-(\mathbf{q}), \mathbf{r}_+(\mathbf{q})]|}. \quad (32)$$

Later on we will always denote universal constants by E_i ($i = 1, 2, \dots$). Note that $\mathbf{q} \in \mathbb{N}^d$ and $\mathbf{q} \neq \mathbf{0}$. From (24) and (30), for all $q > 0$, there exists a positive constant E_1 and E_2 , such that $|\lambda_{\mathbf{q}}^{\pm}/\alpha q^2| \leq E_1$, and

$$|\mathbf{r}_{\pm}(\mathbf{q})| \leq E_2 q^2. \quad (33)$$

By (24), (31), (32), and (33), we deduce that

$$|w_{\mathbf{q}}^{\pm}| \leq E_3 q |\mathbf{w}_{\mathbf{q}}|. \quad (34)$$

Thus, it is clear from (33) and (34) that

$$|w_{\mathbf{q}}^{\pm} \mathbf{r}_{\pm}(\mathbf{q}) \exp(\lambda_{\mathbf{q}}^{\pm} t)| \leq E_2 E_3 q^3 |\mathbf{w}_{\mathbf{q}}| \exp(\lambda_{\mathbf{q}}^{\pm} t). \quad (35)$$

For $t \geq 1$, it is not hard to verify that there exists a constant $E_4 > 0$, such that

$$q^3 \exp(\lambda_{\mathbf{q}}^{\pm} t) \leq E_4. \quad (36)$$

It follows from (35) and (36) that

$$|w_{\mathbf{q}}^{\pm} \mathbf{r}_{\pm}(\mathbf{q}) \exp(\lambda_{\mathbf{q}}^{\pm} t)| \leq E_5 |\mathbf{w}_{\mathbf{q}}|. \quad (37)$$

Denote by $\langle \cdot, \cdot \rangle$ and (\cdot, \cdot) the inner product of $[L^2(\mathbb{T}^d)]^2$ and the scalar product of \mathbb{R}^2 , respectively. A simple computation shows that

$$\|\mathbf{w}(\mathbf{x}, 0)\|^2 = \left(\frac{\pi}{2}\right)^d \sum_{\mathbf{q} \in \Omega} |\mathbf{w}_{\mathbf{q}}|^2. \quad (38)$$

From (28), (37) and (38), we have

$$\|\mathbf{w}(\mathbf{x}, t)\| \leq 2E_5 \exp(\lambda_{\max} t) \|\mathbf{w}(\mathbf{x}, 0)\|, \quad \text{for } t \geq 1. \quad (39)$$

On the other hand, for $t < 1$, it is sufficient to derive the standard energy estimate in L^2 . By (19), we have

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int_{\mathbb{T}^d} \{|u|^2 + A|v|^2\} \mathbf{d}\mathbf{x} \\ & + \int_{\mathbb{T}^d} \{D_u |\nabla u|^2 + AD_v |\nabla v|^2 - \chi \bar{U} \nabla u \nabla v\} \mathbf{d}\mathbf{x} \\ & + (3b_3 \bar{U}^2 - 2b_2 \bar{U} - b_1) \int_{\mathbb{T}^d} u^2 \mathbf{d}\mathbf{x} \\ & + A\beta \int_{\mathbb{T}^d} v^2 \mathbf{d}\mathbf{x} = A\alpha \int_{\mathbb{T}^d} uv \mathbf{d}\mathbf{x}. \end{aligned} \quad (40)$$

Let

$$A = \frac{(\chi \bar{U})^2}{D_u D_v}. \quad (41)$$

Then the integrand of the second term on the left side of (40) satisfies

$$\begin{aligned} & D_u |\nabla u|^2 + AD_v |\nabla v|^2 - \chi \bar{U} \nabla u \nabla v \\ & \geq \frac{D_u}{2} |\nabla u|^2 + \frac{(\chi \bar{U})^2}{2D_u} |\nabla v|^2 \geq 0. \end{aligned} \quad (42)$$

By (40), Young's inequality, and $A \geq 1$, we deduce that

$$\frac{1}{2} \frac{d}{dt} \int_{\mathbb{T}^d} \{|u|^2 + A|v|^2\} \mathbf{d}\mathbf{x} \leq \frac{A\alpha}{2} \int_{\mathbb{T}^d} \{|u|^2 + A|v|^2\} \mathbf{d}\mathbf{x}. \quad (43)$$

Using Grownwall's inequality and noticing $A \geq 1$ and $t < 1$, we can obtain

$$\|\mathbf{w}(\mathbf{x}, t)\| \leq (A \exp(A\alpha))^{1/2} \exp(\lambda_{\max} t) \|\mathbf{w}(\mathbf{x}, 0)\|. \quad (44)$$

If $0 < A < 1$, by (43), $t < 1$, and Grownwall inequality, we have

$$\|\mathbf{w}(\mathbf{x}, t)\| \leq \left(\frac{\exp \alpha}{A}\right)^{1/2} \exp(\lambda_{\max} t) \|\mathbf{w}(\mathbf{x}, 0)\|. \quad (45)$$

Let $C_1 = \max\{2E_5, (A \exp(A\alpha))^{1/2}\} \geq 1$ if $0 < A < 1$. Then $\|\mathbf{w}(\mathbf{x}, t)\| \leq C_1 \exp(\lambda_{\max} t) \|\mathbf{w}(\mathbf{x}, 0)\|$ if $A > 1$ and $C_1 = \max\{2E_5, (\exp \alpha/A)^{1/2}\} \geq 1$. \square

4. Bootstrap Lemma

By a standard PDE theory [14], we can establish the existence of local solutions for (18).

Lemma 3 (local existence). *For $s \geq 1$ ($d = 1$) and $s \geq 2$ ($d = 2, 3$), there exists $T_0 > 0$, such that (18) with $u(\cdot, 0), v(\cdot, 0) \in H^s$ has a unique solution $\mathbf{w}(\cdot, t)$ on $(0, T_0)$ which satisfies*

$$\|\mathbf{w}(t)\|_{H^s} \leq C \|\mathbf{w}(0)\|_{H^s}, \quad 0 < t < T_0, \quad (46)$$

where C is a positive constant depending on $D_u, \chi, \bar{U}, b_1, b_2, b_3, D_v, \alpha$, and β .

It is not hard to verify the following result.

Lemma 4. *Let $\mathbf{w}(\mathbf{x}, t)$ be a solution of (18). Then the even extension of $\mathbf{w}(\mathbf{x}, t)$ on $2\mathbb{T}^d = (-\pi, \pi)^d$ ($d = 1, 2, 3$) is also the solution of (18) which satisfies homogeneous Neumann boundary conditions and periodical boundary conditions on $2\mathbb{T}^d = (-\pi, \pi)^d$ ($d = 1, 2, 3$).*

Lemma 5. Let $[u(\mathbf{x}, t), v(\mathbf{x}, t)]$ be a solution of (18). Then

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \sum_{|\alpha|=2} \int_{\mathbb{T}^d} \left\{ |D^\alpha u|^2 + \frac{(\chi \bar{U})^2}{D_u D_v} |D^\alpha v|^2 \right\} \mathbf{d}\mathbf{x} \\ & + \sum_{|\alpha|=2} \int_{\mathbb{T}^d} \left\{ \frac{D_u}{4} |\nabla D^\alpha u|^2 + \frac{(\chi \bar{U})^2}{2D_u} |\nabla D^\alpha v|^2 \right\} \mathbf{d}\mathbf{x} \\ & + \frac{A\beta}{2} \sum_{|\alpha|=2} \int_{\mathbb{T}^d} |D^\alpha v|^2 \mathbf{d}\mathbf{x} + \frac{3b_3 \bar{U}^2 - 2b_2 \bar{U} - b_1}{2} \\ & \times \sum_{|\alpha|=2} \int_{\mathbb{T}^d} |D^\alpha u|^2 \mathbf{d}\mathbf{x} \leq C_0 \left(\frac{\chi}{2} + |b_2 - 3b_3 \bar{U}| + \frac{3}{2} b_3 \right) \\ & \times \left(\|\mathbf{w}\|_{H^2} + \|\mathbf{w}\|_{H^2}^2 \right) \|\nabla^3 \mathbf{w}\|^2 + C_2 \|u\|^2, \end{aligned} \quad (47)$$

where $C_2 = E_{15} \chi^{12} \bar{U}^{12} \alpha^{12} / 2D_u^8 D_v^6 \beta^6 (3b_3 \bar{U}^2 - 2b_2 \bar{U} - b_1)^3$.

Proof. It is known by Lemma 4 that

$$\begin{aligned} \tilde{u}_t &= D_u \nabla^2 \tilde{u} - \chi \bar{U} \nabla^2 \tilde{v} - (3b_3 \bar{U}^2 - 2b_2 \bar{U} - b_1) \tilde{u} - \chi \nabla (\tilde{u} \nabla \tilde{v}) \\ & + (b_2 - 3b_3 \bar{U}) \tilde{u}^2 - b_3 \tilde{u}^3, \\ \tilde{v}_t &= D_v \nabla^2 \tilde{v} + \alpha \tilde{u} - \beta \tilde{v}, \\ \frac{\partial \tilde{u}}{\partial x_i} &= \frac{\partial \tilde{v}}{\partial x_i} = 0, \quad \text{at } x_i = -\pi, 0, \pi, \text{ for } 1 \leq i \leq d, \end{aligned} \quad (48)$$

where $[\tilde{u}(\mathbf{x}, t), \tilde{v}(\mathbf{x}, t)]$ is the even extension of $[u(\mathbf{x}, t), v(\mathbf{x}, t)]$ on $(-\pi, 0)^d$. Taking the second-order derivative of (48) for x_i, x_j and making inner product with $\partial_{x_i x_j} \tilde{u}$ and $A \partial_{x_i x_j} \tilde{v}$, respectively, on both sides then adding the two equations together, we have

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int_{2\mathbb{T}^d} \left\{ |\partial_{x_i x_j} \tilde{u}|^2 + A |\partial_{x_i x_j} \tilde{v}|^2 \right\} \mathbf{d}\mathbf{x} \\ & + \int_{2\mathbb{T}^d} \left\{ D_u |\nabla \partial_{x_i x_j} \tilde{u}|^2 + A D_v |\nabla \partial_{x_i x_j} \tilde{v}|^2 - \chi \bar{U} \nabla \partial_{x_i x_j} \tilde{u} \right. \\ & \quad \left. \cdot \nabla \partial_{x_i x_j} \tilde{v} \right\} \mathbf{d}\mathbf{x} + A\beta \int_{2\mathbb{T}^d} |\partial_{x_i x_j} \tilde{v}|^2 \mathbf{d}\mathbf{x} \\ & + (3b_3 \bar{U}^3 - 2b_2 \bar{U} - b_1) \int_{2\mathbb{T}^d} |\partial_{x_i x_j} \tilde{u}|^2 \mathbf{d}\mathbf{x} \\ & = \chi \int_{2\mathbb{T}^d} \nabla \partial_{x_i x_j} \tilde{u} \cdot \partial_{x_i x_j} (\tilde{u} \cdot \nabla \tilde{v}) \mathbf{d}\mathbf{x} \end{aligned}$$

$$\begin{aligned} & + A\alpha \int_{2\mathbb{T}^d} \partial_{x_i x_j} \tilde{u} \cdot \partial_{x_i x_j} \tilde{v} \mathbf{d}\mathbf{x} + 2(b_2 - 3b_3 \bar{U}) \\ & \times \int_{2\mathbb{T}^d} \partial_{x_i x_j} \tilde{u} \left(\partial_{x_i} \tilde{u} \cdot \partial_{x_j} \tilde{u} + \tilde{u} \cdot \partial_{x_i x_j} \tilde{u} \right) \mathbf{d}\mathbf{x} \\ & - 3b_3 \int_{2\mathbb{T}^d} \partial_{x_i x_j} \tilde{u} \left(2\tilde{u} \cdot \partial_{x_i} \tilde{u} \cdot \partial_{x_j} \tilde{u} + \tilde{u}^2 \cdot \partial_{x_i x_j} \tilde{u} \right) \mathbf{d}\mathbf{x} \\ & \equiv I_1 + I_2 + I_3 + I_4. \end{aligned} \quad (49)$$

Clearly

$$\begin{aligned} & D_u |\nabla \partial_{x_i x_j} \tilde{u}|^2 + A D_v |\nabla \partial_{x_i x_j} \tilde{v}|^2 - \chi \bar{U} \nabla \partial_{x_i x_j} \tilde{u} \cdot \nabla \partial_{x_i x_j} \tilde{v} \\ & \geq \frac{D_u}{2} |\nabla \partial_{x_i x_j} \tilde{u}|^2 + \frac{(\chi \bar{U})^2}{2D_u} |\nabla \partial_{x_i x_j} \tilde{v}|^2. \end{aligned} \quad (50)$$

The nonlinear term I_1 is bounded by

$$\begin{aligned} I_1 &\leq \chi \left\{ \|\nabla \tilde{v}\|_{L^\infty} \|\nabla \partial_{x_i x_j} \tilde{u}\| \|\partial_{x_i x_j} \tilde{u}\| + 2 \sum_{i=1}^d \|\nabla \tilde{u}\|_{L^\infty} \|\partial_{x_i x_j} \tilde{v}\| \right. \\ & \quad \left. \times \|\nabla \partial_{x_i x_j} \tilde{u}\| + \|\tilde{u}\|_{L^\infty} \|\nabla \partial_{x_i x_j} \tilde{u}\| \|\nabla \partial_{x_i x_j} \tilde{v}\| \right\}. \end{aligned} \quad (51)$$

We know that

$$\begin{aligned} \|\mathbf{g}\|_{L^\infty(2\mathbb{T}^d)} &\leq E_6 \|\mathbf{g}\|_{H^2(2\mathbb{T}^d)}, \\ \|\mathbf{g}\|_{L^4(2\mathbb{T}^d)} &\leq E_7 \|\mathbf{g}\|_{H^2(2\mathbb{T}^d)}, \\ \|\mathbf{g}\|_{L^6(2\mathbb{T}^d)} &\leq E_8 \|\mathbf{g}\|_{H^2(2\mathbb{T}^d)}, \end{aligned} \quad (52)$$

for $d \leq 3$, and

$$\int_{2\mathbb{T}^d} \nabla \tilde{u} \mathbf{d}\mathbf{x} = \int_{2\mathbb{T}^d} \nabla \tilde{v} \mathbf{d}\mathbf{x} = 0, \quad (53)$$

$$\int_{2\mathbb{T}^d} \partial_{x_i x_j} \tilde{u} \mathbf{d}\mathbf{x} = \int_{2\mathbb{T}^d} \partial_{x_i x_j} \tilde{v} \mathbf{d}\mathbf{x} = 0.$$

Applying the Poincaré inequality, we have

$$\|\mathbf{g}\| \leq E_9 \|\mathbf{g}\|_{L^4(2\mathbb{T}^d)} \leq E_{10} \|\mathbf{g}\|_{H^1} \leq E_{11} \|\nabla \mathbf{g}\|, \quad d \leq 3. \quad (54)$$

It follows from (53) and (54) that $\|\partial_{x_i} \mathbf{g}\| \leq E_{11} \|\nabla \partial_{x_i} \mathbf{g}\|$. Thus

$$\|\nabla \mathbf{g}\| \leq E_{11} \left(\sum_{i,j=1}^d \|\partial_{x_i x_j} \mathbf{g}\|^2 \right)^{1/2}. \quad (55)$$

Furthermore,

$$\frac{1}{d^2} \sum_{i,j=1}^d \|\partial_{x_i x_j} \mathbf{g}\|^2 \leq \sum_{|\alpha|=2} \|D^\alpha \mathbf{g}\|^2 \leq \sum_{i,j=1}^d \|\partial_{x_i x_j} \mathbf{g}\|^2. \quad (56)$$

This implies that $(\sum_{i,j=1}^d \|\partial_{x_i x_j} g\|^2)^{1/2}$ is equivalent to $(\sum_{|\alpha|=2} \|D^\alpha g\|^2)^{1/2}$. From (53)–(56), we have

$$\begin{aligned} \|\partial_{x_i x_j} g\| &\leq E_{11} \|\nabla \partial_{x_i x_j} g\|, \\ \|\nabla g\| &\leq E_{11} d \left(\sum_{|\alpha|=2} \|D^\alpha g\|^2 \right)^{1/2} \\ &\leq E_{11} d \left(\sum_{|\alpha|=2} \|\nabla D^\alpha g\|^2 \right)^{1/2}. \end{aligned} \quad (57)$$

It follows from (56) and (57) that

$$\begin{aligned} \|\nabla g\|_{H^2} &\leq E_{12} \left(\sum_{|\alpha|=2} \|\nabla D^\alpha g\|^2 \right)^{1/2}, \\ E_{12} &= (E_{11}^4 d^2 + E_{11}^2 d^2 + 1)^{1/2}. \end{aligned} \quad (58)$$

By (51), (52), (57), and (58), we have

$$\sum_{|\alpha|=2} I_1 \leq \chi E_{13} \|\bar{\mathbf{w}}\|_{H^2} \|\nabla^3 \bar{\mathbf{w}}\|^2. \quad (59)$$

Now we consider I_2 . From Gagliardo-Nirenberg inequality and Young inequality, we obtain

$$\|\partial_{x_i x_j} \bar{u}\|^2 \leq \frac{8E_{14}^2}{9} \left(a \|\nabla \partial_{x_i x_j} \bar{u}\|^2 + \frac{\|\bar{u}\|^2}{4a^2} \right). \quad (60)$$

Let $a = 9D_u^3 D_v^2 \beta^2 (3b_3 \bar{U}^2 - 2b_2 \bar{U} - b_1) / 4E_{14}^2 \chi^4 \bar{U}^4 \alpha^4$. Then

$$\begin{aligned} \sum_{|\alpha|=2} I_2 &\leq \frac{3b_3 \bar{U}^2 - 2b_2 \bar{U} - b_1}{2} \sum_{|\alpha|=2} \|D^\alpha \bar{u}\|^2 \\ &+ \frac{A\beta}{2} \sum_{|\alpha|=2} \|D^\alpha \bar{v}\|^2 + \frac{D_u}{4} \sum_{|\alpha|=2} \|\nabla D^\alpha \bar{u}\|^2 \\ &+ \frac{E_{15} \chi^{12} \bar{U}^{12} \alpha^{12}}{D_u^8 D_v^6 \beta^6 (3b_3 \bar{U}^2 - 2b_2 \bar{U} - b_1)^3} \|\bar{u}\|^2. \end{aligned} \quad (61)$$

It follows from (52), (57), and (58) that

$$\begin{aligned} \sum_{|\alpha|=2} I_3 &\leq 2 |b_2 - 3b_3 \bar{U}| E_{16} \|\bar{\mathbf{w}}\|_{H^2} \|\nabla^3 \bar{\mathbf{w}}\|^2, \\ \sum_{|\alpha|=2} I_4 &\leq 3b_3 E_{17} \|\bar{\mathbf{w}}\|^2 \|\nabla^3 \bar{\mathbf{w}}\|. \end{aligned} \quad (62)$$

Combining (49), (50), (59), (61), and (62), we have

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \sum_{|\alpha|=2} \int_{\mathbb{T}^d} \left\{ |D^\alpha u|^2 + \frac{(\chi \bar{U})^2}{D_u D_v} |D^\alpha v|^2 \right\} \mathbf{d}\mathbf{x} \\ &+ \sum_{|\alpha|=2} \int_{\mathbb{T}^d} \left\{ \frac{D_u}{4} |\nabla D^\alpha u|^2 + \frac{(\chi \bar{U})^2}{2D_u} |\nabla D^\alpha v|^2 \right\} \mathbf{d}\mathbf{x} \\ &+ \frac{A\beta}{2} \sum_{|\alpha|=2} \int_{\mathbb{T}^d} |D^\alpha v|^2 \mathbf{d}\mathbf{x} + \frac{3b_3 \bar{U}^2 - 2b_2 \bar{U} - b_1}{2} \\ &\times \sum_{|\alpha|=2} \int_{\mathbb{T}^d} |D^\alpha u|^2 \mathbf{d}\mathbf{x} \leq C_0 \frac{\chi + 2 |b_2 - 3b_3 \bar{U}| + 3b_3}{2} \\ &\times (\|\mathbf{w}\|_{H^2} + \|\mathbf{w}\|_{H^2}^2) \|\nabla^3 \mathbf{w}\|^2 + C_2 \|u\|^2, \end{aligned} \quad (63)$$

where $C_0 = \max\{E_{13}, E_{16}, E_{17}\}$, $C_2 = E_{15} \chi^{12} \bar{U}^{12} \alpha^{12} / 2D_u^8 D_v^6 \beta^6 (3b_3 \bar{U}^2 - 2b_2 \bar{U} - b_1)^3$. \square

Lemma 6. Let $\mathbf{w}(\mathbf{x}, t)$ be a solution of (18) such that for $0 \leq t \leq T < T_0$

$$\begin{aligned} &\|\mathbf{w}(\cdot, t)\|_{H^2} + \|\mathbf{w}(\cdot, t)\|_{H^2}^2 \\ &\leq \frac{1}{C_0} \min \left\{ \frac{D_u}{2(\chi + 2 |b_2 - 3b_3 \bar{U}| + 3b_3)}, \right. \\ &\quad \left. \frac{(\chi \bar{U})^2}{D_u (\chi + 2 |b_2 - 3b_3 \bar{U}| + 3b_3)} \right\}, \end{aligned} \quad (64)$$

$$\|\mathbf{w}(\cdot, t)\| \leq 2C_1 \exp(\lambda_{\max} t) \|\mathbf{w}(\cdot, 0)\|. \quad (65)$$

Then

$$\begin{aligned} \|\mathbf{w}(\cdot, t)\|_{H^2}^2 &\leq C_3 \{ \|\mathbf{w}(\cdot, 0)\|_{H^2}^2 \\ &+ \exp(2\lambda_{\max} t) \|\mathbf{w}(\cdot, 0)\|^2 \}, \quad 0 \leq t \leq T, \end{aligned} \quad (66)$$

where $C_3 = \max\{(E_{11}^2 d^2 + 1)((\chi \bar{U})^2 / D_u D_v), 4C_1^2\{1 + (E_{11}^2 d^2 + 1)(C_2 / \lambda_{\max})\}\}$ if $(\chi \bar{U})^2 / D_u D_v \geq 1$, $C_3 = \max\{(E_{11}^2 d^2 + 1)(D_u D_v / (\chi \bar{U})^2), 4C_1^2\{1 + (E_{11}^2 d^2 + 1)(C_2 D_u D_v / \lambda_{\max} (\chi \bar{U})^2)\}\}$ if $(\chi \bar{U})^2 / D_u D_v < 1$.

Proof. It is clear from (57) that

$$\|\nabla \mathbf{w}(\cdot, t)\|^2 \leq E_{11}^2 d^2 \sum_{|\alpha|=2} \|D^\alpha \mathbf{w}(\cdot, t)\|^2. \quad (67)$$

It follows from (67) that

$$\|\mathbf{w}(\cdot, t)\|_{H^2}^2 \leq \|\mathbf{w}(\cdot, 0)\|^2 + (E_{11}^2 d^2 + 1) \sum_{|\alpha|=2} \|D^\alpha \mathbf{w}(\cdot, t)\|^2. \quad (68)$$

Now we estimate the second-order derivatives of $\mathbf{w}(\cdot, t)$. By (65) and Lemma 5, we immediately see that

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \sum_{|\alpha|=2} \int_{\mathbb{T}^d} \left\{ |D^\alpha u|^2 + \frac{(\chi \bar{U})^2}{D_u D_v} |D^\alpha v|^2 \right\} \mathbf{d}\mathbf{x} \\ & \leq C_2 \|u\|^2 \leq C_2 \|\mathbf{w}(\cdot, t)\|^2. \end{aligned} \quad (69)$$

Integrating on both sides of (69) from 0 to t and from (65), we have

$$\begin{aligned} & \sum_{|\alpha|=2} \int_{\mathbb{T}^d} \left\{ |D^\alpha u(\cdot, t)|^2 + \frac{(\chi \bar{U})^2}{D_u D_v} |D^\alpha v(\mathbf{x}, t)|^2 \right\} \mathbf{d}\mathbf{x} \\ & \leq \sum_{|\alpha|=2} \int_{\mathbb{T}^d} \left\{ |D^\alpha u(\cdot, 0)|^2 + \frac{(\chi \bar{U})^2}{D_u D_v} |D^\alpha v(\cdot, 0)|^2 \right\} \mathbf{d}\mathbf{x} \\ & + \frac{4C_1^2 C_2}{\lambda_{\max}} \|\mathbf{w}(\cdot, 0)\|^2 \exp(2\lambda_{\max} t). \end{aligned} \quad (70)$$

We will proceed in the following two cases: $(\chi \bar{U})^2 / D_u D_v \geq 1$, $(\chi \bar{U})^2 / D_u D_v < 1$.

(1) If $(\chi \bar{U})^2 / D_u D_v \geq 1$, it follows from (70) that

$$\begin{aligned} & \sum_{|\alpha|=2} \|D^\alpha \mathbf{w}(\cdot, t)\|^2 \leq \frac{(\chi \bar{U})^2}{D_u D_v} \sum_{|\alpha|=2} \|D^\alpha \mathbf{w}(\cdot, 0)\|^2 \\ & + \frac{4C_1^2 C_2}{\lambda_{\max}} \|\mathbf{w}(\cdot, 0)\|^2 \exp(2\lambda_{\max} t). \end{aligned} \quad (71)$$

By (68) and (71), we have

$$\|\mathbf{w}(\cdot, t)\|_{H^2}^2 \leq C_3 \left\{ \|\mathbf{w}(\cdot, 0)\|_{H^2}^2 + \|\mathbf{w}(\cdot, 0)\|^2 \exp(2\lambda_{\max} t) \right\}, \quad (72)$$

where $C_3 = \max\{(E_{11}^2 d^2 + 1)((\chi \bar{U})^2 / D_u D_v), 4C_1^2\{1 + (E_{11}^2 d^2 + 1)(C_2 / \lambda_{\max})\}\}$.

(2) If $(\chi \bar{U})^2 / D_u D_v < 1$, it follows from (71) that

$$\begin{aligned} & \sum_{|\alpha|=2} \|D^\alpha \mathbf{w}(\cdot, t)\|^2 \leq \frac{D_u D_v}{(\chi \bar{U})^2} \sum_{|\alpha|=2} \|D^\alpha \mathbf{w}(\cdot, 0)\|^2 \\ & + \frac{D_u D_v}{(\chi \bar{U})^2} \cdot \frac{4C_1^2 C_2}{\lambda_{\max}} \|\mathbf{w}(\cdot, 0)\|^2 \exp(2\lambda_{\max} t). \end{aligned} \quad (73)$$

By (68) and (73), we have

$$\|\mathbf{w}(\cdot, t)\|_{H^2}^2 \leq C_3 \left\{ \|\mathbf{w}(\cdot, 0)\|_{H^2}^2 + \|\mathbf{w}(\cdot, 0)\|^2 \exp(2\lambda_{\max} t) \right\}, \quad (74)$$

where $C_3 = \max\{(E_{11}^2 d^2 + 1)(D_u D_v / (\chi \bar{U})^2), 4C_1^2\{1 + (E_{11}^2 d^2 + 1)(C_2 D_u D_v / \lambda_{\max} (\chi \bar{U})^2)\}\}$. \square

5. Main Result

Let θ be a small fixed constant, and λ_{\max} be the dominant eigenvalue which is the maximal growth rate. We also denote the gap between the largest growth rate λ_{\max} and the rest by $\nu > 0$. Then for $\delta > 0$ arbitrary small, we define the escape time T^δ by

$$\theta = \delta \exp(\lambda_{\max} T^\delta), \quad (75)$$

or equivalently

$$T^\delta = \frac{1}{\lambda_{\max}} \ln \frac{\theta}{\delta}. \quad (76)$$

Our main result is as follows.

Theorem 7. Assume that the set of $q^2 = \sum_{i=1}^d q_i^2$ satisfying instability criterion (22) is not empty for given parameters $D_u, \chi, \bar{U}, b_1, b_2, b_3, D_v, \alpha, \beta$. Let

$$\mathbf{w}_0(\mathbf{x}) = \sum_{\mathbf{q} \in \Omega} \{w_{\mathbf{q}}^- \mathbf{r}_-(\mathbf{q}) + w_{\mathbf{q}}^+ \mathbf{r}_+(\mathbf{q})\} e_{\mathbf{q}}(\mathbf{x}) \in H^2, \quad (77)$$

such that $\|\mathbf{w}_0\| = 1$. Then there exist constants $\delta_0 > 0$, $C > 0$ and $\theta > 0$ depending on $D_u, \chi, \bar{U}, b_1, b_2, b_3, D_v, \alpha$, and β , such that for all $0 < \delta \leq \delta_0$, if the initial perturbation of the steady state $[\bar{U}, \bar{V}]$ is $\mathbf{w}^\delta(\cdot, 0) = \delta \mathbf{w}_0$, then its nonlinear evolution $\mathbf{w}^\delta(\cdot, t)$ satisfies

$$\begin{aligned} & \left\| \mathbf{w}^\delta(\cdot, t) - \delta e^{\lambda_{\max} t} \sum_{\mathbf{q} \in \Omega_{\max}} w_{\mathbf{q}}^+ \mathbf{r}_+(\mathbf{q}) e_{\mathbf{q}}(\mathbf{x}) \right\| \\ & \leq C \left\{ e^{-\nu t} + \delta \|\mathbf{w}_0\|_{H^2}^2 + \delta^2 \|\mathbf{w}_0\|_{H^2}^3 + \delta e^{\lambda_{\max} t} \right. \\ & \quad \left. + \delta^2 e^{2\lambda_{\max} t} \right\} \delta e^{\lambda_{\max} t} \end{aligned} \quad (78)$$

for $0 \leq t \leq T^\delta$, and $\nu > 0$ is the gap between λ_{\max} and the rest of $\lambda_{\mathbf{q}}$ in (21).

Proof. Let $\mathbf{w}^\delta(\mathbf{x}, t)$ be the solutions to (18) with initial data $\mathbf{w}^\delta(\cdot, 0) = \delta \mathbf{w}_0$. We define

$$T^* = \sup \left\{ t \mid \|\mathbf{w}^\delta(\cdot, t) - \delta e^{\lambda_{\max} t} \mathbf{w}_0\| \leq \frac{C_1}{2} \delta \exp(\lambda_{\max} t) \right\}. \quad (79)$$

We also define

$$\begin{aligned} & T^{**} = \sup \\ & \left\{ t \mid \left\| \mathbf{w}^\delta(\cdot, t) \right\|_{H^2} + \left\| \mathbf{w}^\delta(\cdot, t) \right\|_{H^2}^2 \right. \\ & \left. \leq \frac{1}{C_0^*} \min \left\{ \frac{D_u}{2(\chi + 2|b_2 - 3b_3 \bar{U}| + 3b_3)}, \frac{(\chi \bar{U})^2}{D_u(\chi + 2|b_2 - 3b_3 \bar{U}| + 3b_3)} \right\} \right\}. \end{aligned} \quad (80)$$

Choose θ such that

$$C_0^* C_3 \theta (1 + 2C_3^{1/2} \theta) < \min \left\{ \frac{\lambda_{\max}}{4(\chi + |b_2 - 3b_3\bar{U}| + b_3)}, \frac{D_u}{4(\chi + 2|b_2 - 3b_3\bar{U}| + 3b_3)}, \frac{(\chi\bar{U})^2}{2D_u(\chi + 2|b_2 - 3b_3\bar{U}| + 3b_3)} \right\}. \quad (81)$$

We now establish a sharper L^2 estimate of $\mathbf{w}^\delta(\cdot, t)$ for $0 \leq t \leq \min\{T^\delta, T^*, T^{**}\}$. First of all, by the definition of T^* and Lemma 2, for $0 < t \leq T^*$, it is not hard to see that

$$\|\mathbf{w}^\delta(\cdot, t)\| \leq \frac{3}{2} C_1 \delta \exp(\lambda_{\max} t). \quad (82)$$

Applying Lemma 6 and the bootstrap argument, one can prove

$$\|\mathbf{w}^\delta(\cdot, t)\|_{H^2} \leq \sqrt{C_3} \{\delta \|\mathbf{w}_0\|_{H^2} + \delta \exp(\lambda_{\max} t)\}. \quad (83)$$

From this and $(a + b)^p \leq 2^{p-1}(a^p + b^p)$ ($a \geq 0, b \geq 0, p \geq 1$), it follows that

$$\|\mathbf{w}^\delta(\cdot, t)\|_{H^2}^3 \leq 4C_3^{3/2} \{\delta^3 \|\mathbf{w}_0\|_{H^2}^3 + \delta^3 \exp(3\lambda_{\max} t)\}. \quad (84)$$

Applying Duhamel's principle, we can obtain

$$\begin{aligned} \mathbf{w}^\delta(\cdot, t) &= \delta e^{\mathfrak{L}t} \mathbf{w}_0 \\ &\quad - \int_0^t e^{\mathfrak{L}(t-\tau)} \left[\chi \nabla (u^\delta(\tau) \nabla v^\delta(\tau)) \right. \\ &\quad \left. - (b_2 - 3b_3\bar{U}) (u^\delta(\tau))^2 \right. \\ &\quad \left. + b_3 (u^\delta(\tau))^3, 0 \right] d\tau. \end{aligned} \quad (85)$$

By Lemma 2, (52), (54), and Lemma 6, for $0 \leq t \leq \min\{T^\delta, T^*, T^{**}\}$, we deduce that

$$\begin{aligned} \|\mathbf{w}^\delta(\cdot, t) - \delta e^{\mathfrak{L}t} \mathbf{w}_0\| &\leq C_1 C_0^* \{\chi + |b_2 - 3b_3\bar{U}| + b_3\} \\ &\quad \times \int_0^t e^{\lambda_{\max}(t-\tau)} \left\{ \|\mathbf{w}^\delta(\tau)\|_{H^2}^2 + \|\mathbf{w}^\delta(\tau)\|_{H^2}^3 \right\} d\tau, \end{aligned} \quad (86)$$

where $C_0^* = \max\{E_7^2, (E_{11}^2/E_9^2) + E_6, E_8^3\}$. By our choice of $t \leq \min\{T^\delta, T^*, T^{**}\}$, it is further bounded by

$$\begin{aligned} \|\mathbf{w}^\delta(t) - \delta e^{\mathfrak{L}t} \mathbf{w}_0\| &\leq C_1 C_0^* C_3 (\chi + |b_2 - 3b_3\bar{U}| + b_3) \\ &\quad \times \left\{ \frac{\delta \|\mathbf{w}_0\|_{H^2}^2 + 4C_3^{1/2} \delta^2 \|\mathbf{w}_0\|_{H^2}^3}{\lambda_{\max}} \right. \\ &\quad \left. + \frac{\delta e^{\lambda_{\max} t} + 2C_3^{1/2} \delta^2 e^{2\lambda_{\max} t}}{\lambda_{\max}} \right\} \delta e^{\lambda_{\max} t}. \end{aligned} \quad (87)$$

We now prove by contradiction that for δ sufficiently small, $T^\delta = \min\{T^\delta, T^*, T^{**}\}$. If T^{**} is the smallest, we can let $t = T^{**} \leq T^\delta$ in (83) and (84). If θ satisfies (81) with $C_3 \geq 1$ and δ is sufficiently small such that $C_3 \delta^2 \|\mathbf{w}_0\|_{H^2}^2 + \sqrt{C_3} \delta \|\mathbf{w}_0\|_{H^2} \leq (1/2C_0^*) \min\{D_u/(2(\chi + 2|b_2 - 3b_3\bar{U}| + 3b_3)), (\chi\bar{U})^2/D_u(\chi + 2|b_2 - 3b_3\bar{U}| + 3b_3)\}$, we immediately see that

$$\begin{aligned} \|\mathbf{w}^\delta(T^{**})\|_{H^2} + \|\mathbf{w}^\delta(T^{**})\|_{H^2}^2 &\leq C_3 \delta^2 \|\mathbf{w}_0\|_{H^2}^2 + \sqrt{C_3} \delta \|\mathbf{w}_0\|_{H^2} + C_3^{1/2} \theta (1 + C_3^{1/2} \theta) \\ &< \frac{1}{C_0^*} \min \left\{ \frac{D_u}{2(\chi + 2|b_2 - 3b_3\bar{U}| + 3b_3)}, \frac{(\chi\bar{U})^2}{D_u(\chi + 2|b_2 - 3b_3\bar{U}| + 3b_3)} \right\}. \end{aligned} \quad (88)$$

This is a contradiction to the definition of T^{**} . On the other hand, if T^* is the smallest, we can let $t = T^*$ in (87). If θ satisfies (81) and δ is sufficiently small such that $C_0^* C_3 (\chi + |b_2 - 3b_3\bar{U}| + b_3) (\delta \|\mathbf{w}_0\|_{H^2}^2 + 4C_3^{1/2} \delta^2 \|\mathbf{w}_0\|_{H^2}^3 / \lambda_{\max}) < 1/4$, we also can see that

$$\begin{aligned} \|\mathbf{w}^\delta(\cdot, T^*) - \delta e^{\mathfrak{L}T^*} \mathbf{w}_0\| &\leq C_1 C_0^* C_3 (\chi + |b_2 - 3b_3\bar{U}| + b_3) \\ &\quad \times \left\{ \frac{\delta \|\mathbf{w}_0\|_{H^2}^2 + 4C_3^{1/2} \delta^2 \|\mathbf{w}_0\|_{H^2}^3}{\lambda_{\max}} \right. \\ &\quad \left. + \frac{\theta(1 + 2C_3^{1/2} \theta)}{\lambda_{\max}} \right\} \delta e^{\lambda_{\max} T^*} < \frac{C_1}{2} \delta e^{\lambda_{\max} T^*}. \end{aligned} \quad (89)$$

This again contradicts the definition of T^* . Hence, if δ is sufficiently small, we have

$$T^\delta = \min\{T^\delta, T^*, T^{**}\}. \quad (90)$$

From (28), we have

$$\begin{aligned}
& \left\| \mathbf{w}^\delta(\cdot, t) - \delta e^{2t} \mathbf{w}_0 \right\| \\
& \geq \left\| \mathbf{w}^\delta(\cdot, t) - \delta e^{\lambda_{\max} t} \sum_{\mathbf{q} \in \Omega_{\max}} w_{\mathbf{q}}^+ \mathbf{r}_+(\mathbf{q}) e_{\mathbf{q}}(\mathbf{x}) \right\| \\
& \quad - \left\| \delta \sum_{\mathbf{q} \in \Omega_{\max}} w_{\mathbf{q}}^- \mathbf{r}_-(\mathbf{q}) \exp(\lambda_{\mathbf{q}}^- t) e_{\mathbf{q}}(\mathbf{x}) \right\| \\
& \quad - \left\| \delta \sum_{\mathbf{q} \in \Omega \setminus \Omega_{\max}} \{w_{\mathbf{q}}^- \mathbf{r}_-(\mathbf{q}) \exp(\lambda_{\mathbf{q}}^- t) \right. \\
& \quad \quad \left. + w_{\mathbf{q}}^+ \mathbf{r}_+(\mathbf{q}) \exp(\lambda_{\mathbf{q}}^+ t)\} e_{\mathbf{q}}(\mathbf{x}) \right\| \\
& \equiv \left\| \mathbf{w}^\delta(\cdot, t) - \delta e^{\lambda_{\max} t} \sum_{\mathbf{q} \in \Omega_{\max}} w_{\mathbf{q}}^+ \mathbf{r}_+(\mathbf{q}) e_{\mathbf{q}}(\mathbf{x}) \right\| - I_1 - I_2;
\end{aligned} \tag{91}$$

that is,

$$\begin{aligned}
& \left\| \mathbf{w}^\delta(\cdot, t) - \delta e^{\lambda_{\max} t} \sum_{\mathbf{q} \in \Omega_{\max}} w_{\mathbf{q}}^+ \mathbf{r}_+(\mathbf{q}) e_{\mathbf{q}}(\mathbf{x}) \right\| \\
& \leq \left\| \mathbf{w}^\delta(\cdot, t) - \delta e^{2t} \mathbf{w}_0 \right\| + I_1 + I_2.
\end{aligned} \tag{92}$$

Using (33) and (34), we have

$$I_1^2 \leq \delta^2 e^{2(\lambda_{\max} - \nu)t} \left(\frac{\pi}{2}\right)^d \sum_{\mathbf{q} \in \Omega_{\max}} E_2^2 E_3^2 q^6 |\mathbf{w}_{\mathbf{q}}|^2. \tag{93}$$

We know that there is one (or two) q^2 satisfying $\lambda^+(q^2) = \lambda_{\max}$. If there is only one q^2 satisfying $\lambda^+(q^2) = \lambda_{\max}$, we denote it by q_{\max}^2 . If there are q_1^2 and q_2^2 satisfying $\lambda^+(q^2) = \lambda_{\max}$, we let $q_{\max}^2 = \max\{q_1^2, q_2^2\}$. From (93), we have

$$I_1 \leq E_2 E_3 q_{\max}^3 \delta e^{(\lambda_{\max} - \nu)t} = C^* \delta e^{(\lambda_{\max} - \nu)t}, \tag{94}$$

where $C^* = E_2 E_3 q_{\max}^3$. Now we consider I_2 . By (38), we have

$$I_2 \leq \delta e^{(\lambda_{\max} - \nu)t}. \tag{95}$$

From (87), (92), (94), and (95), it follows that

$$\begin{aligned}
& \left\| \mathbf{w}^\delta(\cdot, t) - \delta e^{\lambda_{\max} t} \sum_{\mathbf{q} \in \Omega_{\max}} w_{\mathbf{q}}^+ \mathbf{r}_+(\mathbf{q}) e_{\mathbf{q}}(\mathbf{x}) \right\| \\
& \leq C \left\{ e^{-\nu t} + \delta \|\mathbf{w}_0\|_{H^2}^2 + \delta^2 \|\mathbf{w}_0\|_{H^2}^3 \right. \\
& \quad \left. + \delta e^{\lambda_{\max} t} + \delta^2 e^{2\lambda_{\max} t} \right\} \delta e^{\lambda_{\max} t},
\end{aligned} \tag{96}$$

where $C = \max\{C^* + 1, (4C_1 C_0^* C_3^{3/2} / \lambda_{\max})(\chi + |b_2 - 3b_3 \bar{U}| + b_3)\}$. Notice that for $0 \leq t \leq T^\delta$, $\delta e^{\lambda_{\max} t} \leq \theta$ is sufficiently small. As long as $w_{\mathbf{q}_0}^+ \neq 0$ for at least one $\mathbf{q}_0 \in \Omega_{\max}$, which is

generic for perturbations, the corresponding fastest growing modes

$$\left\| \delta e^{\lambda_{\max} t} \sum_{\mathbf{q} \in \Omega_{\max}} w_{\mathbf{q}}^+ \mathbf{r}_+(\mathbf{q}) e_{\mathbf{q}}(\mathbf{x}) \right\| \geq \delta e^{\lambda_{\max} t} |w_{\mathbf{q}_0}^+| |\mathbf{r}_+(\mathbf{q}_0)| \tag{97}$$

have the dominant leading order of $\delta e^{\lambda_{\max} t}$. \square

Our Theorem 7 implies that the dynamics of a general perturbation is characterized by such linear dynamics over a long time period of $\varepsilon T^\delta \leq t \leq T^\delta$, for any $\varepsilon > 0$. In particular, choose a fixed $\mathbf{q}_0 = (q_{01}, q_{02}, \dots, q_{0d}) \in \Omega_{\max}$ and let $\mathbf{w}_0(\mathbf{x}) = \mathbf{r}_+(\mathbf{q}_0) / |\mathbf{r}_+(\mathbf{q}_0)| e_{\mathbf{q}_0}(\mathbf{x})$; then

$$\|\mathbf{w}_0(\mathbf{x})\|_{H^2} = \left\{ \left(\frac{\pi}{2}\right)^d (1 + |\mathbf{q}_0|^2 + |\mathbf{q}_0|^4) \right\}^{1/2}. \tag{98}$$

Note that $\delta \leq \theta$, θ , and ν are fixed constants and \mathbf{q}_0 is a fixed vector. From (96) and (98), if $t \leq T^\delta$, we have

$$\begin{aligned}
& \left\| \mathbf{w}^\delta(\cdot, t) - \delta e^{\lambda_{\max} T^\delta} \frac{\mathbf{r}_+(\mathbf{q}_0)}{|\mathbf{r}_+(\mathbf{q}_0)|} e_{\mathbf{q}_0}(\mathbf{x}) \right\| \\
& \leq C^{**} \left\{ \delta^{\nu/\lambda_{\max}} + \theta^2 + \theta^3 \right\},
\end{aligned} \tag{99}$$

where $C^{**} = CC_4$, $C_4 = \max\{\theta^{1-(\nu/\lambda_{\max})}, (\pi/2)^{3d/2} (1 + |\mathbf{q}_0|^2 + |\mathbf{q}_0|^4)^{3/2} + 1\}$. Moreover,

$$\|\mathbf{w}^\delta(\cdot, t)\| \geq \theta - C^{**} \left\{ \delta^{\nu/\lambda_{\max}} + \theta^2 + \theta^3 \right\}. \tag{100}$$

Let $0 < \theta < (\sqrt{1 + (2/C^{**})} - 1)/2$, and $\delta_0 = ((\theta/2C^{**}) - \theta^2 - \theta^3)^{\lambda_{\max}/\nu}$. Then

$$\|\mathbf{w}^\delta(\cdot, t)\| \geq \theta - C^{**} \left\{ \delta^{\nu/\lambda_{\max}} + \theta^2 + \theta^3 \right\} \geq \frac{\theta}{2} > 0, \quad 0 < \delta \leq \delta_0. \tag{101}$$

This implies nonlinear instability as $\delta \rightarrow 0$. In particular, instability occurs before the possible blow-up time.

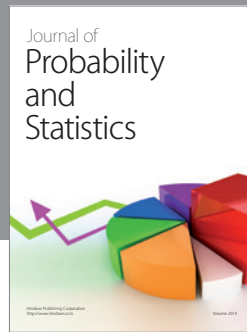
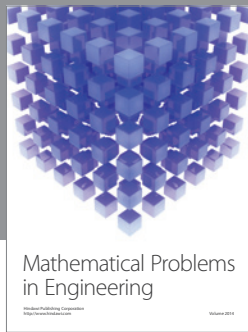
Let us point out that although our proof is based on Guo-Strauss' bootstrap argument, the adaptation to the procedure to our problem is not trivial at all, since the appearance of a growth restriction of a cubic type, we need more delicate estimates. Notice in our theorem that each initial perturbation can be drastically different from another, which gives rise to the richness of the pattern; on the other hand, the finite number maximal growing modes determines the common characteristics of the pattern, over the time scale of $\ln(1/\delta)$. Therefore, our result indeed provides a mathematical description for the pattern formation in the Keller-Segel model with a cubic source term.

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