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Research Article

A Note on Fourth Order Method for Doubly Singular Boundary Value Problems

R. K. Pandey and G. K. Gupta

Department of Mathematics, Indian Institute of Technology, Kharagpur 721302, India

Correspondence should be addressed to R. K. Pandey, rkp@maths.iitkgp.ernet.in

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We present a fourth order finite difference method for doubly singular boundary value problem $(p(x)y'(x))' = q(x)f(x,y)$, $0 < x \leq 1$ with boundary conditions $y(0) = A$ (or $y'(0) = 0$ or $\lim_{x\to 0}p(x)y'(x) = 0$) and $\alpha y(1) + \beta y'(1) = \gamma$, where $\alpha(> 0)$, $\beta(\geq 0)$, γ and A are finite constants. Here $p(0) = 0$ and $q(x)$ is allowed to be discontinuous at the singular point $x = 0$. The method is based on uniform mesh. The accuracy of the method is established under quite general conditions and also corroborated through one numerical example.

1. Introduction

Consider the following class of singular two point boundary value problems:

$$(p(x)y')' = q(x)f(x,y), \quad 0 < x \leq 1, \tag{1.1}$$

with boundary conditions

$$y(0) = A, \quad \alpha y(1) + \beta y'(1) = \gamma \tag{1.2}$$

$$\text{or,} \quad y'(0) = 0 \quad \left(\text{or } \lim_{x\to 0} p(x)y'(x) = 0\right), \quad \alpha y(1) + \beta y'(1) = \gamma, \tag{1.3}$$

where $\alpha > 0$, $\beta \geq 0$ and A, γ are finite constants. Here $p(0) = 0$ and $q(x)$ is unbounded near $x = 0$. The condition $p(0) = 0$ says that the problem is singular and as $q(x)$ is unbounded near $x = 0$, so the problem is doubly singular [1]. Let $p(x)$, $q(x)$, and $f(x,y)$ satisfy the following conditions:

(A-1)

(i) $p(0) = 0,\ p(x) > 0$ in $(0,1]$,
(ii) $p(x) \in C[0,1] \cap C^1(0,1]$,
(iii) $p(x) = x^{b_0} g(x)$, $(0 \le b_0 < 1)$, $g(x) > 0$, and
(iv) $G \in C^4[0,1]$ and $G^v(x)$ exists on $(0,1)$,

where $G(x) = 1/g(x)$ on $[0,1]$.

(A-2)

(i) $q(x) > 0$ in $(0,1]$, $q(x)$ is unbounded near $x = 0$,
(ii) $q(x) \in L^1(0,1)$,
(iii) $q(x) = x^{a_0} H(x)$, $H(0) > 0$ with $(a_0 > -1)$,
(iv) $H(x) \in C^4[0,1]$ and $H^v(x)$ exists on $(0,1)$, and
(v) $1 + a_0 - b_0 \ge 0$.

(A-3) $f(x,y)$ is continuous on $\{[0,1] \times \mathbb{R}\}$, $\partial f/\partial y$ exists, and is continuous and $\partial f/\partial y \ge 0$ for all $0 \le x \le 1$ and for all real y.

Thomas [2] and Fermi [3] independently derived a boundary value problem for determining the electrical potential in an atom. The analysis leads to the nonlinear singular second order problem

$$y'' = x^{-1/2} y^{3/2} \tag{1.4}$$

with a set of boundary conditions. The following are of our interest:

(i) the neutral atom with Bohr radius b given by $y(0) = 1,\ by'(b) - y(b) = 0$;
(ii) the ionized atom given by $y(0) = 1,\ y(b) = 0$.

Furthermore, Chan and Hon [4] have considered the generalized Thomas-Fermi equation

$$\left(x^b y'\right)' = cq(x)y^e, \qquad y(0) = 1, \qquad y(a) = 0 \tag{1.5}$$

for parameter values $0 \le b < 1$, $c > 0$, $d > -2$, $e > 1$, and $q(x) = x^{b+d}$.

Such singular problems have been the concern of several researchers [5–8]. The existence-uniqueness of the solution of the boundary value problem (1.1) with boundary condition (1.2) or (1.3) is established in [1, 9–13]. Bobisud [1] has mentioned that in case $\lim_{x\to 0^+}(q(x)/p'(x)) \ne 0$, the condition $y'(0) = 0$ is quite severe, that is, it is sufficient but not necessary for forcing the solution to be differentiable at $x = 0$. In fact if $\lim_{x\to 0^+} p(x)y'(x) = 0$, then

$$\lim_{x\to 0^+} y'(x) = \lim_{x\to 0^+} \frac{q(x)}{p'(x)} \cdot \lim_{x\to 0^+} f(x, y(x)). \tag{1.6}$$

Thus $\lim_{x\to 0^+} y'(x) = 0$ if either $\lim_{x\to 0^+}(q(x)/p'(x)) = 0$ or $f(0,y(0)) = 0$. But $\lim_{x\to 0^+}(q(x)/p'(x)) \ne 0$ if $q(x)$ has discontinuity at $x = 0$; thus it is natural to consider the weaker boundary condition $\lim_{x\to 0^+} p(x)y'(x) = 0$.

There is a considerable literature on numerical methods for $q(x) = 1$ but to the best of our knowledge very few numerical methods are available to tackle doubly singular boundary value problems. Reddien [14] has considered the linear form of (1.1) and derived numerical methods for $q(x) \in L^2[0,1]$ which is stronger assumption than (A-2)(ii).

Some second order methods (Chawla and Katti [15], Pandey and Singh [16, 17]) as well as fourth order methods (Chawla et al. [18–21], Pandey and Singh [22–24]) have been developed for $q(x) = 1$, $p(x)$. Most of the researchers have developed methods for the function $p(x) = x^{b_0}$, $0 \le b_0 < 1$ and the boundary conditions $y(0) = A$ and $y(1) = B$.

Chawla [19] has given fourth order method for the problem (1.1)-(1.2) with $q(x) = 1$ and $p(x) = x^{b_0}$, $0 \le b_0 < 1$. The method is extended by Pandey and Singh [22] to a class of function $p(x) = x^{b_0}g(x)$ satisfying (A-1)(i–iii) with $G(x) = 1/g(x)$ analytic in the neighborhood of the singular point.

In this work we extend the fourth order accuracy method developed in [22] to doubly singular boundary value problems (1.1)-(1.2) and (1.1), (1.3) both. That is, we allow the data function to be discontinuous at the singular point $x = 0$. Further, we do not require analyticity of the function $G(x) = 1/g(x)$. The convergence of the method is established under quite general conditions on the functions $p(x)$, $q(x)$, and $f(x, y)$. Fourth order convergence of the method is corroborated through one example and maximum absolute errors are displayed in Table 1.

The work is organized in the following manner. In Section 2, first the method is described and then its construction is explained. In Section 3, convergence of the method is established and in Section 4, the order of the method is corroborated through one example.

2. Finite Difference Method

This section is divided in to two parts: (i) description of the method and (ii) derivation of the method. The coefficients not specified explicitly in this section are specified in the appendices.

2.1. Description of the Method

In this section we first state the method; detailed derivation is given in Section 2.2.

For positive integer $N \ge 2$, we consider the mesh $w_h = \{x_k\}_{k=0}^{N}$ over $[0,1] : 0 = x_0 < x_1 < x_2 < \cdots x_N = 1$, $h = 1/N$. For uniform mess $x_k = kh$, where b_0 is mentioned in condition (A-1). Let $r(x) := f(x, y(x))$ ($y(x)$ is the solution), $y_k = y(x_k)$, and so forth. Now we approximate the differential equation (1.1) with boundary condition (1.2) on the grid w_h by the following difference equations:

$$\frac{\tilde{y}_{k-1}}{J_{k-1}} - \left(\frac{1}{J_k} + \frac{1}{J_{k-1}}\right)\tilde{y}_k + \frac{\tilde{y}_{k+1}}{J_k} = a_{0,k}\tilde{r}_k + a_{1,k}\tilde{r}_{k+1} - a_{-1,k}\tilde{r}_{k-1}, \quad k = 1(1)(N-1), \tag{2.1}$$

$$\frac{\tilde{y}_{N-1}}{J_{N-1}} - \left(\frac{1}{J_{N-1}} + \frac{\alpha}{(\beta G_N)}\right)\tilde{y}_N + \frac{\gamma}{(\beta G_N)} = a_{0,N}\tilde{r}_N - a_{-1,N}\tilde{r}_{N-1} + a_{1,N}\tilde{r}_{N-\theta}, \tag{2.2}$$

for boundary value problem (1.1)-(1.2). In the case of boundary condition (1.3) we use the following difference equation:

$$\frac{-\tilde{y}_1}{J_1} + \frac{\tilde{y}_2}{J_1} = a_{-1,1}\tilde{r}_0 + a_{0,1}\tilde{r}_1 + a_{1,1}\tilde{r}_2. \tag{2.3}$$

Here $\tilde{y} = (\tilde{y}_k)$ denotes the approximate solution, $\tilde{y}_k \approx y_k$, $G_k = G(x_k)$, and

$$J_k = \int_{x_k}^{x_{k+1}} (p(\tau))^{-1} d\tau. \tag{2.4}$$

Note that the functions $p(x)$ and $G(x)$ are defined in assumption (A-1). Thus the method for boundary value problem (1.1)-(1.2) is given by difference equations (2.1)-(2.2). Further, for boundary value problem (1.1) and (1.3) the method is given by difference equations (2.1) (with $k = 2(1)(N-1)$) and (2.2)-(2.3).

2.2. Construction of the Method

In this section we describe the derivation of the method. Local truncation errors are mentioned without proof.

With $z(x) = p(x)y'(x)$ the differential equation (1.1) becomes $z'(x) = q(x)f(x,y(x))$. Now integrating $z'(x) = q(x)f(x,y(x))$ twice, first from x_k to x and then from x_k to x_{k+1} and changing the order of integration we get

$$y_{k+1} - y_k = z_k J_k + \int_{x_k}^{x_{k+1}} \left(\int_t^{x_{k+1}} (p(\tau))^{-1} d\tau \right) q(t) r(t) dt, \tag{2.5}$$

where $z_k = z(x_k)$, $r(x) := f(t, y(x))$ and $J_k = \int_{x_k}^{x_{k+1}} (p(\tau))^{-1} d\tau$. In an analogous manner, we get

$$y_k - y_{k-1} = z_k J_{k-1} - \int_{x_{k-1}}^{x_k} \left(\int_{x_{k-1}}^{t} (p(\tau))^{-1} d\tau \right) q(t) r(t) dt. \tag{2.6}$$

Eliminating z_k from (2.5) and (2.6) we obtain the Chawla's identity

$$\frac{y_{k+1} - y_k}{J_k} - \frac{y_k - y_{k-1}}{J_{k-1}} = \frac{I_k^+}{J_k} + \frac{I_k^-}{J_{k-1}}, \quad k = 1(1)(N-1), \tag{2.7}$$

where

$$\begin{aligned} I_k^+ &= \int_{x_k}^{x_{k+1}} \left(\int_t^{x_{k+1}} (p(\tau))^{-1} d\tau \right) q(t) r(t) dt, \\ I_k^- &= \int_{x_{k-1}}^{x_k} \left(\int_{x_{k-1}}^{t} (p(\tau))^{-1} d\tau \right) q(t) r(t) dt. \end{aligned} \tag{2.8}$$

Via Taylor expansion of $G(x), r(x)$ and $H(x)$ about x_k in $I_k^{\pm}$ and taking the following approximations for r_k' and r_k''

$$r_k' = \frac{r_{k+1} - r_{k-1}}{2h} - \frac{1}{6}h^2 r'''(\eta_k), \quad x_{k-1} < \eta_k < x_{k+1},$$
$$r_k'' = \frac{r_{k+1} - 2r_k + r_{k-1}}{h^2} - \frac{1}{12}h^2 r^{iv}(\xi_k), \quad x_{k-1} < \xi_k < x_{k+1}, \tag{2.9}$$

we get the approximation for the smooth solution $y(x)$ as:

$$\frac{\tilde{y}_{k-1}}{J_{k-1}} - \left(\frac{1}{J_k} + \frac{1}{J_{k-1}}\right)\tilde{y}_k + \frac{\tilde{y}_{k+1}}{J_k} = a_{0,k}r_k + a_{1,k}r_{k+1} - a_{-1,k}r_{k-1} + t_k, \quad k = 1(1)(N-1), \tag{2.10}$$

where t_k is local truncation error given by

$$\begin{aligned} t_k = {} & \left(H_k G_k''' b_{03,k} + H_k G^{iv}(\xi_k) b_{04,k}\right) f_k + \left(H_k f_k' + H_k' f_k\right)\left(G_k'' b_{12,k} + G_k''' b_{13,k}\right) \\ & + \left(\frac{1}{2}H_k f_k'' + H_k' f_k' + \frac{1}{2}H_k'' f_k\right)\left(G_k' b_{21,k} + G_k'' b_{22,k}\right) \\ & + \left(\frac{1}{6}H_k f_k''' + \frac{1}{2}H_k' f_k'' + \frac{1}{2}H_k'' f_k' + \frac{1}{6}H_k''' f_k\right)\left(G_k b_{30,k} + G_k' b_{31,k}\right) \\ & + \left(\frac{1}{24}H_k f^{iv}(\xi_k) + \frac{1}{6}H_k' f_k''' + \frac{1}{4}H_k'' f_k'' + \frac{1}{6}H_k''' f_k' + \frac{1}{24}H^{iv}(\eta_k) f_k\right) G_k b_{40,k} \\ & - \frac{1}{6}h^2 f'''(\eta_k)\left(b_{10,k} G_k H_k + b_{11,k} G_k' H_k + b_{20,k} G_k H_k'\right) \\ & - \frac{1}{24} b_{20,k} h^2 G_k H_k f^{iv}(\xi_k), \quad k = 1(1)(N-1),\ x_{k-1} < \xi_k,\ \eta_k < x_{k+1}. \end{aligned} \tag{2.11}$$

The coefficients $b_{03,k}$, $b_{04,k}$, $b_{10,k}$, $b_{11,k}$, $b_{12,k}$, $b_{13,k}$, $b_{20,k}$, $b_{21,k}$, $b_{22,k}$, $b_{30,k}$, $b_{31,k}$, $b_{40,k}$, and so forth, are specified in the appendices and the functions $G(x)$, $H(x)$ are defined in the assumptions (A-1), (A-2).

2.2.1. Discretization of the Boundary Condition at $x = 1$

We write (2.6) for $k = N$

$$\frac{y_N}{J_{N-1}} - \frac{y_{N-1}}{J_{N-1}} = p_N y_N' - \frac{I_N^-}{J_{N-1}}. \tag{2.12}$$

Now, we use the boundary condition at $x = 1$, Taylor expansion of $r(x)$ and following approximations for r'_N and r''_N

$$r'_N = \frac{1}{h}\left[\frac{(1+\theta)}{\theta}r_N + \frac{\theta}{(1-\theta)}r_{N-1} + \frac{1}{\theta(\theta-1)}r_{N-\theta}\right] - \frac{h^2\theta}{6}r'''(\eta_N), \quad x_{N-1} < \eta_N < x_N,$$
$$r''_N = \frac{2}{h^2}\left[\frac{1}{\theta}r_N + \frac{1}{(1-\theta)}r_{N-1} + \frac{1}{\theta(\theta-1)}r_{N-\theta}\right] - \frac{h(1+\theta)}{3}r'''(\eta'_N), \quad x_{N-1} < \eta'_N < x_N;\ 0<\theta<1 \tag{2.13}$$

in (2.12) and get the discretization at $x = 1$ as follows:

$$\frac{\tilde{y}_{N-1}}{J_{N-1}} - \left(\frac{1}{J_{N-1}} + \frac{\alpha}{(\beta G_N)}\right)\tilde{y}_N + \frac{\gamma}{(\beta G_N)} = a_{0,N}r_N - a_{-1,N}r_{N-1} + a_{1,N}r_{N-\theta} + t_N, \tag{2.14}$$

where the local truncation error t_N is given by

$$\begin{aligned} t_N = \frac{1}{J_{N-1}}\Bigg[& H_N f_N G'''_N a^-_{03,N} + H_N f_N G(\xi^-_N)^{iv} a^-_{04,N} + (H_N f'_N + H'_N f_N)\left\{G''_N a^-_{12,N} + G'''_N a^-_{13,N}\right\} \\ & + \left(\frac{1}{2}H_N f''_N + H'_N f'_N + \frac{1}{2}H''_N f_N\right)\left(G'_N a^-_{21,N} + G''_N a^-_{22,N}\right) \\ & + \left(\frac{1}{6}H_N f'''_N + \frac{1}{2}H'_N f''_N + \frac{1}{2}H''_N f'_N + \frac{1}{6}H'''_N f_N\right)\left(G_N a^-_{30,N} + G'_N a^-_{31,N}\right) \\ & + \left(\frac{1}{24}H_N f(\xi_N)^{iv} + \frac{1}{6}H'_N f'''_N + \frac{1}{4}H''_N f''_N + \frac{1}{6}H'''_N f'_N + \frac{1}{24}H(\eta_N)^{iv} f_N\right)G_N a^-_{40,k} \\ & - \frac{\theta}{6}h^2 f'''(\eta^-_N)\left(a^-_{10,N}G_N H_N + a^-_{11,N}G'_N H_N + a^-_{20,N}G_N H'_N\right) \\ & - \frac{h(1+\theta)}{3}a^-_{20,N}h^2 G_N H_N f'''(\eta^-_N)\Bigg], \quad x_{N-1} < \xi^-_N,\ \eta^-_N < x_N. \end{aligned} \tag{2.15}$$

The coefficients $a^-_{ij,N}$ are specified in the appendices.

The discretization (2.14) involves unknown $y_{N-\theta}$, which we approximate in the following way:

$$y_{N-\theta} = \overline{y}_{N-\theta} - \frac{h^3\theta^2(1-\theta)}{6}y'''_N, \tag{2.16}$$

where

$$\overline{y}_{N-\theta} = \theta(1-\theta)\left[\frac{\theta}{(1-\theta)}y_{N-1} + \left(\frac{(1+\theta)}{\theta} + \frac{h\alpha}{\beta}\right) - \frac{h\gamma}{\beta}\right]. \tag{2.17}$$

Now, let $\overline{r}_{N-\theta} = f(x_{N-\theta}, \overline{y}_{N-\theta})$ then replacing $r_{N-\theta}$ by $\overline{r}_{N-\theta}$ in (2.14) we obtain the following discretization for a smooth solution $y(x)$ at $k = N$

$$\frac{\widetilde{y}_{N-1}}{J_{N-1}} - \left(\frac{1}{J_{N-1}} + \frac{\alpha}{(\beta G_N)}\right)\widetilde{y}_N + \frac{\gamma}{(\beta G_N)} = a_{0,N} r_N - a_{-1,N} r_{N-1} + a_{1,N}\overline{r}_{N-\theta} + \overline{t_N}, \tag{2.18}$$

where

$$\begin{aligned}\overline{t_N} &= t_N^{(1)} - \frac{h^3}{6}\theta^2(1-\theta)a_{1,N} y_N''' \frac{\partial f}{\partial y}(x_{N-\theta}, y^*_{N-\theta}),\\ y^*_{N-\theta} &\in \left(\min\{y_{N-\theta}, \overline{y}_{N-\theta}\}, \max\{y_{N-\theta}, \overline{y}_{N-\theta}\}\right).\end{aligned} \tag{2.19}$$

2.2.2. Discretization of the Boundary Condition at $x = 0$

Integrating the differential equation (1.1) twice, first from x_1 to x; then from x_1 to x_2 and interchanging the order of integration we get

$$y_2 - y_1 = J_1 \int_0^{x_1} q(t) r(t) dt + \int_{x_1}^{x_2} \left(\int_t^{x_2} (p(\tau))^{-1} d\tau\right) q(t) r(t) dt. \tag{2.20}$$

Via Taylor expansion of $r(x)$, $H(x)$, and $G(x)$ about $x = x_1$ we get the following discretization for $k = 1$:

$$\frac{-\widetilde{y}_1}{J_1} + \frac{\widetilde{y}_2}{J_1} = a_{-1,1} r_0 + a_{0,1} r_1 + a_{1,1} r_2 + t_1, \tag{2.21}$$

where the local truncation error t_1 is given by

$$\begin{aligned} t_1 = {} & \left(\frac{1}{6}H_1 f_1''' + \frac{1}{2}H_1' f_1'' + \frac{1}{2}H_1'' f_1' + \frac{1}{6}H_1''' f_1\right)\left(-\frac{6x_1^{4+a_0}}{\psi(4)}\right)\\ & + \left(\frac{1}{24}H_1 f_1^{iv} + \frac{1}{6}H_1' f_1''' + \frac{1}{4}H_1'' f_1'' + \frac{1}{6}H_1''' f_1' + \frac{1}{24}H_1^{iv} f_1\right)\left(\frac{24x_1^{4+a_0}}{\psi(5)}\right)\\ & + \frac{1}{J_1}\left\{H_1 f_1\left(\frac{1}{6}a^+_{03,1}G_1''' + \frac{1}{24}a^+_{04,1}G_1^{iv}\right)\right.\\ & \qquad + (H_1 f_1' + H_1' f_1)\left(\frac{1}{2}a^+_{12,1}G_1'' + \frac{1}{6}a^+_{13,1}G_1'''\right) \end{aligned}$$

$$
\begin{aligned}
&+\left(\frac{1}{2}H_1 f_1'' + H_1' f_1' + \frac{1}{2}H_1'' f_1\right)\left(a_{21,1}^{+}G_1' + \frac{1}{2}a_{22,1}^{+}G_1''\right)\\
&+\left(\frac{1}{6}H_1 f_1''' + \frac{1}{2}H_1' f_1'' + \frac{1}{2}H_1'' f_1' + \frac{1}{6}H_1''' f_1\right)\left(a_{30,1}^{+}G_1 + a_{31,1}^{+}G_1'\right)\\
&+\left(\frac{1}{24}H_1 f_1^{iv} + \frac{1}{6}H_1' f_1''' + \frac{1}{4}H_1'' f_1'' + \frac{1}{6}H_1''' f_1' + \frac{1}{24}H_1^{iv} f_1\right)a_{40,1}^{+}G_1\Bigg\}\\
&+ -\frac{h^2}{6}f'''(\eta_1)\Bigg[-\frac{x_1^{2+a_0}}{\psi(2)}H_1 + \frac{2x_1^{3+a_0}}{\psi(3)}H_1'\\
&\qquad + \frac{1}{J_1}\left\{\left(a_{10,1}^{+}G_1 + a_{11,1}^{+}G_1'\right)H_1 + a_{20,1}^{+}G_1 H_1'\right\}\Bigg]\\
&- \frac{h^2}{12}f^{iv}(\xi_1)\left[\frac{x_1^{3+a_0}}{\psi(3)}H_1 + \frac{1}{2J_1}a_{20,1}^{+}G_1 H_1\right], \quad 0 < \xi_1 < x_2.
\end{aligned}
\tag{2.22}
$$

The coefficients $a_{i,j}$ are specified in the appendices and the functions $G(x)$, $H(x)$ are defined in the assumptions (A-1) and (A-2).

The discretization (2.22) involves y_0; so for y_0 we use the following approximation:

$$
\begin{aligned}
\overline{y}_0 = y_1 &+ \frac{x_1^{2+a_0-b_0}}{(1+a_0)(2+a_0-b_0)}H_1 G_1 f_1\\
&+ x_1^{3+a_0-b_0}\Bigg\{\left(-\frac{1}{(1+a_0)(2-b_0)} + \frac{1}{(2+a_0-b_0)(3+a_0-b_0)} + \frac{1}{(2-b_0)(3+a_0-b_0)}\right)G_1' H_1 f_1\\
&\qquad + \left(-\frac{1}{(1+a_0)(2+a_0)} + \frac{1}{(3+a_0-b_0)}\right)G_1 H_1' f_1\Bigg\}\\
&+ \frac{x_1^{2+a_0-b_0}}{2}\left(-\frac{1}{(1+a_0)(2+a_0)} + \frac{1}{(3+a_0-b_0)}\right)(f_2 - f_0),
\end{aligned}
\tag{2.23}
$$

where

$$
\begin{aligned}
y_0 &= \overline{y}_0 + \tau_0,\\
\tau_0 &= c^* h^{4+a_0-b_0}, \quad c^* \text{ is some constant.}
\end{aligned}
\tag{2.24}
$$

Let $\overline{r}_0 = f(x_0, \overline{y}_0)$; then in (2.21) replacing r_0 by $\overline{r}_0$ we obtain the following discretization for $k = 1$:

$$
\frac{1}{J_1}y_1 - \frac{1}{J_1}y_2 + b_{-1,1}\overline{r}_0 + b_{0,1}r_1 + b_{1,1}r_2 + \overline{t}_1^{(1)} = 0, \tag{2.25}
$$

where $\bar{t}_1 = t_1 + b_{-1,1}\tau_0(\partial f/\partial y)(x_0, y_0^*)$, $y_0^* \in (\min\{y_0, \overline{y}_0\}, \max\{y_0, \overline{y}_0\})$, and coefficients $b_{i,j}$, are specified in appendices.

3. Convergence of the Method

In this section we show that under suitable conditions the method (2.1)-(2.2) for the boundary value problems (1.1)-(1.2) described in Section 2 is of fourth order accuracy.

Let $\widetilde{Y} = (\widetilde{y}_1, \ldots, \widetilde{y}_N)^T$, $R(\widetilde{Y}) = (\widetilde{r}_1, \ldots, \widetilde{r}_N)^T$, $Q = (q_1, \ldots, q_N)^T$, and $T = (t_1, \ldots, t_N)^T$, then the finite difference method given by (2.1)-(2.2) for the boundary value problems (1.1)-(1.2) can be expressed in matrix form as

$$B\widetilde{Y} + PR\left(\widetilde{Y}\right) + G\widetilde{Y} = Q, \tag{3.1}$$

and $Y = (y_1, \ldots, y_N)^T$ corresponding to smooth solution $y(x)$ satisfies the perturbed system

$$BY + PR(Y) + GY + T = Q, \tag{3.2}$$

where $B = (b_{ij})$ and $P = (p_{ij})$ are $(N \times N)$ tridiagonal matrices.

Let $E = \widetilde{Y} - Y = (e_1, \ldots, e_N)^T$, then from Mean Value Theorem $R(\widetilde{Y}) - R(Y) = ME$, $M = \operatorname{diag}\{U_1, \ldots, U_N\}(U_k = \partial f_k/\partial y_k \geq 0)$, then from (3.1) and (3.2) we get the error equation as

$$(B + PM + W)E = T. \tag{3.3}$$

We recall that by the notation $Z \geq 0$ we mean that all components z_i of the vector Z satisfy $z_i \geq 0$. Similarly by $B \geq 0$ we mean all the elements b_{ij} of the matrix B satisfy $b_{ij} \geq 0$.

From Corollary of Theorems 7.2 and 7.4 in [25] an (irreducible) tridiagonal matrix B is monotone if

$$b_{k,k+1} \leq 0, \quad b_{k,k-1} \leq 0, \quad \sum_{j=1}^{N} b_{k,j} \begin{cases} \geq 0, & k = 1, 2, \ldots, N, \\ > 0, & \text{for at least one } i. \end{cases} \tag{3.4}$$

Furthermore, from Theorem 7.3 in [25], B^{-1} exists and the elements of B^{-1} are nonnegative. Now, if $B + PM + W$ is monotone and $B + PM + W \geq B$, then from Theorem 7.5 in [25]

$$(B + PM + W)^{-1} \leq B^{-1}. \tag{3.5}$$

Let vector norm $\|E\|$ and matrix norm $\|B\|$ be defined by

$$\|E\| = \max_{1 \leq k \leq N} |e_k|, \qquad \|B\| = \max_{1 \leq k \leq N} \sum_{j=1}^{N} |b_{kj}|, \tag{3.6}$$

then from (3.3) we get

$$\|E\| = \left\|(B + PM + W)^{-1}|T|\right\|, \tag{3.7}$$

where $|T| \equiv (|t_1|, |t_2|, \ldots, |t_N|)$.

Furthermore, if $B + PM + W \geq B$ and B is also monotone matrix, then from (3.5) we get

$$\|E\| \leq \left\|B^{-1}|T|\right\|. \tag{3.8}$$

Now from [22], $B^{-1} = (b_{i,j}^{-1})$ of matrix B is given by

$$b_{i,j}^{-1} = \begin{cases} \dfrac{P(x_i)\left[P(x_N) - P(x_j) + \beta G_N/\alpha\right]}{\left[P(x_N) + \beta G_N/\alpha\right]}, & i \leq j, \\ \dfrac{P(x_j)\left[P(x_N) - P(x_i) + \beta G_N/\alpha\right]}{\left[P(x_N) + \beta G_N/\alpha\right]}, & i \geq j, \end{cases} \tag{3.9}$$

where

$$P(x) = \int_0^x (p(\tau))^{-1} d\tau. \tag{3.10}$$

Let xr^{iv}, $r^{(i)}$ for $i = 0\ (1)\ 3$, and y''' be bounded on $(0, 1]$. Now, as b_0 is fixed in $[0, 1)$ and a_0 is chosen such that $1 + a_0 - b_0 > 0$, for sufficiently small h we get

$$|t_k| \leq Ch^5 x_k^{-1+a_0}, \quad \left|\bar{t}_N\right| \leq \widetilde{C}h^4, \tag{3.11}$$

for suitable constants C and $\widetilde{C}$.

Now from (3.8)–(3.11) we get

$$\begin{aligned} |e_i| \leq C^* h^5 \Bigg[& \left\{1 - \frac{P(x_i)}{\left[P(x_N) + \beta G_N/\alpha\right]}\right\} \sum_{j=1}^{i} P(x_j) x_j^{-1+a_0} \\ & + \frac{P(x_i)}{\left[P(x_N) + \beta G_N/\alpha\right]} \left\{ \sum_{j=i+1}^{N-1} \left[P(x_N) - P(x_j) + \frac{\beta G_N}{\alpha}\right] x_j^{-1+a_0} + \frac{\beta G_N}{h\alpha} \right\} \Bigg], \end{aligned} \tag{3.12}$$

where $C^* = \max\{C, \overline{C}\}$. It is easy to see that

$$h\sum_{j=1}^{i} P(x_j)x_j^{-1+a_0} < \int_0^{x_i} P(x)x^{-1+a_0}dx,$$
$$h\sum_{j=i+1}^{N-1}\left[P(x_N) - P(x_j) + \frac{\beta G_N}{\alpha}\right]x_j^{-1+a_0} < \int_{x_i}^{x_N}\left[P(x_N) - P(x) + \frac{\beta G_N}{\alpha}\right]x^{-1+a_0}dx. \tag{3.13}$$

From (3.12)-(3.13) we obtain

$$|e_i| \le \frac{C^* h^4}{(1-b_0)}\begin{cases}\dfrac{(1-b_0)x_i^{1-b_0}}{-a_0(1-b_0+a_0)}\left[x_i^{a_0} - \dfrac{\alpha + (1-b_0+a_0)(1-a_0)\beta}{\alpha + (1-b_0)\beta}\right] \\ \quad\times\sup\limits_{[0,1]}|G(x)| + (1-b_0)d_1, & a_0 \ne 0 \\ x_i^{1-b_0}\left[\ln\left(\dfrac{1}{x_i}\right) + \dfrac{(2-b_0)\beta}{\alpha + (1-b_0)\beta}\right]\sup\limits_{[0,1]}|G(x)| + d_2, & a_0 = 0,\end{cases} \tag{3.14}$$

where

$$\begin{aligned}d_1 = {} & \frac{\alpha(1-b_0+a_0)^{(1-b_0+a_0)/-a_0}}{(\alpha + (1-b_0)\beta)(1-b_0)^{(1-b_0)/-a_0}} \\ & \times\left[\frac{(2-b_0+a_0)^{(1-b_0+a_0)/-a_0}}{(2-b_0)^{(1-b_0)/-a_0}}\sup_{[0,1]}\left|G'(x)\right| + \frac{(3-b_0+a_0)^{(1-b_0+a_0)/-a_0}}{2(3-b_0)^{(1-b_0)/-a_0}}\sup_{[0,1]}\left|G''(x)\right|\right. \\ & \left. + \frac{(4-b_0+a_0)^{(1-b_0+a_0)/-a_0}}{3!(4-b_0)^{(1-b_0)/-a_0}}\sup_{[0,1]}\left|G'''(x)\right| + \frac{(5-b_0+a_0)^{(1-b_0+a_0)/-a_0}}{4!(5-b_0)^{(1-b_0)/-a_0}}\sup_{[0,1]}\left|G^{'v}(x)\right|\right] \\ d_2 = {} & \frac{\alpha}{(\alpha + (1-b_0)\beta)e^2}\left[\frac{1}{(2-b_0)}e^{1/(2-b_0)}\sup_{[0,1]}\left|G'(x)\right| + \frac{1}{2(3-b_0)}e^{2/(3-b_0)}\sup_{[0,1]}\left|G''(x)\right|\right. \\ & \left. + \frac{1}{3!(4-b_0)}e^{3/(4-b_0)}\sup_{[0,1]}\left|G'''(x)\right| + \frac{1}{4!(5-b_0)}e^{4/(5-b_0)}\sup_{[0,1]}\left|G^{'v}(x)\right|\right].\end{aligned} \tag{3.15}$$

In view of the following inequalities

$$x^{1-b_0}\left(x^{a_0} - \frac{\alpha + (1-b_0+a_0)(1-a_0)\beta}{\alpha + (1-b_0)\beta}\right) \le \left(\frac{\alpha + (1-b_0)\beta}{\alpha + (1-b_0+a_0)(1-a_0)\beta}\right)^{(1-b_0+a_0)/-a_0},$$
$$x^{1-b_0}\left(\ln\left(\frac{1}{x}\right) + \frac{\beta}{\alpha + (1-b_0)\beta}\right) \le \frac{1}{(1-b_0)}e^{-[\alpha-(1-b_0)\beta]/[\alpha+(1-b_0)\beta]} \tag{3.16}$$

for $x \in (0,1]$, $(B + PM + W)^{-1} \le B^{-1}$ and from (3.14) it is easy to establish that

$$\|E\|_\infty = O\left(h^4\right). \tag{3.17}$$

Thus we have established the following result.

Theorem 3.1. *Assume that $p(x)$, $q(x)$, and $f(x,y)$ satisfy assumptions given in $(A$-1$)$, $(A$-2$)$, and $(A$-3$)$, respectively. Let $r(x) := f(x, y(x))$, then the method is of fourth order accuracy for sufficiently small mesh size h provided, $xr^{(iv)}(x)$, $r^{(i)}$, $i = 0\ (1)\ 3$, and y''' are bounded on $(0,1]$.*

Remark 3.2. We have developed the numerical method for the boundary condition $y'(0) = 0$ but could not establish the fourth order convergence although the order of the accuracy is verified through one example in the next section.

4. Numerical Illustrations

To illustrate the convergence of the method and to corroborate their order of the accuracy, we apply the method to following example. The maximum absolute errors are displayed in Table 1.

Example 4.1. Consider

$$\begin{gathered}\left(x^{b_0}\frac{1}{1+x^{3.5}}y'\right)' = (1.9-b_0)x^{-0.1}\frac{\left((1.9-b_0)x^{1.9-b_0}e^{y} - 0.9 + 3.5x^{3.5}/\left(1+x^{3.5}\right)\right)}{\left(4+x^{1.9-b_0}\right)\left(1+x^{3.5}\right)},\\ y(0) = \ln\left(\frac{1}{4}\right) \quad \text{or } \left(y'(0) = 0\right), \qquad y(1) + 5y'(1) = \ln\left(\frac{1}{5}\right) - (1.9-b_0),\end{gathered} \tag{4.1}$$

with exact solution $y(x) = \ln(1/(4 + x^{1.9-b_0}))$.

The method is applied on the example for $b_0 = 0.1,\ 0.5$. Maximum absolute errors and order of accuracy for Example 4.1 are displayed in Table 1.

We use the following approximation of J_k in our numerical program:

$$\begin{aligned}J_k = {}& \frac{x_{k+1}^{1-b_0} - x_k^{1-b_0}}{1-b_0}G_k + \left(\frac{x_{k+1}^{2-b_0} - x_k^{2-b_0}}{2-b_0} - \frac{G_k\left(x_{k+1}^{1-b_0} - x_k^{1-b_0}\right)}{1-b_0}\right)G_k'\\ &+ \frac{1}{2}G_k''\left(\frac{x_{k+1}^{3-b_0} - x_k^{3-b_0}}{3-b_0} + x_k^2\frac{x_{k+1}^{1-b_0} - x_k^{1-b_0}}{1-b_0} - 2x_k\frac{x_{k+1}^{2-b_0} - x_k^{2-b_0}}{2-b_0}\right)\\ &+ \frac{1}{6}G_k'''\left(\frac{x_{k+1}^{4-b_0} - x_k^{4-b_0}}{4-b_0} - 3x_k\frac{x_{k+1}^{3-b_0} - x_k^{3-b_0}}{3-b_0} + 3x_k^2\frac{x_{k+1}^{2-b_0} - x_k^{2-b_0}}{2-b_0} - x_k^3\frac{x_{k+1}^{1-b_0} - x_k^{1-b_0}}{1-b_0}\right).\end{aligned} \tag{4.2}$$

Table 1: Maximum absolute errors and order of the methods for Example 4.1.

N	$b_0 = 0.1$	Order	$b_0 = 0.5$	Order	$b_0 = 0.1$	Order	$b_0 = 0.5$	Order
	$y(0) = \ln(1/4)$ and $\theta = 0.01$				$y(0) = \ln(1/4)$ and $\theta = 0.98$			
32	8.23 (−7)[a]		1.10 (−6)		6.11 (−7)		8.11 (−7)	
64	5.05 (−8)	4.03	6.80 (−8)	4.02	3.50 (−8)	4.13	4.60 (−8)	4.14
128	3.13 (−9)		4.22 (−9)		2.09 (−9)		2.72 (−9)	
256	1.94 (−10)	4.01	2.64 (−10)	4.00	1.27 (−10)	4.04	1.67 (−10)	4.03
	$y'(0) = 0$ and $\theta = 0.01$				$y'(0) = 0$ and $\theta = 0.98$			
32	3.37 (−7)		4.03 (−7)		1.72 (−7)		1.95 (−7)	
64	2.04 (−8)	4.05	2.46 (−8)	4.03	9.13 (−9)	4.24	9.80 (−9)	4.31
128	1.25 (−9)		1.52 (−9)		5.20 (−10)		5.39 (−10)	
266	7.74 (−11)	4.02	9.56 (−11)	3.99	3.09 (−11)	4.08	3.23 (−11)	4.06

[a] $8.23\ (-7) = 8.23 \times 10^{-7}$.

4.1. Numerical Results for Uniform Mesh Case

Remark 4.2. In the discretization (2.18), θ may take any value in $(0,1)$ but from the Table 1, maximum absolute errors are less for the value of θ close to one. So it is better to take value of θ close to one.

Appendices

The coefficients involved in the method and their approximations are mentioned below.

A. Coefficients for Method

$$
\begin{aligned}
a^{(1)}_{\pm 1,k} &= \frac{1}{2h}\left[\pm\left(b_{10,k}G_kH_k + b_{11,k}G'_kH_k + b_{20,k}G_kH'_k\right) + \frac{1}{h}b_{20,k}G_kH_K\right],\\
a^{(1)}_{0,k} &= \left(b_{00,k} - \frac{1}{h^2}b_{20,k}\right)G_kH_k + b_{01,k}H_kG'_k + \frac{1}{2}b_{02,k}G''_kH_k\\
&\quad + b_{10,k}G_kH'_k + b_{11,k}G'_kH'_k + \frac{1}{2}b_{20,k}G_kH''_k,\\
a^{(1)}_{-1,N} &= \frac{1}{J_{N-1}}\left[\frac{\theta}{h(1-\theta)}\left\{a^{-}_{10,N}G_NH_N + a^{-}_{11,N}G'_NH_N + a^{-}_{20,N}G_NH'_N\right\}\right.\\
&\qquad\left. + \frac{1}{h^2(1-\theta)}a^{-}_{20,N}G_NH_N\right],\\
a^{(1)}_{0,N} &= \frac{1}{J_{N-1}}\left[a^{-}_{00,N}G_NH_N + a^{-}_{10,N}G_NH'_N + a^{-}_{01,N}G'_NH_N + \frac{1}{2}a^{-}_{02,N}G''_NH_N + a^{-}_{11,N}G'_NH'_N\right.\\
&\qquad + \frac{1}{2}a^{-}_{20,N}G_NH''_N + \frac{(1+\theta)}{h\theta}\left\{a^{-}_{10,N}G_NH_N + a^{-}_{11,N}G'_NH_N + a^{-}_{20,N}G_NH'_N\right\}
\end{aligned}
$$

$$+\frac{1}{\theta h^2}a^{-}_{20,N}G_N H_N\Big],$$

$$a^{(1)}_{1,N}=\frac{1}{J_{N-1}}\left[\frac{1}{h\theta(1-\theta)}\left\{a^{-}_{10,N}G_N H_N+a^{-}_{11,N}G'_N H_N+a^{-}_{20,N}G_N H'_N\right\}+\frac{1}{h^2\theta(1-\theta)}a^{-}_{20,N}G_N H_N\right], \tag{A.1}$$

where

$$a^{\pm}_{00,k}=\frac{1}{(1-b_0)}\left[-\frac{\left(x^{2+a_0-b_0}_{k\pm1}-x^{2+a_0-b_0}_{k}\right)}{\phi(2)}+\frac{x_{k\pm1}}{\varphi(1)}\left(x^{1+a_0}_{k\pm1}-x^{1+a_0}_{k}\right)\right],$$

$$a^{\pm}_{01,k}=\frac{1}{(1-b_0)}\left[\frac{(x_k-x_{k\pm1})x^{2+a_0-b_0}_{k\pm1}}{\phi(2)}+\frac{1}{\phi(3)}\left(x^{3+a_0-b_0}_{k\pm1}-x^{3+a_0-b_0}_{k}\right)+\frac{(x_{k\pm1}-x_k)x^{1-b_0}_{k\pm1}}{\varphi(1)}\left(x^{1+a_0}_{k\pm1}-x^{1+a_0}_{k}\right)-\frac{x^{2-b_0}_{k\pm1}}{\varphi(1)\varphi(2)}\left(x^{1+a_0}_{k\pm1}-x^{1+a_0}_{k}\right)+\frac{1}{\varphi(2)(3+a_0-b_0)}\left(x^{3+a_0-b_0}_{k\pm1}-x^{3+a_0-b_0}_{k}\right)\right],$$

$$a^{\pm}_{02,k}=\frac{1}{(1-b_0)}\left[-\frac{(x_{k\pm1}-x_k)^2x^{2+a_0-b_0}_{k\pm1}}{\phi(2)}+\frac{1}{\phi(3)}\left(2(x_{k\pm1}-x_k)x^{3+a_0-b_0}_{k\pm1}\right)-\frac{2}{\phi(4)}\left(x^{4+a_0-b_0}_{k\pm1}-x^{4+a_0-b_0}_{k}\right)+\frac{(x_{k\pm1}-x_k)^2x^{1-b_0}_{k\pm1}}{(1+a_0)}\left(x^{1+a_0}_{k\pm1}-x^{1+a_0}_{k}\right)+\frac{2(x_{k\pm1}-x_k)}{\varphi(2)(3+a_0-b_0)}x^{3+a_0-b_0}_{k\pm1}-\frac{2\phi(2)}{\varphi(2)\phi(4)}\left(x^{4+a_0-b_0}_{k\pm1}-x^{4+a_0-b_0}_{k}\right)-\frac{2(x_{k\pm1}-x_k)}{\varphi(2)\varphi(1)}x^{2-b_0}_{k\pm1}\left(x^{1+a_0}_{k\pm1}-x^{1+a_0}_{k}\right)+\frac{2}{\varphi(3)\varphi(1)}\left(x^{1+a_0}_{k\pm1}-x^{1+a_0}_{k}\right)-\frac{2}{\varphi(3)(4+a_0-b_0)}\left(x^{4+a_0-b_0}_{k\pm1}-x^{4+a_0-b_0}_{k}\right)\right],$$

$$a^{\pm}_{03,k}=\frac{1}{1-b_0}\left[\pm\frac{h^3}{(1+a_0)}x^{1-b_0}_{k\pm1}\left(x^{1+a_0}_{k\pm1}-x^{1+a_0}_{k}\right)\mp\frac{h^3}{\phi(2)}x^{2+a_0-b_0}_{k\pm1}+\frac{3h^2}{\phi(3)}x^{3+a_0-b_0}_{k\pm1}\mp\frac{6h}{\phi(4)}x^{4+a_0-b_0}_{k\pm1}+\frac{6}{\phi(5)}\left(x^{5+a_0-b_0}_{k\pm1}-x^{5+a_0-b_0}_{k}\right)-\frac{3h^2}{(1+a_0)(2-b_0)}x^{2-b_0}_{k\pm1}\left(x^{1+a_0}_{k\pm1}-x^{1+a_0}_{k}\right)\right.$$

$$+\frac{3}{(2-b_0)}\left\{\frac{h^2}{3+a_0-b_0}x_{k\pm1}^{3+a_0-b_0}\mp\frac{h}{(3+a_0-b_0)(4+a_0-b_0)}x_{k\pm1}^{4+a_0-b_0}\right.$$

$$\left.+\frac{2\phi(2)}{\phi(5)}\left(x_{k\pm1}^{5+a_0-b_0}-x_k^{5+a_0-b_0}\right)\right\}$$

$$\pm\frac{6}{(2-b_0)(3-b_0)(1+a_0)}x_{k\pm1}^{3-b_0}\left(x_{k\pm1}^{1+a_0}-x_k^{1+a_0}\right)-\frac{6}{(2-b_0)(3-b_0)}$$

$$\left\{\pm\frac{h}{(4+a_0-b_0)}x_{k\pm1}^{4+a_0-b_0}-\frac{1}{(4+a_0-b_0)(5+a_0-b_0)}\left(x_{k\pm1}^{5+a_0-b_0}-x_k^{5+a_0-b_0}\right)\right\}$$

$$-\frac{6}{\varphi(4)(1+a_0)}x_{k\pm1}^{4-b_0}\left(x_{k\pm1}^{1+a_0}-x_k^{1+a_0}\right)$$

$$\left.\frac{6}{\varphi(4)(5+a_0-b_0)}\left(x_{k\pm1}^{5+a_0-b_0}-x_k^{5+a_0-b_0}\right)\right],$$

$$a_{04,k}^{\pm}=\frac{1}{1-b_0}\left[\frac{h^4}{\varphi(1)}x_{k\pm1}^{1-b_0}\left(x_{k\pm1}^{1+a_0}-x_k^{1+a_0}\right)-\frac{h^4}{\phi(2)}x_{k\pm1}^{2+a_0-b_0}\pm\frac{4h^3}{\phi(3)}x_{k\pm1}^{3+a_0-b_0}\right.$$

$$-\frac{12h^2}{\phi(4)}x_{k\pm1}^{4+a_0-b_0}\pm\frac{24h}{\phi(5)}x_{k\pm1}^{5+a_0-b_0}+\frac{24}{\phi(6)}\left(x_{k\pm1}^{6+a_0-b_0}-x_k^{6+a_0-b_0}\right)$$

$$\mp\frac{4h^3}{\varphi(2)\varphi(1)}x_{k\pm1}^{2-b_0}\left(x_{k\pm1}^{1+a_0}-x_k^{1+a_0}\right)$$

$$+\frac{4}{(2-b_0)}\left\{\mp\frac{h^3}{(3+a_0-b_0)}x_{k\pm1}^{3+a_0-b_0}-\frac{3h^2\phi(2)}{\phi(4)}x_{k\pm1}^{4+a_0-b_0}\right.$$

$$\left.\pm\frac{6h\phi(2)}{\phi(5)}x_{k\pm1}^{5+a_0-b_0}-\frac{6\phi(2)}{\phi(6)}\left(x_{k\pm1}^{6+a_0-b_0}-x_k^{6+a_0-b_0}\right)\right\}$$

$$+\frac{12h^2}{\varphi(2)\varphi(1)}x_{k\pm1}^{3-b_0}\left(x_{k\pm1}^{1+a_0}-x_k^{1+a_0}\right)$$

$$-\frac{12}{\varphi(3)}\left\{\frac{h^2}{(4+a_0-b_0)}x_{k\pm1}^{4+a_0-b_0}\mp\frac{2h\phi(3)}{\phi(5)}x_{k\pm1}^{5+a_0-b_0}\right.$$

$$\left.+\frac{2\phi(3)}{\phi(5)}\left(x_{k\pm1}^{6+a_0-b_0}-x_k^{6+a_0-b_0}\right)\right\}$$

$$\mp\frac{24h}{\varphi(4)\varphi(1)}x_{k\pm1}^{4-b_0}\left(x_{k\pm1}^{1+a_0}-x_k^{1+a_0}\right)$$

$$+\frac{24}{\varphi(4)}\left\{\pm\frac{h}{(5+a_0-b_0)}x_{k\pm1}^{5+a_0-b_0}\right.$$

$$\left.-\frac{1}{(5+a_0-b_0)(6+a_0-b_0)}\left(x_{k\pm1}^{6+a_0-b_0}-x_k^{6+a_0-b_0}\right)\right\}$$

$$+\frac{24}{\varphi(5)\psi(1)}x_{k\pm1}^{5-b_0}\left(x_{k\pm1}^{1+a_0}-x_k^{1+a_0}\right)$$

$$-\frac{24}{\varphi(6)(6+a_0-b_0)}\left(x_{k\pm1}^{6+a_0-b_0}-x_k^{6+a_0-b_0}\right)\Bigg],$$

$$a_{10,k}^{\pm}=\frac{1}{(1-b_0)}\left[-\frac{(x_{k\pm1}-x_k)}{\phi(1)}x_{k\pm1}^{2+a_0-b_0}+\frac{(x_{k\pm1}-x_k)}{\psi(1)}x_{k\pm1}^{2+a_0-b_0}-\frac{1}{\psi(2)}x_{k\pm1}^{1-b_0}\left(x_{k\pm1}^{2+a_0}-x_k^{2+a_0}\right)\right.$$

$$\left.+\frac{1}{\phi(3)}\left(x_{k\pm1}^{3+a_0-b_0}-x_k^{3+a_0-b_0}\right)\right],$$

$$a_{11,k}^{\pm}=\frac{1}{(1-b_0)}\left[-\frac{(x_{k\pm1}-x_k)^2x_{k\pm1}^{2+a_0-b_0}}{\phi(2)}+\frac{(x_{k\pm1}-x_k)^2}{\psi(1)}x_{k\pm1}^{2+a_0-b_0}\right.$$

$$-\frac{(x_{k\pm1}-x_k)}{\psi(2)}x_{k\pm1}^{1-b_0}\left(x_{k\pm1}^{1-b_0}-x_k^{1-b_0}\right)$$

$$+\frac{2(x_{k\pm1}-x_k)}{\phi(3)}x_{k\pm1}^{3+a_0-b_0}-\frac{2}{\phi(4)}\left(x_{k\pm1}^{4+a_0-b_0}-x_k^{4+a_0-b_0}\right)$$

$$+\frac{1}{(2-b_0)}\left[\frac{(x_{k\pm1}-x_k)}{(3+a_0-b_0)}x_{k\pm1}^{3+a_0-b_0}-\frac{(x_{k\pm1}-x_k)}{\psi(1)}x_{k\pm1}^{3+a_0-b_0}+\frac{1}{\psi(2)}x_{k\pm1}^{2-b_0}\right.$$

$$\left.\left.\times\left(x_{k\pm1}^{2+a_0}-x_k^{2+a_0}\right)-\frac{\phi(2)}{\phi(4)}\left(x_{k\pm1}^{4+a_0-b_0}-x_k^{4+a_0-b_0}\right)\right]\right],$$

$$a_{12,k}^{\pm}=\frac{1}{1-b_0}\left[h^2x_{k\pm1}^{1-b_0}\left\{\pm\frac{h}{\psi(1)}x_{k\pm1}^{1+a_0}-\frac{1}{\psi(2)}\left(x_{k\pm1}^{2+a_0}-x_k^{2+a_0}\right)\right\}\right.$$

$$\mp\frac{h^3}{\phi(2)}x_{k\pm1}^{2+a_0-b_0}+\frac{3h^2}{\phi(3)}x_{k\pm1}^{3+a_0-b_0}\mp\frac{6h}{\phi(4)}x_{k\pm1}^{4+a_0-b_0}$$

$$+\frac{6}{\phi(5)}\left(x_{k\pm1}^{5+a_0-b_0}-x_k^{5+a_0-b_0}\right)$$

$$\mp\frac{2h}{\varphi(2)}x_{k\pm1}^{2-b_0}\left\{\frac{h}{\psi(1)}x_{k\pm1}^{1+a_0}-\frac{1}{\psi(2)}\left(x_{k\pm1}^{2+a_0}-x_k^{2+a_0}\right)\right\}$$

$$+\frac{2}{\varphi(2)}\left\{\frac{h^2}{(3+a_0-b_0)}x_{k\pm1}^{3+a-0-b_0}\mp\frac{2h\phi(2)}{\phi(4)}x_{k\pm1}^{4+a-0-b_0}\right.$$

$$\left.+\frac{2\phi(2)}{\phi(5)}\left(x_{k\pm1}^{5+a_0-b_0}-x_k^{5+a_0-b_0}\right)\right\}$$

$$+\frac{2}{\varphi(3)}\left\{x_{k\pm1}^{3-b_0}\left\{\pm\frac{h}{\psi(1)}x_{k\pm1}^{1+a_0}-\frac{1}{\psi(2)}\left(x_{k\pm1}^{2+a_0}-x_k^{2+a_0}\right)\right\}\right.$$

$$\left.\mp\frac{h}{(4+a_0-b_0)}x_{k\pm1}^{4+a_0-b_0}+\frac{1}{(4+a_0-b_0)(5+a_0-b_0)}\left(x_{k\pm1}^{5+a_0-b_0}-x_k^{5+a_0-b_0}\right)\right],$$

$$
\begin{aligned}
a^{\pm}_{13,k} = \frac{1}{1-b_0}\Bigg[& \pm h^3 x_{k\pm1}^{1-b_0}\left\{\pm\frac{h}{\varphi(1)}x_{k\pm1}^{1+a_0} - \frac{1}{\varphi(2)}\left(x_{k\pm1}^{2+a_0} - x_k^{2+a_0}\right)\right\} \\
& - \frac{h^4}{\phi(2)}x_{k\pm1}^{2+a_0-b_0} \pm \frac{4h^3}{\phi(3)}x_{k\pm1}^{3+a_0-b_0} - \frac{12h^2}{\phi(4)}x_{k\pm1}^{4+a_0-b_0} \pm \frac{24h}{\phi(5)}x_{k\pm1}^{5+a_0-b_0} \\
& - \frac{24}{\phi(6)}\left(x_{k\pm1}^{6+a_0-b_0} - x_k^{6+a_0-b_0}\right) \\
& - \frac{3h^2}{\varphi(2)}x_{k\pm1}^{2-b_0}\left\{\pm\frac{h}{\varphi(1)}x_{k\pm1}^{1+a_0} - \frac{1}{\varphi(2)}\left(x_{k\pm1}^{2+a_0} - x_k^{2+a_0}\right)\right\} \\
& + \frac{3\phi(2)}{\varphi(2)}\Bigg\{\pm\frac{h^3}{\phi(3)}x_{k\pm1}^{3+a_0-b_0} - \frac{3h^2}{\phi(4)}x_{k\pm1}^{4+a_0-b_0} \pm \frac{6h}{\phi(5)}x_{k\pm1}^{5+a_0-b_0} \\
& \qquad - \frac{6}{\phi(6)}\left(x_{k\pm1}^{6+a_0-b_0} - x_k^{6+a_0-b_0}\right)\Bigg\} \\
& \pm \frac{6h}{\varphi(3)}x_{k\pm1}^{3-b_0}\left\{\pm\frac{h}{\varphi(1)}x_{k\pm a}^{1+a_0} - \frac{1}{\varphi(2)}\left(x_{k\pm1}^{2+a_0} - x_k^{2+a_0}\right)\right\} \\
& - \frac{6\phi(3)}{\varphi(3)}\left\{\frac{h^2}{\phi(4)}x_{k\pm1}^{4+a_0-b_0} \mp \frac{2h}{\phi(5)}x_{k\pm1}^{5+a_0-b_0} + \frac{2}{\phi(6)}\left(x_{k\pm1}^{6+a_0-b_0} - x_k^{6+a_0-b_0}\right)\right\} \\
& - \frac{6}{\varphi(4)}\Bigg\{x_{k\pm1}^{4-b_0}\left\{\pm\frac{h}{\varphi(1)}x_{k\pm1}^{1+a_0} - \frac{1}{\varphi(2)}\left(x_{k\pm1}^{2+a_0} - x_k^{2+a_0}\right)\right\} \mp \frac{h}{(5+a_0-b_0)}x_{k\pm1}^{5+a_0-b_0} \\
& + \frac{1}{(5+a_0-b_0)(6+a_0-b_0)}\left(x_{k\pm1}^{6+a_0-b_0} - x_k^{6+a_0-b_0}\right)\Bigg\}\Bigg],
\end{aligned}
$$

$$
\begin{aligned}
a^{\pm}_{20,k} = \frac{1}{(1-b_0)}\Bigg[& -\frac{(x_{k\pm1}-x_k)^2}{\phi(2)}x_{k\pm1}^{2+a_0-b_0} + \frac{2(x_{k\pm1}-x_k)}{\phi(3)}x_{k\pm1}^{3+a_0-b_0} + \frac{(x_{k\pm1}-x_k)^2}{\varphi(1)}x_{k\pm1}^{2+a_0-b_0} \\
& - \frac{2}{\phi(4)}\left(x_{k\pm1}^{4+a_0-b_0} - x_k^{4+a_0-b_0}\right) - \frac{2(x_{k\pm1}-x_k)}{\varphi(2)}x_{k\pm1}^{3+a_0-b_0} \\
& + \frac{2}{\varphi(3)}x_{k\pm1}^{1-b_0}\left(x_{k\pm1}^{3+a_0} - x_k^{3+a_0}\right)\Bigg],
\end{aligned}
$$

$$
\begin{aligned}
a^{\pm}_{21,k} = \frac{1}{1-b_0}\Bigg[& \pm h x_{k\pm1}^{1-b_0}\left\{\frac{h^2}{\varphi(1)}x_{k\pm1}^{1+a_0} \mp \frac{2}{\varphi(2)}x_{k\pm1}^{2+a_0} + \frac{2}{\varphi(3)}\left(x_{k\pm1}^{3+a_0} - x_k^{3+a_0}\right)\right\} \\
& \mp \frac{h^3}{\phi(2)}x_{k\pm1}^{2+a_0-b_0} + \frac{3h^2}{\phi(3)}x_{k\pm1}^{3+a_0-b_0} \mp \frac{6h}{\phi(4)}x_{k\pm1}^{4+a_0-b_0} \\
& + \frac{6}{\phi(5)}\left(x_{k\pm1}^{5+a_0-b_0} - x_k^{5+a_0-b_0}\right) \\
& - \frac{1}{\varphi(2)}\left\{x_{k\pm1}^{2-b_0}\left\{\frac{h^2}{\varphi(1)}x_{k\pm1}^{1+a_0} \mp \frac{2h}{\varphi(2)}x_{k\pm1}^{2+a_0} + \frac{2}{\varphi(3)}\left(x_{k\pm1}^{3+a_0} - x_k^{3+a_0}\right)\right\}\right.
\end{aligned}
$$

$$+\phi(2)\left\{-\frac{h^2}{\phi(3)}x_{k\pm1}^{3+a_0-b_0}\right.$$

$$\left.\left.\left.\pm\frac{2h}{\phi(4)}x_{k\pm1}^{4+a_0-b_0}-\frac{2}{\phi(5)}\left(x_{k\pm1}^{5+a_0-b_0}-x_k^{5+a_0-b_0}\right)\right\}\right\}\right],$$

$$a_{22,k}^{\pm}=\frac{1}{1-b_0}\left[x_{k\pm1}^{1-b_0}\left(h^2+\frac{\mp hx+2x^2}{(2-b_0)}\right)\left\{\frac{h^2}{\psi(1)}x_{k\pm1}^{1+a_0}\mp\frac{2h}{\psi(2)}x_{k\pm1}^{2+a_0}\right.\right.$$

$$\left.+\frac{1}{\psi(3)}\left(x_{k\pm1}^{3+a_0}-x_k^{3+a_0}\right)\right\}$$

$$-\frac{h^4}{\phi(2)}x_{k\pm1}^{2+a_0-b_0}\pm\frac{4h^3}{\phi(3)}x_{k\pm1}^{3+a_0-b_0}+\frac{12h^2}{\phi(4)}x_{k\pm1}^{4+a_0-b_0}\mp\frac{24h}{\phi(5)}x_{k\pm1}^{5+a_0-b_0}$$

$$+\frac{24}{\phi(6)}\left(x_{k\pm1}^{6+a_0-b_0}-x_k^{6+a_0-b_0}\right)$$

$$+\frac{2\phi(2)}{(2-b_0)}\left\{\pm\frac{h^3}{\phi(3)}x_{k\pm1}^{3+a_0-b_0}-\frac{3h^2}{\phi(4)}x_{k\pm1}^{4+a_0-b_0}\right.$$

$$\left.\mp\frac{6h}{\phi(5)}x_{k\pm1}^{5+a_0-b_0}-\frac{6}{\phi(6)}\left(x_{k\pm1}^{6+a_0-b_0}-x_k^{6+a_0-b_0}\right)\right\}$$

$$\left.-\frac{2\varphi(1)}{\varphi(3)}\phi(3)\left\{\frac{h^2}{\phi(4)}x_{k\pm1}^{4+a_0-b_0}\mp\frac{h}{\phi(5)}x_{k\pm1}^{5+a_0-b_0}+\frac{1}{\phi(6)}\left(x_{k\pm1}^{6+a_0-b_0}-x_k^{6+a_0-b_0}\right)\right\}\right],$$

$$a_{30,k}^{\pm}=\frac{1}{1-b_0}\left[x_{k\pm1}^{1-b_0}\left\{\pm\frac{h^3}{\psi(1)}x_{k\pm1}^{1+a_0}-\frac{3h^2}{\psi(2)}x_{k\pm1}^{2+a_0}\pm\frac{6h}{\psi(3)}x_{k\pm1}^{3+a_0}\right.\right.$$

$$\left.-\frac{1}{\psi(4)}\left(x_{k\pm1}^{4+a_0}-x_k^{4+a_0}\right)\right\}$$

$$\mp\frac{h^3}{\phi(2)}x_{k\pm1}^{2+a_0-b_0}+\frac{3h^2}{\phi(3)}x_{k\pm1}^{3+a_0-b_0}\mp\frac{6h}{\phi(4)}x_{k\pm1}^{4+a_0-b_0}$$

$$\left.+\frac{6}{\phi(5)}\left(x_{k\pm1}^{5+a_0-b_0}-x_k^{5+a_0-b_0}\right)\right],$$

$$a_{31,k}^{\pm}=\frac{1}{1-b_0}\left[\left(\pm hx_{k\pm1}^{1-b_0}-\frac{1}{(2-b_0)}x_{k\pm1}^{2-b_0}\right)\left\{\pm\frac{h^3}{\psi(1)}x_{k\pm1}^{1+a_0}-\frac{3h^2}{\psi(2)}x_{k\pm1}^{2+a_0}\pm\frac{6h}{\psi(3)}x_{k\pm1}^{3+a_0}\right.\right.$$

$$\left.-\frac{6}{\psi(4)}\left(x_{k\pm1}^{4+a_0}-x_k^{4+a_0}\right)\right\}$$

$$-\frac{h^4}{\phi(2)}x_{k\pm1}^{2+a_0-b_0}\pm\frac{4h^3}{\phi(3)}x_{k\pm1}^{3+a_0-b_0}-\frac{12h^2}{\phi(4)}x_{k\pm1}^{4+a_0-b_0}\pm\frac{24h}{\phi(5)}x_{k\pm1}^{5+a_0-b_0}$$

$$
\begin{aligned}
&- \frac{24}{\phi(6)}\left(x_{k\pm1}^{6+a_0-b_0} - x_k^{6+a_0-b_0}\right) \\
&+ \frac{\phi(2)}{(2-b_0)}\left\{\pm\frac{h^3}{\phi(3)}x_{k\pm1}^{3+a_0-b_0} - \frac{3h^2}{\phi(4)}x_{k\pm1}^{4+a_0-b_0}\right. \\
&\qquad\left.\pm\frac{6h}{\phi(5)}x_{k\pm1}^{5+a_0-b_0} - \frac{6}{\phi(6)}\left(x_{k\pm1}^{6+a_0-b_0} - x_k^{6+a_0-b_0}\right)\right\}\Bigg], \\
a_{40,k}^{\pm} &= \frac{1}{1-b_0}\left[x_{k\pm1}^{1-b_0}\left\{\frac{h^4}{\psi(1)}x_{k\pm1}^{1+a_0} \mp \frac{4h^3}{\psi(2)}x_{k\pm1}^{2+a_0} + \frac{12h^2}{\psi(3)}x_{k\pm1}^{3+a_0} \mp \frac{24h}{\psi(4)}x_{k\pm1}^{4+a_0}\right.\right. \\
&\qquad\left. + \frac{24}{\psi(5)}\left(x_{k\pm1}^{5+a_0} - x_k^{5+a_0}\right)\right\} \\
&\quad - \frac{h^4}{\phi(2)}x_{k\pm1}^{2+a_0-b_0} \pm \frac{4h^3}{\phi(3)}x_{k\pm1}^{3+a_0-b_0} - \frac{12h^2}{\phi(4)}x_{k\pm1}^{4+a_0-b_0} \\
&\quad\left. \pm \frac{24h}{\phi(5)} - \frac{24}{\phi(6)}\left(x_{k\pm1}^{6+a_0-b_0} - x_k^{6+a_0-b_0}\right)\right], \\
b_{ij,k} &= \left(\frac{a_{ij,k}^{+}}{J_k} + \frac{a_{ij,k}^{-}}{J_{k-1}}\right), \\
\phi(i) &= \prod_{j=2}^{i}(j+a_0-b_0), \qquad \psi(i) = \prod_{j=1}^{i}(j+a_0), \qquad \varphi(i) = \prod_{j=2}^{i}(j-b_0).
\end{aligned}
\tag{A.2}
$$

B. Approximations of the Coefficients

For uniform mesh $x_k = kh,\ k = 0(1)N$ and $h = 1/N$ then for fixed x_k as $h \to 0$, we get the following approximations for the coefficients:

$$
\begin{gathered}
b_{00,k} \sim \frac{hx_k^{a_0}}{G_k}, \quad b_{01,k} \sim (a_0-2b_0)\frac{h^3x_k^{a_0-1}}{4G_k}, \quad b_{02,k} \sim \frac{h^3x_k^{a_0}}{2G_k}, \quad b_{10,k} \sim (2a_0-3b_0)\frac{h^3x_k^{a_0-1}}{12G_k}, \\
b_{11,k} \sim \frac{h^3x_k^{a_0}}{4G_k}, \quad b_{20,k} \sim \frac{h^3x_k^{a_0}}{6G_k}, \quad b_{03,k} \sim (2a_0-3b_0)\frac{h^3x_k^{a_0-1}}{3G_k}, \quad b_{04,k} \sim \frac{2h^5x_k^{a_0}}{3G_k}, \\
b_{12,k} \sim (3a_0-4b_0)\frac{h^5x_k^{a_0-1}}{9G_k}, \quad b_{13,k} \sim \frac{h^5x_k^{a_0}}{3G_k}, \quad b_{21,k} \sim (5a_0-6b_0)\frac{h^5x_k^{a_0-1}}{18G_k}, \quad b_{22,k} \sim \frac{2h^5x_k^{a_0}}{9G_k}, \\
b_{30,k} \sim (6a_0-7b_0)\frac{h^5x_k^{a_0-1}}{30G_k}, \quad b_{31,k} \sim \frac{h^5x_k^{a_0}}{6G_k}, \quad b_{40,k} \sim \frac{2h^5x_k^{a_0}}{15G_k},
\end{gathered}
$$

$$\left|a^{-}_{00,N}\right| < \frac{h^2}{2}, \quad \left|a^{-}_{01,N}\right| < \frac{2h^3}{3}, \quad \left|a^{-}_{02,N}\right| < \frac{h^4}{6}, \quad \left|a^{-}_{10,N}\right| < \frac{h^3}{3}, \quad \left|a^{-}_{11,N}\right| < \frac{h^4}{4}, \quad \left|a^{-}_{20,N}\right| < \frac{h^4}{6},$$

$$\left|a^{-}_{03,N}\right| < \frac{2h^5}{5}, \quad \left|a^{-}_{04,N}\right| < \frac{h^6}{3}, \quad \left|a^{-}_{12,N}\right| < \frac{h^5}{5}, \quad \left|a^{-}_{13,N}\right| < \frac{h^6}{6}, \quad \left|a^{-}_{21,N}\right| < \frac{2h^5}{15}, \quad \left|a^{-}_{22,N}\right| < \frac{h^6}{9},$$

$$\left|a^{-}_{30,N}\right| < \frac{h^5}{10}, \quad \left|a^{-}_{31,N}\right| < \frac{h^6}{12}, \quad \left|a^{-}_{40,N}\right| < \frac{h^6}{15}. \tag{B.1}$$

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