*Research Article*

# Discontinuous Time Relaxation Method for the Time-Dependent Navier-Stokes Equations

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A high-order family of time relaxation models based on approximate deconvolution is considered. A fully discrete scheme using discontinuous finite elements is proposed and analyzed. Optimal velocity error estimates are derived. The dependence of these estimates with respect to the Reynolds number Re is $\mathcal{O}(\mathrm{Re}\, e^{\mathrm{Re}})$, which is an improvement with respect to the continuous finite element method where the dependence is $\mathcal{O}(\mathrm{Re}\, e^{\mathrm{Re}^3})$.

## 1. Introduction

Turbulence is a phenomenon that appears in many processes in the nature, and it is connected with many industrial applications. Based on the Kolmogorov theory [1], Direct Numerical Simulation (DNS) of turbulent flow, where all the scales/structures are captured, requires the number of mesh points in space per each time step to be $\mathcal{O}(\mathrm{Re}^{9/4})$ in three-dimensional problems, where Re is the Reynolds number. This is not computationally economical and sometimes not even feasible. One approach is to regularize the flow and one such type of regularization is the time relaxation method, where an additional term, the so-called time relaxation term, is added to the Navier-Stokes equations (cf. Adams and Stolz [2] and Layton and Neda [3]). The contribution to the Navier-Stokes equations from the time relaxation term induces an action on the small scales of the flow in which these scales are driven to zero. This time relaxation term is based on filtering and deconvolution methodology. In general, many spacial filtering operators associated with a length-scale $\delta$ are possible (cf. Berselli et al. [4], John [5], Geurts [6], Sagaut [7] and Germano [8]). First, consider the equations of differential filter (cf. Germano [8])

$$\overline{\phi} = G\phi \quad \text{in } \Omega, \tag{1.1}$$

$$\overline{\phi} = 0 \quad \text{on } \partial\Omega, \tag{1.2}$$

where $G^{-1} = -\delta^2\Delta + I$, $\Omega$ is the domain and $\partial\Omega$ is its boundary. Here $\delta > 0$ represents the averaging radius, in general, chosen to be of the order of the mesh size.

The deconvolution algorithm that it is considered herein was studied by van Cittert in 1931, and its use in Large Eddy Simulation (LES) pioneered by Stolz and Adams (cf. Stolz and Adams [2, 9]). For each $N = 0, 1, \ldots$, it computes an approximate solution $\mathbf{u}_N$ by $N$ steps of a fixed point iteration for the fixed point problem (cf. Bertero and Boccacci [10])

$$\text{given } \overline{\mathbf{u}} \text{ solve } \mathbf{u} = \mathbf{u} + \{\overline{\mathbf{u}} - G\mathbf{u}\}, \quad \text{for } \mathbf{u}. \tag{1.3}$$

The deconvolution approximation is then computed as follows.

*Algorithm 1.1* (van Cittert approximate deconvolution algorithm). Consider that $\mathbf{u}_0 = \overline{\mathbf{u}}$, for $n = 1, 2, \ldots, N - 1$, perform

$$\mathbf{u}_{n+1} = \mathbf{u}_n + \{\overline{\mathbf{u}} - G\mathbf{u}_n\}. \tag{1.4}$$

By eliminating the intermediate steps, it is easy to find an explicit formula for the $N$th deconvolution operator $G_N$ given by

$$G_N\overline{\mathbf{u}} := \sum_{n=0}^{N}(I - G)^n\overline{\mathbf{u}}. \tag{1.5}$$

For example, the approximate deconvolution operator corresponding to $N = 0, 1, 2$ is

$$G_0\overline{\mathbf{u}} = \overline{\mathbf{u}},$$

$$G_1\overline{\mathbf{u}} = 2\overline{\mathbf{u}} - \overline{\overline{\mathbf{u}}}, \tag{1.6}$$

$$G_2\overline{\mathbf{u}} = 3\overline{\mathbf{u}} - 3\overline{\overline{\mathbf{u}}} + \overline{\overline{\overline{\mathbf{u}}}}.$$

Many fluid models that are based on numerical regularization and computational stabilizations have been explored in computational fluid dynamics. One such regularization and the most recent has been proposed by Stolz et al. [11, 12] and arises by adding a linear, lower order time regularization term, $\chi\mathbf{u}' = \chi(\mathbf{u} - G_N\overline{\mathbf{u}})$, to the Navier-Stokes equations (NSE). This term involves $\mathbf{u}'$ which represents the part of the velocity that fluctuates on scales less than order $\delta$ and it is added to the NSE with the aim of driving the unresolved fluctuations of the velocity field to zero. The time relaxation family of models, under the no-slip boundary condition, is then defined by

$$\mathbf{u}_t + \mathbf{u} \cdot \nabla\mathbf{u} - \nu\Delta\mathbf{u} + \nabla p + \chi(\mathbf{u} - G_N\overline{\mathbf{u}}) = \mathbf{f} \quad \text{in } (0, T] \times \Omega, \tag{1.7}$$

$$\nabla \cdot \mathbf{u} = 0 \quad \text{in } [0, T] \times \Omega, \tag{1.8}$$

$$\mathbf{u} = \mathbf{0} \quad \text{in } [0, T] \times \partial\Omega, \tag{1.9}$$

$$\mathbf{u}(0, \cdot) = \mathbf{u}_0(\cdot) \quad \text{in } \Omega, \tag{1.10}$$

where $\Omega \subset \mathbb{R}^2$, is a convex bounded regular domain with boundary $\partial\Omega$, $\mathbf{u}$ is the fluid velocity, $p$ is the fluid pressure and $\mathbf{f}$ is the body force driving the flow. The kinematic viscosity $\nu > 0$ is inversely proportional to the Reynolds number of the flow. The initial velocity is given by $\mathbf{u}_0$. A pressure normalization condition $\int_\Omega p = 0$ is also needed for uniqueness of the pressure. The time relaxation coefficient $\chi$ has units $1/time$. The domain is two-dimensional, but the numerical methods and the analysis can be generalized to three-dimensional domains, as stated in [13] for the case of Stokes and Navier-Stokes problems.

Existence, uniqueness and regularity of strong solutions of these models are discussed in [3]. Even though there are papers on the simulation of the models for incompressible and compressible flows, there is little published work in the literature on the numerical analysis of the models. In [14], a fully discrete scheme using continuous finite elements and Crank-Nicolson for time discretization is analyzed and the energy cascade and joint helicity-energy cascades are studied in [3, 15], respectively.

In this work, a class of discontinuous finite element methods for solving high-order time relaxation family of fluid models (1.7)–(1.10) is formulated and analyzed. The approximations of the averaged velocity $\mathbf{u}$ and pressure $p$ are discontinuous piecewise polynomials of degree $r$ and $r - 1$, respectively. Because of the lack of continuity constraint between elements, the Discontinuous Galerkin (DG) methods offer several advantages over the classical continuous finite element methods: (i) local mesh refinement and derefinement are easily implemented (several hanging nodes per edge are allowed); (ii) the incompressibility condition is satisfied locally on each mesh element; (iii) unstructured meshes and domains with complicated geometries are easily handled. In the case of DNS, DG methods have been applied to the steady-state NSE (cf. Girault et al. [16]) and to the time-dependent NSE (cf. Girault et al. [17]) where they are combined with an operator-splitting technique. Another discontinuous Galerkin method for the NSE based on a mixed formulation are considered in [18] by Cockburn et al.. For high Reynolds numbers, the numerical analysis of a DG scheme combined with a large eddy simulation turbulence model (subgrid eddy viscosity model) is derived in [19] by Kaya and Rivière.

This paper is organized in the following way. Section 2 introduces some notation and mathematical properties. In Section 3, the fully discrete schemes are introduced and it is proved that the schemes solutions are computable. A priori velocity error estimates are derived in Section 4. The family of models (1.7)–(1.10) is regularization of the NSE. Thus, the correct question is to study convergence of discretizations of (1.7)–(1.10) to solutions of the NSE as $h$ and $\delta \to 0$ (rather than to solution of (1.7)–(1.10)). This is the problem studied herein. Conclusions are given in the last section.

## 2. Notation and Mathematical Preliminaries

To obtain a discretization of the model a regular family of triangulations $\mathcal{E}_h$ of $\overline{\Omega}$, consisting of triangles of maximum diameter $h$, is introduced. Let $h_E$ denote the diameter of a triangle $E$ and $\rho_E$ the diameter of its inscribed circle. Regulary, it is meant that there exists a parameter $\zeta > 0$, independent of $h$, such that

$$\frac{h_E}{\rho_E} = \zeta_E \leq \zeta, \quad \forall E \in \mathcal{E}_h. \tag{2.1}$$

This assumption will be used throughout this work. $\Gamma_h$ denotes the set of all interior edges of $\mathcal{E}_h$. Let $e$ denote a segment of $\Gamma_h$ shared by two triangles $E^k$ and $E^l$ ($k < l$) of $\mathcal{E}_h$; it is associated with $e$ a specific unit normal vector $\mathbf{n}_e$ directed from $E^k$ to $E^l$ and the jump and average of a function $\phi$ on $e$ is formally defined by

$$[\phi] = (\phi|_{E^k})|_e - (\phi|_{E^l})|_e, \qquad \{\phi\} = \frac{1}{2}(\phi|_{E^k})|_e + \frac{1}{2}(\phi|_{E^l})|_e. \tag{2.2}$$

If $e$ belongs to the boundary $\partial\Omega$, then $\mathbf{n}_e$ is the unit normal $\mathbf{n}$ exterior to $\Omega$ and the jump and the average of $\phi$ on $e$ coincide with the trace of $\phi$ on $e$.

Here, for any domain $\mathcal{O}$, $L^2(\mathcal{O})$ is the classical space of square-integrable functions with inner-product $(f,g)_{\mathcal{O}} = \int_{\mathcal{O}} fg$ and norm $\|\cdot\|_{0,\mathcal{O}}$. The space $L_0^2(\Omega)$ is the subspace of functions of $L^2(\Omega)$ with zero mean value

$$L_0^2(\Omega) = \left\{ \mathbf{v} \in L^2(\Omega) : \int_{\Omega} \mathbf{v} = 0 \right\}. \tag{2.3}$$

The standard Sobolev spaces are denoted by $W_p^r(\Omega)$ where $W_2^r(\Omega)$ is the $H^r(\Omega)$, with norm $\|\cdot\|_{r,\Omega}$ and seminorm $|\cdot|_{r,\Omega}$.

Next, the discrete velocity and pressure spaces are defined to be consisting of discontinuous piecewise polynomials. For any positive integer $r$, the corresponding finite-dimensional spaces are

$$\mathbf{X}^h = \left\{ \mathbf{v} \in \left(L^2(\Omega)\right)^2 : \forall E \in \mathcal{E}_h, \; \mathbf{v} \in (\mathbb{P}_r(E))^2 \right\},$$
$$Q^h = \left\{ q \in L_0^2(\Omega) : \forall E \in \mathcal{E}_h, \; q \in \mathbb{P}_{r-1}(E) \right\}, \tag{2.4}$$

where $P_r(E) = \text{span}\{x^i y^j : i + j \leq r\}$ is defined as the span of polynomials of order $r$ over triangle $E$.

Denoting by $|e|$ the measure of $e$, the following norms are associated for the spaces $\mathbf{X}^h$ and $Q^h$

$$\|\mathbf{v}\|_X = \left( \|\|\nabla\mathbf{v}\|\|_{0,\Omega}^2 + \sum_{e \in \Gamma_h \cup \partial\Omega} \frac{1}{|e|} \|[\mathbf{v}]\|_{0,e}^2 \right)^{1/2},$$
$$\|q\|_Q = \|q\|_{0,\Omega}, \tag{2.5}$$

where $\|\|\mathbf{v}\|\|_{0,\Omega}$ is the broken norm defined by

$$\|\|\mathbf{v}\|\|_{0,\Omega} = \left( \sum_{E \in \mathcal{E}_h} \|\mathbf{v}\|_{0,E}^2 \right)^{1/2}. \tag{2.6}$$

Finally, some trace and inverse inequalities are recalled, that hold true on each element $E$ in $\mathcal{E}_h$, with diameter $h_E$, the constant $C$ is independent of $h_E$

$$\|\mathbf{v}\|_{0,e} \leq C\left(h_E^{-1/2}\|\mathbf{v}\|_{0,E} + h_E^{1/2}\|\nabla\mathbf{v}\|_{0,E}\right), \quad \forall e \in \partial E, \; \forall \mathbf{v} \in \left(H^1(E)\right)^2, \tag{2.7}$$

$$\|\nabla\mathbf{v}\|_{0,e} \leq C\left(h_E^{-1/2}\|\nabla\mathbf{v}\|_{0,E} + h_E^{1/2}\left\|\nabla^2\mathbf{v}\right\|_{0,E}\right), \quad \forall e \in \partial E, \; \forall \mathbf{v} \in \left(H^2(E)\right)^2, \tag{2.8}$$

$$\|\mathbf{v}\|_{0,e} \leq C h_E^{-1/2}\|\mathbf{v}\|_{0,E}, \quad \forall e \in \partial E, \; \forall \mathbf{v} \in \mathbf{X}^h, \tag{2.9}$$

$$\|\nabla\mathbf{v}\|_{0,e} \leq C h_E^{-1/2}\|\nabla\mathbf{v}\|_{0,E}, \quad \forall e \in \partial E, \; \forall \mathbf{v} \in \mathbf{X}^h. \tag{2.10}$$

## 3. Numerical Methods

In this section, the DG scheme is introduced and the existence of the numerical solution is shown. First, the bilinear forms are defined $a : \mathbf{X}^h \times \mathbf{X}^h \to \mathbb{R}$, and $J_0 : \mathbf{X}^h \times \mathbf{X}^h \to \mathbb{R}$ by

$$a(\mathbf{z}, \mathbf{v}) = \sum_{E \in \mathcal{E}_h} \int_E \nabla\mathbf{z} : \nabla\mathbf{v} - \sum_{e \in \Gamma_h \cup \partial\Omega} \int_e \{\nabla\mathbf{z}\}\mathbf{n}_e \cdot [\mathbf{v}] + \epsilon_a \sum_{e \in \Gamma_h \cup \partial\Omega} \int_e \{\nabla\mathbf{v}\}\mathbf{n}_e \cdot [\mathbf{z}],$$

$$J_0(\mathbf{z}, \mathbf{v}) = \sum_{e \in \Gamma_h \cup \partial\Omega} \frac{\sigma}{|e|} \int_e [\mathbf{z}] \cdot [\mathbf{v}]. \tag{3.1}$$

The parameter $\epsilon_a$ takes the value $-1, 0$ or $1$: this will yield different schemes that are slight variations of each other. It will be showed that all the resulting schemes are convergent with optimal convergence rate in the energy norm $\|\cdot\|_X$. In the case where $\epsilon_a = -1$, the bilinear form $a$ is symmetric; otherwise it is nonsymmetric. We remark that the form $a(\mathbf{u}, \mathbf{v})$ is the standard primal DG discretization of the operator $-\Delta\mathbf{u}$. Finally, if $\epsilon_a$ is either $-1$ or $0$, the jump parameter $\sigma$ should be chosen sufficiently large to obtain coercivity of $a$ (see Lemma 3.1). If $\epsilon_a = 1$, then the jump parameter $\sigma$ is taken equal to $1$.

The incompressibility condition (1.8) is enforced by means of the bilinear form $b : \mathbf{X}^h \times Q^h \to \mathbb{R}$ defined by

$$b(\mathbf{v}, q) = -\sum_{E \in \mathcal{E}_h} \int_E q\nabla \cdot \mathbf{v} + \sum_{e \in \Gamma_h \cup \partial\Omega} \int_e \{q\}[\mathbf{v}] \cdot \mathbf{n}_e. \tag{3.2}$$

Finally, the DG discretization of the nonlinear convection term $\mathbf{w} \cdot \nabla\mathbf{w}$, which was introduced in [16] by Girault et al. and studied extensively in [16, 17] by the same authors, is recalled as follows:

$$c^z(\mathbf{u}; \mathbf{v}, \mathbf{t}) = \sum_{E \in \mathcal{E}_h} \left( \int_E (\mathbf{u} \cdot \nabla\mathbf{v}) \cdot \mathbf{t} + \frac{1}{2} \int_E (\nabla \cdot \mathbf{u})\mathbf{v} \cdot \mathbf{t} \right) - \frac{1}{2} \sum_{e \in \Gamma_h \cup \partial\Omega} \int_e [\mathbf{u}] \cdot \mathbf{n}_e \{\mathbf{v} \cdot \mathbf{t}\}$$

$$+ \sum_{E \in \mathcal{E}_h} \int_{\partial E_-} |\{\mathbf{u}\} \cdot \mathbf{n}_E| \left(\mathbf{v}^{\text{int}} - \mathbf{v}^{\text{ext}}\right) \cdot \mathbf{t}^{\text{int}}, \tag{3.3}$$

where

$$\partial E_- = \{\mathbf{x} \in \partial E : \{\mathbf{z}\} \cdot \mathbf{n}_E < 0\}, \tag{3.4}$$

the superscript $\mathbf{z}$ denotes the dependence of $\partial E_-$ on $\mathbf{z}$ and the superscript int (resp., ext) refers to the trace of the function on a side of $E$ coming from the interior of $E$ (resp., coming from the exterior of $E$ on that side). When the side of $E$ belongs to $\partial \Omega$, the convention is the same as for defining jumps and average, that is, the jump and average coincide with the trace of the function. Note that the form c is not linear with respect to $\mathbf{z}$, but linear with respect to $\mathbf{u}, \mathbf{v}$ and $\mathbf{t}$.

Some important properties satisfied by the forms $a, b, c$ (cf. Wheeler [20], and Girault et al. [16, 17]) are recalled.

**Lemma 3.1** (Coercivity). *If $\epsilon_a = 1$, assume that $\sigma = 1$. If $\epsilon_a \in \{-1, 0\}$, assume that $\sigma$ is sufficiently large. Then, there is a constant $\kappa > 0$, independent of $h$, such that*

$$a(\mathbf{v}, \mathbf{v}) + J_0(\mathbf{v}, \mathbf{v}) \geq \kappa \|\mathbf{v}\|_X^2, \quad \forall \mathbf{v} \in \mathbf{X}^h. \tag{3.5}$$

It is clear that $\kappa = 1$ if $\epsilon_a = 1$. Otherwise, $\kappa$ is a constant that depends on the polynomial degree of $\mathbf{v}$ and of the smallest angle in the mesh. A precise lower bound for $\sigma$ is given in [21] by Epshteyn and Rivière.

**Lemma 3.2** (Inf-sup condition). *There exists a positive constant $\beta$, independent of $h$ such that*

$$\inf_{q \in Q^h} \sup_{\mathbf{v} \in \mathbf{X}^h} \frac{b(\mathbf{v}, q)}{\|\mathbf{v}\|_X \|q\|_{0,\Omega}} \geq \beta. \tag{3.6}$$

**Lemma 3.3** (Positivity). *One has*

$$c^{\mathbf{v}}(\mathbf{v}, \mathbf{z}, \mathbf{z}) \geq 0, \quad \forall \mathbf{v}, \mathbf{z} \in \left\{ \mathbf{t} \in \left( L^2(\Omega) \right)^2 : \mathbf{t}|_E \in \left( H^2(E) \right)^2 \forall E \in \mathcal{E}_h \right\}. \tag{3.7}$$

The discrete form of the differential filter (1.1) is defined following the work of Manica and Merdan [22].

*Definition 3.4* (Discrete differential filter). Given $v \in L^2(\Omega)$, for a given filtering radius $\delta > 0$, $G^h : L^2(\Omega) \to \mathbf{X}^h$, $\overline{\mathbf{v}}^h = G^h\mathbf{v}$ where $\overline{\mathbf{v}}^h$ is the unique solution in $\mathbf{X}^h$ of

$$\delta^2 \left( a\left(\overline{\mathbf{v}}^h, \phi\right) + J_0\left(\overline{\mathbf{v}}^h, \phi\right) \right) + \left(\overline{\mathbf{v}}^h, \phi\right) = (\mathbf{v}, \phi) \quad \forall \phi \in \mathbf{X}^h. \tag{3.8}$$

*Remark 3.5.* An attractive alternative is to define the differential filter by a discrete Stokes problem so as to preserve incompressibility approximately [23–25].

*Definition 3.6.* The discrete van Cittert deconvolution operators $G_N^h$ are

$$G_N^h \mathbf{v} := \sum_{n=0}^{N} (\Pi_h - G_h)^n \mathbf{v}, \tag{3.9}$$

where $\Pi_h : L^2(\Omega) \to \mathbf{X}^h$ is the $L^2$ projection.

For $\mathbf{v} \in \mathbf{X}^h$, the discrete deconvolution operator for $N = 0, 1, 2$ is

$$G_0^h \mathbf{v} = \mathbf{v},$$

$$G_1^h \mathbf{v} = 2\mathbf{v} - \overline{\mathbf{v}}^h, \tag{3.10}$$

$$G_2^h \mathbf{v} = 3\mathbf{v} - 3\overline{\mathbf{v}}^h + \overline{\overline{\mathbf{v}}^h}^h.$$

$G_N$ was shown to be an $O(\delta^{2N+2})$ approximate inverse to the filter operator $G$ in Lemma 2.1 of Dunca and Epshteyn [26], recalled next.

**Lemma 3.7.** *$G_N$ is a bounded, self-adjoint positive operator. $G_N$ is an $O(\delta^{2N+2})$ asymptotic inverse to the filter $G$. Specifically, for smooth $\phi$ and as $\delta \to 0$,*

$$\phi = G_N \overline{\phi} + (-1)^{(N+1)} \delta^{2N+2} \Delta^{N+1} G^{(N+1)} \phi. \tag{3.11}$$

Some basic facts about discrete differential filters and deconvolution operators are presented next.

**Lemma 3.8.** *For $\mathbf{v} \in L^2(\Omega)$, one has the following bounds for the discretely filtered and approximately deconvolved $\mathbf{v}$:*

$$\left\| \overline{\mathbf{v}}^h \right\|_{0,\Omega} \leq \|\mathbf{v}\|_{0,\Omega}, \tag{3.12}$$

$$\left\| G_N^h \overline{\mathbf{v}}^h \right\|_{0,\Omega} \leq C(N) \|\mathbf{v}\|_{0,\Omega}. \tag{3.13}$$

*Proof.* The proof of (3.12) follows from the standard finite element techniques applied on the discretized equation (3.8) of the filter problem (1.1). Pick $\phi = \overline{\mathbf{v}}^h$. Then, using coercivity result

$$\delta^2 \kappa \left\| \overline{\mathbf{v}}^h \right\|_X^2 + \left\| \overline{\mathbf{v}}^h \right\|_{0,\Omega}^2 \leq \left( \mathbf{v}, \overline{\mathbf{v}}^h \right)_\Omega \leq \frac{1}{2} \|\mathbf{v}\|_{0,\Omega}^2 + \frac{1}{2} \left\| \overline{\mathbf{v}}^h \right\|_{0,\Omega}^2. \tag{3.14}$$

The term $\delta^2 \kappa \|\overline{\mathbf{v}}^h\|_X^2$ is positive, so it will be dropped, which yields

$$\frac{1}{2} \left\| \overline{\mathbf{v}}^h \right\|_{0,\Omega}^2 \leq \frac{1}{2} \|\mathbf{v}\|_{0,\Omega}^2. \tag{3.15}$$

Multiplying by 2 and taking the square root yields the estimate (3.12). Equation (3.13) follows immediately from (3.12) and the definition of $G_N^h$.                                                                              □

**Lemma 3.9.** *For smooth $\phi$ the discrete approximate deconvolution operator satisfies*

$$\left\| \phi - G_N^h \overline{\phi}^h \right\|_{0,\Omega} \leq C_1 \delta^{2N+2} \|G\phi\|_{2N+2,\Omega} + C_2 \left( \delta h^r + h^{r+1} \right) \left( \sum_{n=2}^{N+1} |G^n \phi|_{r+1,\Omega} \right). \tag{3.16}$$

*Proof.* The proof follows the same arguments as in [27] for the case of continuous finite element discretization of the filter problem. The error is decomposed in the following way:

$$\left\| \phi - G_N^h \overline{\phi}^h \right\|_{0,\Omega} \leq \left\| \phi - G_N \overline{\phi} \right\|_{0,\Omega} + \left\| G_N \overline{\phi} - G_N^h \overline{\phi} \right\|_{0,\Omega} + \left\| G_N^h \overline{\phi} - G_N^h \overline{\phi}^h \right\|_{0,\Omega}. \tag{3.17}$$

Lemma 3.7 gives

$$\left\| \phi - G_N \overline{\phi} \right\|_{0,\Omega} \leq C \delta^{2N+2} \left\| \overline{\phi} \right\|_{2N+2,\Omega}. \tag{3.18}$$

The standard discontinuous finite element bound for (3.8) is given by (cf. Rivière [13])

$$\left\| \overline{\phi} - \overline{\phi}^h \right\|_{0,\Omega} \leq C \left( \delta h^r + h^{r+1} \right) \left| \overline{\phi} \right|_{r+1,\Omega}. \tag{3.19}$$

Lemma 3.8 gives for the third term in (3.17) that $\|G_N^h \overline{\phi} - G_N^h \overline{\phi}^h\| \leq C \|\overline{\phi} - \overline{\phi}^h\|$. Then, (3.19) is applied.

Now, it is left to bound the second term from (3.17). First, note that for $N = 0$, $\|G_0 \overline{\phi}^h - G_0^h \overline{\phi}^h\|_{0,\Omega} = 0$. Based on Definition 3.6 of continuous and discrete deconvolution operators and their expansion (see (3.10)), $G_N$ is a polynomial of degree $N$ in $G$ (and $G_N^h$ in $G^h$ as well). Thus, the second term in (3.17) can be written as

$$\left\| G_N \overline{\phi} - G_N^h \overline{\phi} \right\|_{0,\Omega} = \left\| \sum_{n=0}^{N} \alpha_n \left( G^n \overline{\phi} - \left( G^h \right)^n \overline{\phi} \right) \right\|_{0,\Omega} \leq \sum_{n=0}^{N} \alpha_n \left\| G^n \overline{\phi} - \left( G^h \right)^n \overline{\phi} \right\|_{0,\Omega}. \tag{3.20}$$

For $O(1)$ coefficients $\alpha_n$ and for $N = 1$, the result (3.19) gives

$$\left\| G \overline{\phi} - \left( G^h \right) \overline{\phi} \right\|_{0,\Omega} = \left\| \overline{\overline{\phi}} - \overline{\overline{\phi}}^h \right\|_{0,\Omega}$$

$$\leq C \left( \delta h^r + h^{r+1} \right) \left( \left| \overline{\overline{\phi}} \right|_{r+1,\Omega} \right). \tag{3.21}$$

For $N = 2$, the results (3.19) and (3.12) give

$$\left\| G_2\bar{\phi} - \left(G_2^h\right)\bar{\phi} \right\|_{0,\Omega} = \left\| \bar{\bar{\bar{\phi}}} - \overline{\bar{\phi}^h}^h \right\|_{0,\Omega}$$

$$\leq \left\| \bar{\bar{\bar{\phi}}} - \bar{\bar{\bar{\phi}}}^h \right\|_{0,\Omega} + \left\| \bar{\bar{\phi}}^h - \overline{\bar{\phi}^h}^h \right\|_{0,\Omega} \qquad (3.22)$$

$$\leq \left\| \bar{\bar{\bar{\phi}}} - \bar{\bar{\bar{\phi}}}^h \right\|_{0,\Omega} + \left\| \bar{\bar{\phi}} - \bar{\bar{\phi}}^h \right\|_{0,\Omega}$$

$$\leq C\left(\delta h^r + h^{r+1}\right)\left( \left| \bar{\bar{\bar{\phi}}} \right|_{r+1,\Omega} + \left| \bar{\bar{\phi}} \right|_{r+1,\Omega} \right).$$

Inductively,

$$\left\| G_N\bar{\phi} - G_N^h\bar{\phi} \right\|_{0,\Omega} \leq C\left(\delta h^r + h^{r+1}\right)\left( \sum_{n=2}^{N+1} |G^n\phi|_{r+1,\Omega} \right). \qquad (3.23)$$

The proof is completed by combining the derived bounds for the terms in (3.17). □

*Remark 3.10.* There remains the question of uniform in $\delta$ bound of the last term, $|(G)^n\phi|_{r+1,\Omega}$, in (3.16). This is a question about uniform regularity of an elliptic-elliptic singular perturbation problem and some results are proven in [28] by Layton. To summarize, in the periodic case it is very easy to show by Fourier series that for all $k$

$$|G\phi|_{r+1,\Omega} \leq C|\phi|_{r+1,\Omega} \text{ and thus } |G^n\phi|_{r+1,\Omega} \leq C|\phi|_{r+1,\Omega}. \qquad (3.24)$$

The nonperiodic case can be more delicate. Suppose $\partial\Omega \in C^{r+3}$ and $\phi = 0$ on $\partial\Omega$ (i.e., $\phi \in H_0^1(\Omega) \cap H^{r+1}(\Omega)$). Call $G\phi = \bar{\phi}$ so $\bar{\phi}$ satisfies

$$-\delta^2\Delta\bar{\phi} + \bar{\phi} = \phi \quad \text{in } \Omega, \qquad \bar{\phi} = 0 \quad \text{on } \partial\Omega. \qquad (3.25)$$

Then it is known that $\bar{\phi} \in H^{r+3}(\Omega) \cap H_0^1(\Omega)$, and $\Delta\bar{\phi} = 0$ on $\partial\Omega$. Further,

$$\left\| \bar{\phi} \right\|_{j,\Omega} \leq C\|\phi\|_{j,\Omega}, \quad j = 0, 1, 2. \qquad (3.26)$$

So, (3.24) holds for $r = -1, 0, +1$. It also holds for higher values of $r$ provided additionally $\Delta^j\phi = 0$ on $\partial\Omega$ for $0 \leq j \leq [(r+1)/2] - 1$.

Now consider the second term $n = 2$, that is, $G^2\phi = \overline{\overline{\phi}}$. We know from elliptic theory for $\phi \in H^{r+1}(\Omega) \bigcap H_0^1(\Omega)$, that $\overline{\phi} \in H^{r+3}(\Omega) \bigcap H_0^1(\Omega)$, (as noted above) $\Delta\overline{\phi} = 0$ on $\partial\Omega$ and

$$-\delta^2 \Delta\overline{\overline{\phi}} + \overline{\overline{\phi}} = \overline{\phi} \quad \text{in } \Omega, \qquad \overline{\phi} = \Delta\overline{\phi} = 0 \quad \text{on } \partial\Omega. \tag{3.27}$$

Theorem 1.1 in [28] then implies, uniformly in $\delta$,

$$\left\| \overline{\overline{\phi}} \right\|_{j,\Omega} \le C \left\| \overline{\phi} \right\|_{j,\Omega}, \quad j = 0, 1, 2, 3, 4. \tag{3.28}$$

This extends directly to $G^n\phi$.

Extending Lemma 3.9, the following assumption will be made.

*Assumption DG1.* The $|G^n(\phi)|_{r+1,\Omega}$ terms in (3.16) are independent of $\delta$ and

$$\left\| \phi - G_N^h \overline{\phi}^h \right\| \le C_1 \delta^{2N+2} \|\phi\|_{2N+2,\Omega} + C_2 \left( \delta h^r + h^{r+1} \right) |\phi|_{r+1,\Omega}. \tag{3.29}$$

The minimal conditions that are assumed throughout are that the (discrete) filter and (discrete) deconvolution used satisfy the following consistency conditions of Stanculescu [29].

*Assumption DG2.* $G^h$ and $(I - G_N^h G^h)$ are symmetric, positive definite (*SPD*) operators.

These have been proven to hold for van Cittert deconvolution (cf. Stanculescu [29], Manica and Merdan [22] and Layton et al. [27]). For the DG method, the second assumption restricts our parameter $e_a = -1$ for the discretization of the filter problem (3.8), so that the bilinear form $a(\cdot, \cdot)$ is symmetric.

The numerical scheme that uses discontinuous finite elements in space and backward Euler in time is derived next. For this, let $\Delta t$ denote the time step such that $M = T/\Delta t$ is a positive integer. Let $0 = t_0 < t_1 < \cdots < t_M = T$ be a subdivision of the interval $(0, T)$. The function $\phi$ evaluated at the time $t_m$ is denoted by $\phi_m$. With the above forms, the fully discrete scheme is: find $(\mathbf{u}_n^h, p_n^h)_{n \ge 0} \in \mathbf{X}^h \times Q^h$ such that:

$$\frac{1}{\Delta t} \left( \mathbf{u}_{n+1}^h - \mathbf{u}_n^h, \mathbf{v} \right)_\Omega + \nu \left( a\left( \mathbf{u}_{n+1}^h, \mathbf{v} \right) + J_0 \left( \mathbf{u}_{n+1}^h, \mathbf{v} \right) \right) + \chi \left( \mathbf{u}_{n+1}^h - G_N^h \overline{\mathbf{u}_{n+1}^h}^h, \mathbf{v} \right)_\Omega$$
$$+ c^{\mathbf{u}_n^h} \left( \mathbf{u}_n^h; \mathbf{u}_{n+1}^h, \mathbf{v} \right) + b\left( \mathbf{v}, p_{n+1}^h \right) = (\mathbf{f}_{n+1}, \mathbf{v})_\Omega \quad \forall \mathbf{v} \in \mathbf{X}^h, \tag{3.30}$$

$$b\left( \mathbf{u}_{n+1}^h, q \right) = 0 \quad \forall q \in Q^h, \tag{3.31}$$

$$\left( \mathbf{u}_0^h, \mathbf{v} \right)_\Omega = (\mathbf{u}_0, \mathbf{v})_\Omega \quad \forall \mathbf{v} \in \mathbf{X}^h. \tag{3.32}$$

*Remark 3.11.* The time relaxation term can be treated explicitly such that the optimal accuracy and stability are obtained and this would make the scheme much easier to compute [30].

The consistency result of the semidiscrete scheme is showed next.

**Lemma 3.12** (Consistency). *Let $(\mathbf{u}, p)$ be the solution to (1.7)–(1.10), then $(\mathbf{u}, p)$ satisfies*

$$(\mathbf{u}_t, \mathbf{v})_\Omega + \nu\big(a(\mathbf{u}, \mathbf{v}) + J_0(\mathbf{u}, \mathbf{v})\big) + \chi\left(\mathbf{u} - G_N^h \overline{\mathbf{u}}^h, \mathbf{v}\right)_\Omega + c^{\mathbf{u}}(\mathbf{u}; \mathbf{u}, \mathbf{v}) + b(\mathbf{v}, p) \tag{3.33}$$

$$= (\mathbf{f}, \mathbf{v})_\Omega + E(\mathbf{u}, p, \mathbf{f}; \mathbf{v}) \quad \forall \mathbf{v} \in \mathbf{X}^h,$$

$$b(\mathbf{u}, q) = 0 \quad \forall q \in Q^h, \tag{3.34}$$

$$(\mathbf{u}(0), \mathbf{v})_\Omega = (\mathbf{u}_0, \mathbf{v})_\Omega \quad \forall \mathbf{v} \in \mathbf{X}^h, \tag{3.35}$$

*where the consistency error $E(\mathbf{u}, \mathbf{v}) = \chi(\mathbf{u} - G_N^h \overline{\mathbf{u}}^h, \mathbf{v})_\Omega - \chi(\mathbf{u} - G_N \overline{\mathbf{u}}, \mathbf{v})_\Omega$. Furthermore,*

$$E(\mathbf{u}, \mathbf{v}) \le C\left(\delta h^r + h^{r+1}\right) |\mathbf{u}|_{r+1,\Omega} \|\mathbf{v}\|_{0,\Omega}. \tag{3.36}$$

*Proof.* Equations (3.34) and (3.35) are clearly satisfied because of (1.8), (1.9), and (1.10) and the regularity of $\mathbf{u}$. Next, we multiply (1.7) by $\mathbf{v}$ and integrate over one mesh element $E$

$$(\mathbf{u}_t, \mathbf{v})_E + (\nabla \cdot (\mathbf{u}\mathbf{u}), \mathbf{v})_E - \nu(\Delta \mathbf{u}, \mathbf{v})_E + \chi(\mathbf{u} - G_N \overline{\mathbf{u}}, \mathbf{v})_E + \left(\nabla p, \mathbf{v}\right)_E = (\mathbf{f}, \mathbf{v})_E. \tag{3.37}$$

Summing over all elements $E$

$$\sum_{E \in \mathcal{E}_h} (\mathbf{u}_t, \mathbf{v})_E + (\nabla \cdot (\mathbf{u}\mathbf{u}), \mathbf{v})_\Omega - \nu \sum_{E \in \mathcal{E}_h} (\Delta \mathbf{u}, \mathbf{v})_E + \chi(\mathbf{u} - G_N \overline{\mathbf{u}}, \mathbf{v})_\Omega + \left(\nabla p, \mathbf{v}\right)_\Omega = (\mathbf{f}, \mathbf{v})_\Omega. \tag{3.38}$$

By Green's formula

$$-\nu \sum_{E \in \mathcal{E}_h} (\Delta \mathbf{u}, \mathbf{v})_E = -\nu(\Delta \mathbf{u}, \mathbf{v})_\Omega = \nu \sum_{E \in \mathcal{E}_h} (\nabla \mathbf{u}, \nabla \mathbf{v})_E - \nu \sum_{e \in \Gamma_h \cup \partial \Omega} \int_e (\nabla \mathbf{u})\mathbf{n}_e \cdot [\mathbf{v}]. \tag{3.39}$$

The regularity of $\mathbf{u}$ then yields

$$-\nu \sum_{E \in \mathcal{E}_h} (\Delta \mathbf{u}, \mathbf{v})_E = \nu(a(\mathbf{u}, \mathbf{v}) + J_0(\mathbf{u}, \mathbf{v})). \tag{3.40}$$

Note that Green's formula yields

$$\left(\nabla p, \mathbf{v}\right)_\Omega = b(\mathbf{v}, p), \tag{3.41}$$

and that the incompressibility condition with the regularity of $\mathbf{u}$ yields

$$(\mathbf{u} \cdot \nabla \mathbf{u}, \mathbf{v})_\Omega = c^{\mathbf{u}}(\mathbf{u}; \mathbf{u}, \mathbf{v}). \tag{3.42}$$

The final result is obtained by bounding the consistency error $E(\mathbf{u}, \mathbf{v})$. For $N = 0$, we have

$$E(\mathbf{u}, \mathbf{v}) \leq \left\| \overline{\mathbf{u}} - \overline{\mathbf{u}}^h \right\|_{0,\Omega} \|\mathbf{v}\|_{0,\Omega} \leq C\left(\delta h^r + h^{r+1}\right) |\overline{\mathbf{u}}|_{r+1,\Omega} \|\mathbf{v}\|_{0,\Omega}. \tag{3.43}$$

For $N = 1$,

$$
\begin{aligned}
E(\mathbf{u}, \mathbf{v}) &\leq \left\| G\overline{\mathbf{u}} - (G^h)\overline{\mathbf{u}}^h \right\|_{0,\Omega} \|\mathbf{v}\|_{0,\Omega} \\[2mm]
&\leq \left\| \overline{\overline{\mathbf{u}}} - \overline{\overline{\mathbf{u}}^h}^h \right\|_{0,\Omega} \|\mathbf{v}\|_{0,\Omega} \\[2mm]
&\leq \left( \left\| \overline{\overline{\mathbf{u}}} - \overline{\overline{\mathbf{u}}}^h \right\|_{0,\Omega} + \left\| \overline{\overline{\mathbf{u}}}^h - \overline{\overline{\mathbf{u}}^h}^h \right\|_{0,\Omega} \right) \|\mathbf{v}\|_{0,\Omega} \\[2mm]
&\leq \left( \left\| \overline{\overline{\mathbf{u}}} - \overline{\overline{\mathbf{u}}}^h \right\|_{0,\Omega} + \left\| \overline{\mathbf{u}} - \overline{\mathbf{u}}^h \right\|_{0,\Omega} \right) \|\mathbf{v}\|_{0,\Omega} \\[2mm]
&\leq C\left(\delta h^r + h^{r+1}\right) \left( \left| \overline{\overline{\mathbf{u}}} \right|_{r+1,\Omega} + |\overline{\mathbf{u}}|_{r+1,\Omega} \right) \|\mathbf{v}\|_{0,\Omega}.
\end{aligned}
\tag{3.44}
$$

The bound for $E(\mathbf{u}, \mathbf{v})$ is obtained by applying an induction argument and Remark 3.10.    □

The existence and uniqueness of the discrete solution is stated next.

**Proposition 3.13.** *Assume that Lemma 3.1 holds. Then, there exists a unique solution to* (3.30)–(3.32).

*Proof.* The existence of $\mathbf{u}_0^h$ is trivial. Given $\mathbf{u}_n^h$, the problem of finding a unique $\mathbf{u}_{n+1}^h$ satisfying (3.30)-(3.31) is linear and finite-dimensional. Therefore, it suffices to show uniqueness of the solution. Consider the problem restricted to the subspace $\mathbf{V}^h$ defined by

$$\mathbf{V}^h = \left\{ \mathbf{v} \in \mathbf{X}^h : b(\mathbf{v}, q) = 0 \ \forall q \in Q^h \right\}. \tag{3.45}$$

Let $\mathbf{u}_{n+1}^h$ and $\widehat{\mathbf{u}}_{n+1}^h$ be two solutions and let $\boldsymbol{\theta}_{n+1} = \mathbf{u}_{n+1}^h - \widehat{\mathbf{u}}_{n+1}^h$. Then, $\boldsymbol{\theta}_{n+1}$ satisfies:

$$
\begin{aligned}
&\frac{1}{\Delta t}(\boldsymbol{\theta}_{n+1}, \mathbf{v})_\Omega + \nu(a(\boldsymbol{\theta}_{n+1}, \mathbf{v}) + J_0(\boldsymbol{\theta}_{n+1}, \mathbf{v})) + c^{\mathbf{u}_n^h}\left(\mathbf{u}_n^h; \boldsymbol{\theta}_{n+1}, \mathbf{v}\right) \\[2mm]
&\quad + \chi\left(\boldsymbol{\theta}_{n+1} - G_N^h \overline{\boldsymbol{\theta}_{n+1}}^h, \mathbf{v}\right) = 0 \quad \forall \mathbf{v} \in \mathbf{V}^h.
\end{aligned}
\tag{3.46}
$$

Choosing $\mathbf{v} = \boldsymbol{\theta}_{n+1}$ and using the coercivity result (3.5), positivity result (3.7) and positivity of the operator $I - G_N^h G^h$ given in Assumption DG2, we obtain:

$$\frac{1}{\Delta t} \|\boldsymbol{\theta}_{n+1}\|_{0,\Omega}^2 + \nu\kappa \|\boldsymbol{\theta}_{n+1}\|_X^2 \leq 0, \tag{3.47}$$

which yields that $\boldsymbol{\theta}_{n+1} = \mathbf{0}$. The existence and uniqueness of the pressure $p^h_{n+1}$ is then obtained from the inf-sup condition (3.6). □

Some approximation properties of the spaces $\mathbf{X}^h$ and $Q^h$ are recalled next. From Crouzeix and Raviart [31], and Girault et al. [16], for each integer $r \geq 1$, and for any $\mathbf{v} \in (H^1_0(\Omega))^2$, there is a unique discrete velocity $\tilde{\mathbf{v}} \in \mathbf{X}^h$ such that

$$b(\mathbf{v} - \tilde{\mathbf{v}}, q) = 0 \quad \forall q \in Q^h. \tag{3.48}$$

Furthermore, if $\mathbf{v} \in (H^1_0(\Omega))^2 \cap (H^{r+1}(\Omega))^2$, there is a constant $C$ independent of $h$ such that

$$\|\mathbf{v} - \tilde{\mathbf{v}}\|_X \leq Ch^r |\mathbf{v}|_{r+1,\Omega}, \tag{3.49}$$

$$|\mathbf{v} - \tilde{\mathbf{v}}|_{m,\Omega} \leq Ch^{r+1-m} |\mathbf{v}|_{r+1,\Omega}, \quad m = 0, 1. \tag{3.50}$$

For the pressure space, we use the approximation given by the $L^2$ projection. For any $q \in L^2_0(\Omega)$, there exists a unique discrete pressure $\tilde{q} \in Q^h$ such that

$$(q - \tilde{q}, z)_\Omega = 0 \quad \forall z \in Q^h. \tag{3.51}$$

In addition, if $q \in H^r(\Omega)$, then

$$\|q - \tilde{q}\|_{m,E} \leq Ch^{r-m} |q|_{r,E}, \quad \forall E \in \mathcal{E}_h, \ m = 0, 1, 2. \tag{3.52}$$

The discrete Gronwall's lemma plays an important role in the following analysis.

**Lemma 3.14** (Discrete Gronwall's Lemma (cf. Heywood and Rannacher [32])). *Let $\Delta t$, $H$, and $a_n$, $b_n$, $c_n$, $\gamma_n$ (for integers $n \geq 0$) be nonnegative numbers such that*

$$a_l + \Delta t \sum_{n=0}^{l} b_n \leq \Delta t \sum_{n=0}^{l} \gamma_n a_n + \Delta t \sum_{n=0}^{l} c_n + H \quad \text{for } l \geq 0. \tag{3.53}$$

*Suppose that $\Delta t \gamma_n < 1$, for all $n$, and set $\sigma_n = (1 - \Delta t \gamma_n)^{-1}$. Then,*

$$a_l + \Delta t \sum_{n=0}^{l} b_n \leq \exp\left(\Delta t \sum_{n=0}^{l} \sigma_n \gamma_n\right) \left\{\Delta t \sum_{n=0}^{l} c_n + H\right\} \quad \text{for } l \geq 0. \tag{3.54}$$

The family of time relaxation models is a regularization of NSE, and therefore it is natural to investigate the finite element error between the discretized model and the NSE.

To that end, we will assume that the solution to the Navier-Stokes equations $(\mathbf{w}, P)$ that is approximated is a strong solution and in particular satisfies (cf. Rivière [13])

$$(\mathbf{w}_t, \mathbf{v})_\Omega + \nu(a(\mathbf{w}, \mathbf{v}) + J_0(\mathbf{w}, \mathbf{v})) + c^{\mathbf{w}}(\mathbf{w}; \mathbf{w}, \mathbf{v}) + b(\mathbf{v}, P) = (\mathbf{f}, \mathbf{v})_\Omega \quad \forall \mathbf{v} \in \mathbf{X}^h, \tag{3.55}$$

$$b(\mathbf{w}, q) = 0 \quad \forall q \in Q^h, \tag{3.56}$$

$$\left(\mathbf{w}_0^h, \mathbf{v}\right)_\Omega = (\mathbf{u}_0, \mathbf{v})_\Omega \quad \forall \mathbf{v} \in \mathbf{X}^h. \tag{3.57}$$

## 4. A Priori Error Estimates

In this section, convergence of the scheme (3.30)–(3.32) is proved. Optimal error estimates in the energy norm are obtained.

**Theorem 4.1.** *Assume that* $\mathbf{w} \in l^2(0, T; (H^{r+1}(\Omega))^2) \cap l^2(0, T; (H^{2N+2}(\Omega))^2$, $\mathbf{w}_t \in l^2(0, T; (H^{r+1}(\Omega))^2) \cap L^\infty((0, T) \times \Omega)$, $\mathbf{w}_{tt} \in L^2(0, T; (H^1(\Omega))^2)$, $p \in l^2(0, T; H^r(\Omega))$ and $\mathbf{u}_0 \in (H^{r+1}(\Omega))^2$. Assume also that the coercivity Lemma 3.1 holds and that* $\mathbf{w}$ *satisfies* (3.24). *If* $\delta$ *is chosen of the order of* $h$, *and* $\Delta t < 1$, *there exists a constant* $C$, *independent of* $h$ *and* $\Delta t$ *but dependent on* $\nu^{-1}$ *such that the following error bound holds, for any* $1 \le m \le M$:

$$\left\| \mathbf{w}_m - \mathbf{u}_m^h \right\|_{0,\Omega} + \left( \nu \kappa \Delta t \sum_{n=1}^m \left\| \mathbf{w}_n - \mathbf{u}_n^h \right\|_X^2 \right)^{1/2} \le Ch^r \left( \nu^{-1} + \nu + \chi + 1 \right) + C\Delta t + C\chi h^{2N+2}. \tag{4.1}$$

*Remark 4.2.* The dependence of these error estimates with respect to the Reynolds number Re $(\sim 1/\nu)$ is $\mathcal{O}(\text{Re}\, e^{\text{Re}})$, which is an improvement with respect to the continuous finite element method where the dependence is $\mathcal{O}(\text{Re}\, e^{\text{Re}^3})$.

*Proof.* Defining $\mathbf{e}_n = \mathbf{u}^h(t^n) - \mathbf{w}(t^n)$ and subtracting (3.55) from (3.30), we have

$$\frac{1}{\Delta t}(\mathbf{e}_{n+1} - \mathbf{e}_n, \mathbf{v})_\Omega - (\mathbf{w}_t(t_{n+1}), \mathbf{v})_\Omega + \nu(a + J_0)(\mathbf{e}_{n+1}, \mathbf{v}) + \chi\left(\mathbf{e}_{n+1} - G_N^h \overline{\mathbf{e}_{n+1}}^h, \mathbf{v}\right)_\Omega$$

$$+ c^{\mathbf{u}_n^h}\left(\mathbf{u}_n^h; \mathbf{u}_{n+1}^h, \mathbf{v}\right) - c^{\mathbf{w}_{n+1}}(\mathbf{w}_{n+1}; \mathbf{w}_{n+1}, \mathbf{v}) + b\left(\mathbf{v}, p_{n+1}^h - P_{n+1}\right) \tag{4.2}$$

$$= -\frac{1}{\Delta t}(\mathbf{w}_{n+1} - \mathbf{w}_n, \mathbf{v})_\Omega - \chi\left(\mathbf{w}_{n+1} - G_N^h \overline{\mathbf{w}_{n+1}}^h, \mathbf{v}\right)_\Omega \quad \forall \mathbf{v} \in \mathbf{X}^h,$$

$$b(\mathbf{e}_{n+1}, q) = 0 \quad \forall q \in Q^h. \tag{4.3}$$

We now decompose the error $\mathbf{e}_n = \boldsymbol{\phi}_n - \boldsymbol{\eta}_n$, where $\boldsymbol{\phi}_n = \mathbf{u}_n^h - \widetilde{\mathbf{w}}_n$ and $\boldsymbol{\eta}_n$ is the interpolation

error $\eta_n = \mathbf{w}_n - \widetilde{\mathbf{w}}_n$. Choosing $\mathbf{v} = \phi_{n+1}$ in the equation above, using the coercivity result (3.5) and positivity of the operator $I - G_N^h G^h$, we obtain for (4.2)

$$
\begin{aligned}
\frac{1}{2\Delta t}\left(\|\phi_{n+1}\|_{0,\Omega}^2 - \|\phi_n\|_{0,\Omega}^2\right) &+ \nu\kappa\|\phi_{n+1}\|_X^2 \\
&+ c^{\mathbf{u}_n^h}\left(\mathbf{u}_n^h; \mathbf{u}_{n+1}^h, \phi_{n+1}\right) - c^{\mathbf{w}_{n+1}}\left(\mathbf{w}_{n+1}; \mathbf{w}_{n+1}, \phi_{n+1}\right) \le \left(\eta_t(t_{n+1}), \phi_{n+1}\right)_\Omega \\
&+ \nu(a + J_0)\left(\eta_{n+1}, \phi_{n+1}\right) + \left(\widetilde{\mathbf{w}}_t(t_{n+1}) - \frac{1}{\Delta t}(\widetilde{\mathbf{w}}_{n+1} - \widetilde{\mathbf{w}}_n), \phi_{n+1}\right)_\Omega \\
&+ \chi\left(\eta_{n+1} - G_N^h\overline{\eta_{n+1}}^h, \phi_{n+1}\right)_\Omega - \chi\left(\mathbf{w}_{n+1} - G_N^h\overline{\mathbf{w}_{n+1}}^h, \phi_{n+1}\right)_\Omega + b\left(\phi_{n+1}, P_{n+1} - p_{n+1}^h\right).
\end{aligned}
$$
(4.4)

Consider now the nonlinear terms from the above equation. First note that since $\mathbf{w}$ is continuous, we can rewrite

$$
c^{\mathbf{w}_{n+1}}\left(\mathbf{w}_{n+1}; \mathbf{w}_{n+1}, \phi_{n+1}\right) = c^{\mathbf{u}_n^h}\left(\mathbf{w}_{n+1}; \mathbf{w}_{n+1}, \phi_{n+1}\right),
$$
(4.5)

so, for readability, the superscript $\mathbf{w}_n^h$ in the $c$ form is dropped. Therefore, adding and subtracting the interpolant $\widetilde{\mathbf{w}}_{n+1}$ yields

$$
\begin{aligned}
c^{\mathbf{u}_n^h}\left(\mathbf{u}_n^h, \mathbf{u}_{n+1}^h, \phi_{n+1}\right) &- c^{\mathbf{u}_n^h}\left(\mathbf{w}_{n+1}, \mathbf{w}_{n+1}, \phi_{n+1}\right) \\
&= c\left(\mathbf{u}_n^h, \phi_{n+1}, \phi_{n+1}\right) - c(\phi_n, \eta_{n+1}, \phi_{n+1}) + c(\phi_n, \mathbf{w}_{n+1}, \phi_{n+1}) \\
&\quad - c(\eta_n, \widetilde{\mathbf{w}}_{n+1}, \phi_{n+1}) - c(\mathbf{w}_n, \eta_{n+1}, \phi_{n+1}) - c(\mathbf{w}_{n+1} - \mathbf{w}_n, \mathbf{w}_{n+1}, \phi_{n+1}).
\end{aligned}
$$
(4.6)

Thus, the error equation (4.4) is rewritten as

$$
\begin{aligned}
\frac{1}{2\Delta t}\left(\|\phi_{n+1}\|_{0,\Omega}^2 - \|\phi_n\|_{0,\Omega}^2\right) &+ \nu\kappa\|\phi_{n+1}\|_X^2 + c\left(\mathbf{u}_n^h, \phi_{n+1}, \phi_{n+1}\right) \\
\le\ & \left|c(\phi_n, \eta_{n+1}, \phi_{n+1})\right| + \left|c(\phi_n, \mathbf{w}_{n+1}, \phi_{n+1})\right| + \left|c(\eta_n, \widetilde{\mathbf{w}}_{n+1}, \phi_{n+1})\right| \\
&+ \left|c(\mathbf{w}_n, \eta_{n+1}, \phi_{n+1})\right| + \left|c(\mathbf{w}_{n+1} - \mathbf{w}_n, \mathbf{w}_{n+1}, \phi_{n+1})\right| + \left|(\eta_t(t_{n+1}), \phi_{n+1})_\Omega\right| \\
&+ \left|\left(\widetilde{\mathbf{w}}_t(t_{n+1}) - \frac{1}{\Delta t}(\widetilde{\mathbf{w}}_{n+1} - \widetilde{\mathbf{w}}_n), \phi_{n+1}\right)_\Omega\right| + \left|\nu(a + J_0)(\eta_{n+1}, \phi_{n+1})\right| \\
&+ \left|\chi\left(\eta_{n+1} - G_N^h\overline{\eta_{n+1}}^h, \phi_{n+1}\right)_\Omega\right| + \left|\chi\left(\mathbf{w}_{n+1} - G_N^h\overline{\mathbf{w}_{n+1}}^h, \phi_{n+1}\right)_\Omega\right| + \left|b\left(\phi_{n+1}, P_{n+1} - p_{n+1}^h\right)\right| \\
\le\ & |T_0| + |T_1| + \cdots + |T_{10}|.
\end{aligned}
$$
(4.7)

From property (3.7), the term $c(\mathbf{w}_n^h; \phi_{n+1}, \phi_{n+1})$ in the left-hand side of (4.7) is positive and therefore it will be dropped. For the other terms of the form $c(\cdot, \cdot, \cdot)$ that appear on the right-hand side of the above error equation we obtain bounds, exactly as in the proof of Theorem 5.2

in [19] by Kaya and Rivière. The constant $C$ is a generic constant that is independent of $h, \nu$ and $\Delta t$, and that takes different values at different places

$$|T_0| = |c(\phi_n, \boldsymbol{\eta}_{n+1}, \phi_{n+1})| \leq \frac{\nu\kappa}{20}\|\phi_{n+1}\|_X^2 + \frac{C}{\nu}\|\phi_n\|_{0,\Omega}^2,$$

$$|T_1| = |c(\phi_n, \mathbf{w}_{n+1}, \phi_{n+1})| \leq \frac{\nu\kappa}{20}\|\phi_{n+1}\|_X^2 + \frac{C}{\nu}\|\phi_n\|_{0,\Omega}^2,$$

$$|T_2| = |c(\eta_n, \tilde{\mathbf{w}}_{n+1}, \phi_{n+1})| \leq \frac{\nu\kappa}{20}\|\phi_{n+1}\|_X^2 + \frac{C}{\nu}h^{2r}|\mathbf{w}_n|_{2r,\Omega}^2, \quad (4.8)$$

$$|T_3| = |c(\mathbf{w}_n, \boldsymbol{\eta}_{n+1}, \phi_{n+1})| \leq \frac{\nu\kappa}{20}\|\phi_{n+1}\|_X^2 + \frac{C}{\nu}h^{2r}|\mathbf{w}_n|_{2r,\Omega}^2,$$

$$|T_4| = |c(\mathbf{w}_{n+1} - \mathbf{w}_n, \mathbf{w}_{n+1}, \phi_{n+1})| \leq \frac{\nu\kappa}{20}\|\phi_{n+1}\|_X^2 + \frac{C}{\nu}\Delta t^2\|\mathbf{w}_t\|_{L^\infty([t_n,t_{n+1}]\times\Omega)}^2.$$

Therefore, we have

$$|T_0| + \cdots + |T_4| \leq \frac{5\nu\kappa}{20}\|\phi_{n+1}\|_X^2 + C\nu^{-1}\|\phi_n\|_{0,\Omega}^2 + C\nu^{-1}h^{2r}|\mathbf{w}_n|_{r+1,\Omega}^2 + C\nu^{-1}\Delta t^2\|\mathbf{w}_t\|_{L^\infty([t_n,t_{n+1}]\times\Omega)}^2. \quad (4.9)$$

To bound $T_5$, Cauchy-Schwarz's inequality, Young's inequality and the approximation result (3.50) for $\mathbf{w}_t$ are applied

$$|T_5| \leq \|\phi_{n+1}\|_{0,\Omega}\|\boldsymbol{\eta}_t(t_{n+1})\|_{0,\Omega}$$

$$\leq \frac{1}{8}\|\phi_{n+1}\|_{0,\Omega}^2 + Ch^{2r+2}\left|\mathbf{w}_t\left(t^{n+1}\right)\right|_{k+1,\Omega}^2. \quad (4.10)$$

To bound the term $T_6$, a Taylor expansion with integral remainder is used

$$\tilde{\mathbf{w}}_n = \tilde{\mathbf{w}}_{n+1} - \Delta t\tilde{\mathbf{w}}_t(t_{n+1}) + \frac{1}{2}\int_{t_n}^{t_{n+1}}(s - t_n)\tilde{\mathbf{w}}_{tt}(s)ds. \quad (4.11)$$

This implies that

$$\left\|\tilde{\mathbf{w}}_t(t_{n+1}) - \frac{\tilde{\mathbf{w}}_{n+1} - \tilde{\mathbf{w}}_n}{\Delta t}\right\|_{0,\Omega}^2 \leq \frac{\Delta t}{6}\int_{t_n}^{t_{n+1}}\|\tilde{\mathbf{w}}_{tt}(s)\|_{0,\Omega}^2 ds. \quad (4.12)$$

Thus, with (3.50), we have

$$|T_6| \leq \frac{1}{8}\|\phi_{n+1}\|_{0,\Omega}^2 + C\Delta t\int_{t_n}^{t_{n+1}}\|\tilde{\mathbf{w}}_{tt}(s)\|_{0,\Omega}^2 ds$$

$$\leq \frac{1}{8}\|\phi_{n+1}\|_{0,\Omega}^2 + C\Delta t\int_{t_n}^{t_{n+1}}\|\mathbf{w}_{tt}(s)\|_{0,\Omega}^2 ds. \quad (4.13)$$

Next, the term $T_7$ is expanded as

$$
\begin{aligned}
|T_7| &\leq \left| \nu \sum_{E \in \mathcal{E}_h} \int_E \nabla \boldsymbol{\eta}_{n+1} : \nabla \boldsymbol{\phi}_{n+1} \right| + \left| \nu \sum_{e \in \Gamma_h \cup \partial \Omega} \int_e \{ \nabla \boldsymbol{\eta}_{n+1} \} \mathbf{n}_e \cdot [\boldsymbol{\phi}_{n+1}] \right| \\
&\quad + \left| \nu \epsilon_a \sum_{e \in \Gamma_h \cup \partial \Omega} \int_e \{ \nabla \boldsymbol{\phi}_{n+1} \} \mathbf{n}_e \cdot [\boldsymbol{\eta}_{n+1}] \right| + |\nu J_0(\boldsymbol{\eta}_{n+1}, \boldsymbol{\phi}_{n+1})| \\
&= |T_{71}| + |T_{72}| + |T_{73}| + |T_{74}|.
\end{aligned}
\tag{4.14}
$$

The term $T_{71}$ is bounded using Cauchy-Schwarz inequality, Young's inequality and the approximation result (3.49)

$$
\begin{aligned}
|T_{71}| &\leq \nu \| \boldsymbol{\phi}_{n+1} \|_X \| \boldsymbol{\eta}_{n+1} \|_X \\
&\leq \frac{\nu \kappa}{20} \| \boldsymbol{\phi}_{n+1} \|_X^2 + C\nu \| \boldsymbol{\eta}_{n+1} \|_X^2 \\
&\leq \frac{\nu \kappa}{20} \| \boldsymbol{\phi}_{n+1} \|_X^2 + C\nu h^{2r} |\mathbf{w}_{n+1}|_{r+1,\Omega}^2.
\end{aligned}
\tag{4.15}
$$

Using Cauchy-Schwarz's inequality, trace inequality (2.8) and approximation result (3.50), we have

$$
\begin{aligned}
|T_{72}| &\leq \nu \sum_{e \in \Gamma_h \cup \partial \Omega} \| \{ \nabla \boldsymbol{\eta}_{n+1} \} \mathbf{n}_e \|_{0,e} \sum_{e \in \Gamma_h \cup \partial \Omega} \| [\boldsymbol{\phi}_{n+1}] \|_{0,e} \\
&\leq C\nu \left( \sum_{e \in \Gamma_h \cup \partial \Omega} \frac{1}{|e|} \| [\boldsymbol{\phi}_{n+1}] \|_{0,e}^2 \right)^{1/2} \left( \| | \nabla \boldsymbol{\eta}_{n+1} | \|_{0,\Omega} + h \| | \nabla^2 \boldsymbol{\eta}_{n+1} | \|_{0,\Omega} \right) \\
&\leq \frac{\nu \kappa}{20} \| \boldsymbol{\phi}_{n+1} \|_X^2 + C\nu h^{2r} |\mathbf{w}_{n+1}|_{r+1,\Omega}^2.
\end{aligned}
\tag{4.16}
$$

Using Cauchy-Schwarz's inequality, trace inequality (2.10), and approximation result (3.49), we have

$$
\begin{aligned}
|T_{73}| &\leq \nu \left( \sum_{e \in \Gamma_h \cup \partial \Omega} \| \{ \nabla \boldsymbol{\phi}_{n+1} \} \mathbf{n}_e \|_{0,e}^2 \right)^{1/2} \left( \sum_{e \in \Gamma_h \cup \partial \Omega} \| [\boldsymbol{\eta}_{n+1}] \|_{0,e}^2 \right)^{1/2} \\
&\leq C\nu \| \boldsymbol{\phi}_{n+1} \|_X \left( \sum_{e \in \Gamma_h \cup \partial \Omega} \frac{1}{|e|} \| [\boldsymbol{\eta}_{n+1}] \|_{0,e}^2 \right)^{1/2} \\
&\leq \frac{\nu \kappa}{20} \| \boldsymbol{\phi}_{n+1} \|_X^2 + C\nu h^{2r} |\mathbf{w}_{n+1}|_{r+1,\Omega}^2.
\end{aligned}
\tag{4.17}
$$

Using the approximation result (3.49), we have

$$
\begin{aligned}
|T_{74}| &\leq \nu \left( \sum_{e \in \Gamma_h \cup \partial\Omega} \frac{\sigma}{|e|} \left\| [\phi_{n+1}] \right\|_{0,e}^2 \right)^{1/2} \left( \sum_{e \in \Gamma_h \cup \partial\Omega} \frac{\sigma}{|e|} \left\| [\eta_{n+1}] \right\|_{0,e}^2 \right)^{1/2} \\
&\leq C\nu \|\phi_{n+1}\|_X \|\eta_{n+1}\|_X \\
&\leq \frac{\nu\kappa}{20} \|\phi_{n+1}\|_X^2 + C\nu h^{2r} |\mathbf{w}_{n+1}|_{r+1,\Omega}^2.
\end{aligned}
\tag{4.18}
$$

Putting together the bounds (4.15), (4.16), (4.17) and (4.18),

$$
|T_7| \leq 4\frac{\nu\kappa}{20} \|\phi_{n+1}\|_X^2 + C\nu h^{2r} |\mathbf{w}_{n+1}|_{r+1,\Omega}^2.
\tag{4.19}
$$

For the terms $T_8$ and $T_9$, Cauchy-Schwarz's inequality, Young's inequality and bounds (3.13) and (3.29) are applied

$$
\begin{aligned}
|T_8| &= \left| \chi \left( \eta_{n+1} - G_N^h \overline{\eta_{n+1}}^h, \phi_{n+1} \right)_\Omega \right| \\
&\leq \chi \left\| \eta_{n+1} - G_N^h \overline{\eta_{n+1}}^h \right\|_{0,\Omega} \|\phi_{n+1}\|_{0,\Omega} \\
&\leq C\chi^2 \left\| \eta_{n+1} - G_N^h \overline{\eta_{n+1}}^h \right\|_{0,\Omega}^2 + \frac{1}{8} \|\phi_{n+1}\|_{0,\Omega}^2 \\
&\leq C(N)\chi^2 \|\eta_{n+1}\|_{0,\Omega}^2 + \frac{1}{8} \|\phi_{n+1}\|_{0,\Omega}^2 \\
&\leq C(N)\chi^2 h^{2r+2} |\mathbf{w}_{n+1}|_{r+1,\Omega}^2 + \frac{1}{8} \|\phi_{n+1}\|_{0,\Omega}^2,
\end{aligned}
\tag{4.20}
$$

$$
\begin{aligned}
|T_9| &= \left| \chi \left( \mathbf{w}_{n+1} - G_N^h \overline{\mathbf{w}_{n+1}}^h, \phi_{n+1} \right)_\Omega \right| \\
&\leq \chi \left\| \mathbf{w}_{n+1} - G_N^h \overline{\mathbf{w}_{n+1}}^h \right\|_{0,\Omega} \|\phi_{n+1}\|_{0,\Omega} \\
&\leq C\chi^2 \left\| \mathbf{w}_{n+1} - G_N^h \overline{\mathbf{w}_{n+1}}^h \right\|_{0,\Omega}^2 + \frac{1}{8} \|\phi_{n+1}\|_{0,\Omega}^2 \\
&\leq C\chi^2 \delta^{4N+4} \|\mathbf{w}_{n+1}\|_{2N+2,\Omega}^2 + C\chi^2 \left( \delta^2 h^{2r} + h^{2r+2} \right) |\mathbf{w}_{n+1}|_{r+1,\Omega}^2 + \frac{1}{8} \|\phi_{n+1}\|_{0,\Omega}^2.
\end{aligned}
$$

Using (4.3) with (3.48) and (3.51), the pressure term $T_{10}$ is reduced to

$$
\begin{aligned}
|T_{10}| &= \left| b(\phi_{n+1}, p_{n+1} - \tilde{p}_{n+1}) + b\left(\phi_{n+1}, \tilde{p}_{n+1} - p_{n+1}^h\right) \right| \\
&= \left| b(\phi_{n+1}, p_{n+1} - \tilde{p}_{n+1}) \right| \\
&= \left| \sum_{e \in \Gamma_h} \int_e \{p_{n+1} - \tilde{p}_{n+1}\} [\phi_{n+1}] \cdot \mathbf{n}_e \right|,
\end{aligned}
\tag{4.21}
$$

which is bounded by using Cauchy-Schwarz's inequality, Young's inequality, trace inequality (2.7) and approximation result (3.52)

$$
\begin{aligned}
|T_{10}| &\le C \left( \sum_{e \in \Gamma_h \cup \partial \Omega} \frac{1}{|e|} \, \| \, [\phi_{n+1}] \, \|_{0,e}^2 \right)^{1/2} \left( \| p_{n+1} - \widetilde{p}_{n+1} \|_{0,\Omega} + h \, \| \, \| \nabla p_{n+1} - \nabla \widetilde{p}_{n+1} \| \, \|_{0,\Omega} \right) \\
&\le \frac{\nu\kappa}{20} \| \phi_{n+1} \|_X^2 + C\nu^{-1} h^{2r} |p_{n+1}|_{r,\Omega}^2.
\end{aligned}
\tag{4.22}
$$

With the bounds (4.9), (4.10), (4.13), (4.19), (4.20), and (4.22), the error equation becomes

$$
\begin{aligned}
\frac{1}{2\Delta t} &\left( \| \phi_{n+1} \|_{0,\Omega}^2 - \| \phi_n \|_{0,\Omega}^2 \right) + \frac{\nu\kappa}{2} \| \phi_{n+1} \|_X^2 \\
&\le + C\nu^{-1} \| \phi_n \|_{0,\Omega}^2 + \frac{1}{2} \| \phi_{n+1} \|_{0,\Omega}^2 + C\nu^{-1} \Delta t^2 \| \mathbf{w}_t \|_{L^\infty([t_n, t_{n+1}] \times \Omega)}^2 \\
&\quad + C h^{2r+2} \left| \mathbf{w}_t \left( t^{n+1} \right) \right|_{k+1,\Omega}^2 + C\Delta t \int_{t_n}^{t_{n+1}} \| \mathbf{w}_{tt}(s) \|_{0,\Omega}^2 ds + C\nu^{-1} h^{2r} |\mathbf{w}_n|_{r+1,\Omega}^2 \\
&\quad + C\nu h^{2r} |\mathbf{w}_{n+1}|_{r+1,\Omega}^2 + C\chi^2 h^{2r+2} \| \mathbf{w}_{n+1} \|_{r+1,\Omega}^2 + C \left( \chi^2 \left( \delta^2 h^{2r} + h^{2r+2} \right) \right) |\mathbf{w}_{n+1}|_{r+1,\Omega}^2 \\
&\quad + C\chi^2 \delta^{4N+4} \| \mathbf{w}_{n+1} \|_{2N+2,\Omega}^2 + C\nu^{-1} h^{2r} |p_{n+1}|_{r,\Omega}^2,
\end{aligned}
\tag{4.23}
$$

where $C$ are constants independent of $h, \nu$ and $\Delta t$. Now, multiply the equation by $2\Delta t$, sum from $n = 0$ to $n = m - 1$, and use the assumption that $\delta$ is chosen of the order of $h$ to obtain

$$
\begin{aligned}
(1 - \Delta t) &\| \phi_m \|_{0,\Omega}^2 + \nu\kappa\Delta t \sum_{n=0}^{m-1} \| \phi_{n+1} \|_X^2 \\
&\le \| \phi_0 \|_{0,\Omega}^2 + C \left( \nu^{-1} + 1 \right) \sum_{n=0}^{m-1} \| \phi_n \|_{0,\Omega}^2 \\
&\quad + C h^{2r} \Delta t \sum_{n=0}^{m-1} \left( \nu^{-1} |\mathbf{w}_n|_{r+1,\Omega}^2 + \left( \nu + \chi^2 \right) |\mathbf{w}_{n+1}|_{r+1,\Omega}^2 + |\mathbf{w}_t(t_{n+1})|_{r+1,\Omega}^2 + \nu^{-1} |p_{n+1}|_{r,\Omega}^2 \right) \\
&\quad + C\Delta t^2 \int_0^T \| \mathbf{w}_{tt}(s) \|_{1,\Omega}^2 ds + C\Delta t^2 \nu^{-1} \| \mathbf{w}_t \|_{L^\infty([0,T] \times \Omega)}^2 + C\Delta t \chi^2 h^{4N+4} \| \mathbf{w}_{n+1} \|_{2N+2,\Omega}^2.
\end{aligned}
\tag{4.24}
$$

Thus, using Gronwall's lemma with $\Delta t < 1$, there is a constant $C$ independent of $h$ and $\Delta t$, but dependent on $\nu^{-1}$, such that

$$
\| \phi_m \|_{0,\Omega}^2 + \nu\kappa\Delta t \sum_{n=1}^m \| \phi_n \|_X^2 \le \| \phi_0 \|_{0,\Omega}^2 + C h^{2r} \left( \nu^{-1} + \nu + \chi^2 + 1 \right) + C\Delta t^2 + C\chi^2 h^{4N+4}.
\tag{4.25}
$$

The final result is then obtained by noting that the term $\|\phi_0\|_{0,\Omega}^2$ is of order $h^{2r}$ and by using triangle inequality and approximation result.                                                                 □

## 5. Conclusion

In this paper, a numerical scheme for solving the time relaxation family of models based on approximate deconvolution technique for fluid flow problems is formulated and analyzed. The proposed method is convergent with optimal convergence rates with respect to the mesh size. The approximations of the average velocity and pressure are discontinuous piecewise polynomials. One benefit of using discontinuous elements is that the error estimates depend on the Reynolds number as $\mathcal{O}(\mathrm{Re}\,e^{\mathrm{Re}})$, whereas the dependence is $\mathcal{O}(\mathrm{Re}\,e^{\mathrm{Re}^3})$ for continuous finite elements (cf. [14]).

The proposed scheme (3.30)–(3.32) contains parameter $\epsilon_a \in \{-1, 0, 1\}$ in the bilinear form $a(\cdot, \cdot)$ for the discretization of the viscous term that yields different numerical approximations. Only numerical simulations of benchmark problems for high Reynolds numbers, will help determine which choices of $\epsilon_a$ are preferred for a given mesh size. This is the object of a future paper.

## References

[1] A. N. Kolmogorov, "The local structure of turbulence in incompressible viscous fluid for very large Reynolds numbers," *Doklady Akademii Nauk SSSR*, vol. 30, pp. 9–13, 1941.

[2] N. A. Adams and S. Stolz, "Deconvolution methods for subgrid-scale approximation in large eddy simulation," in *Modern Simulation Strategies for Turbulent Flow*, R.T. Edwards, 2001.

[3] W. Layton and M. Neda, "Truncation of scales by time relaxation," *Journal of Mathematical Analysis and Applications*, vol. 325, no. 2, pp. 788–807, 2007.

[4] L. C. Berselli, T. Iliescu, and W. J. Layton, *Mathematics of Large Eddy Simulation of Turbulent Flows*, Scientific Computation, Springer, Berlin, Germany, 2006.

[5] V. John, *Large Eddy Simulation of Turbulent Incompressible Flows*, vol. 34 of *Lecture Notes in Computational Science and Engineering*, Springer, Berlin, Germany, 2004.

[6] B. J. Geurts, "Inverse modeling for large eddy simulation," *Physics of Fluids*, vol. 9, pp. 3585–3587, 1997.

[7] P. Sagaut, *Large Eddy Simulation for Incompressible Flows*, Scientific Computation, Springer, Berlin, Germany, 2001.

[8] M. Germano, "Differential filters of elliptic type," *Physics of Fluids*, vol. 29, no. 6, pp. 1757–1758, 1986.

[9] S. Stolz and N. A. Adams, "An approximate deconvolution procedure for large-eddy simulation," *Physics of Fluids*, vol. 11, pp. 1699–1701, 1999.

[10] M. Bertero and P. Boccacci, *Introduction to Inverse Problems in Imaging*, Institute of Physics, Bristol, UK, 1998.

[11] S. Stolz, N. A. Adams, and L. Kleiser, "The approximate deconvolution model for large-eddy simulation of compressible flows and its application to shock-turbulent-boundary-layer interaction," *Physics of Fluids*, vol. 13, no. 10, pp. 2985–3001, 2001.

[12] S. Stolz, N. A. Adams, and L. Kleiser, "An approximate deconvolution model for large-eddy simulation with application to incompressible wall-bounded flows," *Physics of Fluids*, vol. 13, no. 4, pp. 997–1015, 2001.

[13] B. Rivière, *Discontinuous Galerkin Methods for Solving Elliptic and Parabolic Equations*, vol. 35 of *Frontiers in Applied Mathematics*, SIAM, Philadelphia, Pa, USA, 2008.

[14] V. J. Ervin, W. J. Layton, and M. Neda, "Numerical analysis of a higher order time relaxation model of fluids," *International Journal of Numerical Analysis and Modeling*, vol. 4, no. 3-4, pp. 648–670, 2007.

[15] W. J. Layton, C. C. Manica, M. Neda, and L. G. Rebholz, "Helicity and energy conservation and dissipation in approximate deconvolution LES models of turbulence," *Advances and Applications in Fluid Mechanics*, vol. 4, no. 1, pp. 1–46, 2008.

[16] V. Girault, B. Rivière, and M. F. Wheeler, "A discontinuous Galerkin method with nonoverlapping domain decomposition for the Stokes and Navier-Stokes problems," *Mathematics of Computation*, vol. 74, no. 249, pp. 53–84, 2005.

[17] V. Girault, B. Rivière, and M. F. Wheeler, "A splitting method using discontinuous Galerkin for the transient incompressible Navier-Stokes equations," *Mathematical Modelling and Numerical Analysis*, vol. 39, no. 6, pp. 1115–1147, 2005.

[18] B. Cockburn, G. Kanschat, and D. Schotzau, "A locally conservative LDG method for the incompressible Navier-Stokes equations," *Mathematics of Computation*, vol. 74, no. 251, pp. 1067–1095, 2005.

[19] S. Kaya and B. Rivière, "A discontinuous subgrid eddy viscosity method for the time-dependent Navier-Stokes equations," *SIAM Journal on Numerical Analysis*, vol. 43, no. 4, pp. 1572–1595, 2005.

[20] M. F. Wheeler, "An elliptic collocation-finite element method with interior penalties," *SIAM Journal on Numerical Analysis*, vol. 15, no. 1, pp. 152–161, 1978.

[21] Y. Epshteyn and B. Rivière, "Estimation of penalty parameters for symmetric interior penalty Galerkin methods," *Journal of Computational and Applied Mathematics*, vol. 206, no. 2, pp. 843–872, 2007.

[22] C. C. Manica and S. K. Merdan, "Convergence analysis of the finite element method for a fundamental model in turbulence," Tech. Rep. TR-MATH 06-12, University of Pittsburgh, Pittsburgh, Pa, USA, 2006.

[23] C. C. Manica, M. Neda, M. Olshanskii, and L. Rebholz, "Enabling numerical accuracy of Navier-Stokes-$\alpha$ through deconvolution and enhanced stability," *Mathematical Modelling and Numerical Analysis*. In press.

[24] J. M. Connors, "Convergence analysis and computational testing of the finite element discretization of the Navier-Stokes-alpha model," *Numerical Methods for Partial Differential Equations*, vol. 26, pp. 1328–1350, 2010.

[25] A. Bowers and L. Rebholz, "Increasing accuracy and efficiency in FE computations of the Leray-deconvolution model," *Numerical Methods for Partial Differential Equations*. In press.

[26] A. Dunca and Y. Epshteyn, "On the Stolz-Adams deconvolution model for the large-eddy simulation of turbulent flows," *SIAM Journal on Mathematical Analysis*, vol. 37, no. 6, pp. 1890–1902, 2006.

[27] W. Layton, C. C. Manica, M. Neda, and L. G. Rebholz, "Numerical analysis and computational testing of a high accuracy Leray-deconvolution model of turbulence," *Numerical Methods for Partial Differential Equations*, vol. 24, no. 2, pp. 555–582, 2008.

[28] W. Layton, "A remark on regularity of an elliptic-elliptic singular perturbation problem," Tech. Rep., University of Pittsburgh, Pittsburgh, Pa, USA, 2007.

[29] I. Stanculescu, "Existence theory of abstract approximate deconvolution models of turbulence," *Annali dell'Universitá di Ferrara*, vol. 54, no. 1, pp. 145–168, 2008.

[30] M. Anitescu, F. Pahlevani, and W. J. Layton, "Implicit for local effects and explicit for nonlocal effects is unconditionally stable," *Electronic Transactions on Numerical Analysis*, vol. 18, pp. 174–187, 2004.

[31] M. Crouzeix and P.-A. Raviart, "Conforming and nonconforming finite element methods for solving the stationary Stokes equations. I," *RAIRO: Numerical Analysis*, vol. 7, no. R-3, pp. 33–75, 1973.

[32] J. G. Heywood and R. Rannacher, "Finite-element approximation of the nonstationary Navier-Stokes problem. IV. Error analysis for second-order time discretization," *SIAM Journal on Numerical Analysis*, vol. 27, no. 2, pp. 353–384, 1990.