

Research Article

On Properties of Geodesic η -Preinvex Functions

I. Ahmad,^{1,2} Akhlag Iqbal,¹ and Shahid Ali¹

¹ Department of Mathematics, Aligarh Muslim University, Aligarh 202 002, India

² Department of Mathematics and Statistics, King Fahd University of Petroleum and Minerals, Dhahran 31261, Saudi Arabia

Correspondence should be addressed to Akhlag Iqbal, akhlag6star@gmail.com

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The present paper deals with the properties of geodesic η -preinvex functions and their relationships with η -invex functions and strictly geodesic η -preinvex functions. The geodesic η -pre-pseudo-invex and geodesic η -pre-quasi-invex functions on the geodesic invex set are introduced and some of their properties are discussed.

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1. Introduction

In the recent years, several generalizations have been developed for the classical properties of convexity. This concept in linear topological spaces relies on the possibility of connecting any two points of the space by the line segment between them. In 1981, Hanson [1] introduced the concept of invexity by generalizing the difference $(x - y)$ in the definition of convex function to any function $\eta(x, y)$. Hanson's initial results inspired a great deal of subsequent work, which has greatly expanded the role and applications of invexity in nonlinear optimization and other branches of pure and applied sciences.

Ben-Israel and Mond [2] introduced a new generalization of convex sets and convex functions. Craven [3] called them invex sets and preinvex functions, respectively. Jeyakumar [4] studied the properties of preinvex functions and their role in optimization and mathematical programming.

In general, a manifold is not a linear space, but the extension of concepts and techniques from linear spaces to Riemannian manifold are natural. Rapcsák [5] and Udriște [6] considered a generalization of convexity, called geodesic convexity and extended many results of convex analysis and optimization theory to Riemannian manifolds. In this setting, the linear space has been replaced by Riemannian manifold and the line segment by a geodesic. For more details, readers may see [5, 6] and the references cited therein.

The notion of invex functions on Riemannian manifold was introduced in [7], however its generalization has been investigated by Mititelu [8]. Barani and Pouryayevali [9] introduced the geodesic invex set, geodesic invex function and geodesic preinvex function on Riemannian manifold with respect to particular maps, and studied the relations between them.

In this paper, we discuss various concepts, definitions and properties for the functions on Riemannian manifold. The notion of invexity and its generalization on Riemannian manifold is recalled in Section 2. In Section 3, we discuss some properties of geodesic η -preinvex functions and their relationships with η -invex functions. The relationship between strictly geodesic η -preinvex function and geodesic η -preinvex function is investigated. We also prove that the composite function is η -invex on the geodesic invex set.

The geodesic η -pre-pseudo-invex (p.p.i) and geodesic η -pre-quasi-invex (p.q.i) functions are introduced and their properties are also discussed in Section 4. The results obtained here generalize those which are present in the literature.

2. Preliminaries

In this section, we recall some definitions and known results about Riemannian manifolds, which will be used throughout the paper. For the standard material on differential geometry, one can consult [10].

Suppose that (M, g) is a complete n -dimensional Riemannian manifold. A subset A of M is called totally convex, if A contains every geodesic $\gamma_{x,y}$ of M , whose end points x and y belong to A [6].

Definition 2.1. Let M be a complete n -dimensional Riemannian manifold and A be a totally convex set in M . A function $f : A \rightarrow R$ is said to be geodesic convex if for all geodesic arcs $\gamma : [x, y] \rightarrow M$ and all $t \in [0, 1]$, one has

$$f(\gamma_{x,y}(t)) \leq (1-t)f(y) + tf(x), \quad (2.1)$$

or

$$f(\gamma(y)) - f(\gamma(x)) \geq d(f(\gamma))_x(y - x) \quad (2.2)$$

if f is a differentiable function.

Let M be an n -dimensional differentiable manifold and $T_p M$ be the tangent space to M at p . Also, assume that $TM = \bigcup_{p \in M} T_p M$ is the tangent bundle of M . For any $\epsilon > 0$, $I_\epsilon = (-\epsilon, \epsilon) \subset R$.

Definition 2.2 (see [7, 8]). A differentiable curve $\alpha : I_\epsilon \rightarrow M$ is called a differentiable application.

Let $\alpha(0) = p \in M$. Then tangent vector to the curve α at p is $v = \alpha'(0) \in T_{\alpha(0)} M = T_p M$.

Assume that N is another differentiable manifold and $\varphi : M \rightarrow N$ is a differentiable application.

Definition 2.3 (see [8]). The linear application $d\varphi_p : T_pM \rightarrow T_{\varphi(p)}N$ defined by $d\varphi_p(v) = \beta'(0)$, where $v = \alpha'(0) \in T_pM$ and $\beta = \varphi \circ \alpha$ is called the differential of φ at the point p . But

$$\beta'(0) = \varphi'(\alpha(0))\alpha'(0) = \varphi'(p)v. \quad (2.3)$$

So,

$$d\varphi_p(v) = \varphi'(p)v, \quad v \in T_pM. \quad (2.4)$$

The length of a piecewise C^1 curve $\gamma : [a, b] \rightarrow M$ is defined by

$$L(\gamma) := \int_a^b \|\gamma'(t)\|_{\gamma(t)} dt. \quad (2.5)$$

For any two points $p, q \in M$, we define

$$d(p, q) := \inf \{L(\gamma) : \gamma \text{ is a piecewise } C^1 \text{ curve joining } p \text{ to } q\}. \quad (2.6)$$

Then d is a distance which induces the original topology on M . We know that on every Riemannian manifold there exists exactly one covariant derivation called Levi-Civita connection denoted by $\nabla_X Y$ for any vector fields $X, Y \in M$.

Definition 2.4. A geodesic is a C^∞ smooth path γ whose tangent is parallel along the path γ , that is, γ satisfies the equation $\nabla_{d\gamma(t)/dt} d\gamma(t)/dt = 0$. Any path γ joining p and q in M such that $L(\gamma) = d(p, q)$ is a geodesic and it is called a minimal geodesic.

In other words, a C^∞ curve $\gamma : I \rightarrow M$ whose acceleration vector field vanishes identically is called geodesic [6].

We consider now an application $\eta : M \times M \rightarrow TM$ such that $\eta(p, q) \in T_qM$ for every $q \in M$ and any $p \in M$. For a differentiable function $f : M \rightarrow R$, Pini [7] defined invexity as follows.

Definition 2.5. The differentiable function f is said to be η -invex on M if for any $x, y \in M$,

$$f(x) - f(y) \geq df_y(\eta(x, y)). \quad (2.7)$$

If f is a differentiable map from the manifold M to the manifold N , we will denote by df_x the differential of f at x .

Mititelu [8] generalized the above definition as follows.

Definition 2.6. The differentiable function f is said to be η -pseudoinvex on M if for any $x, y \in M$,

$$df_y(\eta(x, y)) \geq 0 \implies f(x) \geq f(y). \quad (2.8)$$

Definition 2.7. The differentiable function f is said to be η -quasiinvex on M if for any $x, y \in M$,

$$f(x) \leq f(y) \implies df_y(\eta(x, y)) \leq 0. \quad (2.9)$$

Remark 2.8. If f is a differentiable and η -invex function defined on M and $df_y(\eta(x, y)) \geq 0$ for all $x, y \in M$, then f is η -pseudoinvex on M .

In all these definitions,

$$df_y(\eta(x, y)) = [df(y)]\eta(x, y). \quad (2.10)$$

If (M, g) is a Riemannian manifold and f is a differentiable map from M to N , then

$$df_y(\eta(x, y)) = g_y(\text{grad } f(y), \eta(x, y)), \quad (2.11)$$

where $\text{grad } f(y)$ is the gradient of f at the point y .

Barani and Pouryayevali [9] defined the geodesic invex set and the invexity of a function f on an open geodesic invex subset of a Riemannian manifold.

Definition 2.9. Let M be a Riemannian manifold and let $\eta : M \times M \rightarrow TM$ be a function such that for every $x, y \in M$, $\eta(x, y) \in T_y M$. A nonempty subset S of M is said to be geodesic invex set with respect to η if for every $x, y \in S$ there exists a unique geodesic $\gamma_{x,y} : [0, 1] \rightarrow M$ such that

$$\gamma_{x,y}(0) = y, \quad \gamma'_{x,y}(0) = \eta(x, y), \quad \gamma_{x,y}(t) \in S, \quad \forall t \in [0, 1]. \quad (2.12)$$

Definition 2.10. Let M be a Riemannian manifold, let S be an open subset of M which is geodesic invex with respect to $\eta : M \times M \rightarrow TM$, and let f be a real differentiable function defined on S . Then, f is said to be η -invex on S if for every $x, y \in S$,

$$f(x) - f(y) \geq df_y(\eta(x, y)). \quad (2.13)$$

Differentiable convex functions (on an open convex subset A) are invex. Also, the wider class of geodetically convex functions on manifolds is included in the class of invex functions, under some additional hypotheses on manifold [11]. In particular, this is true if the manifold has the property that given any two points x, y there exists a unique geodesic $\gamma_{x,y}(t)$ joining them. In this case a function which is geodetically convex is also geodesic invex with respect to $\eta(x, y) = \gamma'_{x,y}(0)$.

3. Some Properties of Geodesic η -Preinvex Functions

The definition of preinvex function on R^n is given in [12]. See also [13, 14] for the properties of preinvex functions. Barani and Pouryayevali [9] extended this notion to Riemannian manifolds.

Definition 3.1 (see [9]). Let M be a Riemannian manifold and let S be an open subset of M which is geodesic invex with respect to $\eta : M \times M \rightarrow TM$. Then, $f : S \rightarrow R$ is said to be geodesic η -preinvex on S if

$$f(\gamma_{x,y}(t)) \leq tf(x) + (1-t)f(y), \quad (3.1)$$

for every $x, y \in S$, $t \in [0, 1]$. If the above inequality is strict then, f is said to be strictly geodesic η -preinvex on S .

Theorem 3.2. *Let M be a Riemannian manifold and let S be an open subset of M which is geodesic invex with respect to $\eta : M \times M \rightarrow TM$. Let $f : S \rightarrow R$ be a geodesic η -preinvex function and let $g : I \rightarrow R$ be an increasing convex function such that $\text{range}(f) \subseteq I$. Then the composite function $g(f)$ is geodesic η -preinvex on S .*

Proof. Since f is geodesic η -preinvex function, we have

$$f(\gamma_{x,y}(t)) \leq tf(x) + (1-t)f(y). \quad (3.2)$$

Since g is an increasing and convex function, we get

$$\begin{aligned} g[f(\gamma_{x,y}(t))] &\leq g[tf(x) + (1-t)f(y)] \\ &\leq tg(f(x)) + (1-t)g(f(y)). \end{aligned} \quad (3.3)$$

Hence, $g(f)$ is geodesic η -preinvex on S . \square

Theorem 3.3. *Let M be a Riemannian manifold and let S be an open subset of M which is geodesic invex with respect to $\eta : M \times M \rightarrow TM$. Let $f : S \rightarrow R$ be a geodesic η -preinvex function and let $g : I \rightarrow R$ be a strictly increasing convex function such that $\text{range}(f) \subseteq I$. Then the composite function $g(f)$ is strictly geodesic η -preinvex on S .*

Proof. Since f is geodesic η -preinvex function, we have

$$f(\gamma_{x,y}(t)) \leq tf(x) + (1-t)f(y). \quad (3.4)$$

Since g is the strictly increasing and convex function, we get

$$\begin{aligned} g[f(\gamma_{x,y}(t))] &< g[tf(x) + (1-t)f(y)] \\ &\leq tg(f(x)) + (1-t)g(f(y)). \end{aligned} \quad (3.5)$$

Or

$$g[f(\gamma_{x,y}(t))] < tg(f(x)) + (1-t)g(f(y)). \quad (3.6)$$

Hence, $g(f)$ is strictly geodesic η -preinvex on S . \square

Similarly, we can prove the following results.

Theorem 3.4. Let M be a Riemannian manifold and let S be an open subset of M which is geodesic invex with respect to $\eta : M \times M \rightarrow TM$. Let $f : S \rightarrow R$ be a geodesic η -preinvex function and let $g : I \rightarrow R$ be increasing and strictly convex function such that $\text{range}(f) \subseteq I$. Then the composite function $g(f)$ is strictly geodesic η -preinvex on S .

Theorem 3.5. Let S be an open subset of M which is geodesic invex with respect to $\eta : M \times M \rightarrow TM$. Suppose $f_i : S \rightarrow R, i = 1, 2, \dots, p$, be geodesic η -preinvex. Then

$$f = \sum_{i=1}^p \lambda_i f_i, \quad \forall \lambda_i \in R, \lambda_i \geq 0, i = 1, 2, \dots, p, \quad (3.7)$$

is geodesic η -preinvex function on S .

Now, we prove the following proposition, which guarantees that a differentiable and geodesic η -preinvex function f is η -invex.

Proposition 3.6. Let M be a complete manifold and $S \subseteq M$ which is geodesic invex with respect to $\eta : M \times M \rightarrow TM$. Let $f : S \rightarrow R$ be a differentiable function and there exists a sequence $\{t_n\}$ of positive real numbers such that $t_n \rightarrow 0$ as $n \rightarrow \infty$ and

$$f(\gamma(t_n)) \leq t_n f(x) + (1 - t_n) f(y) \quad (3.8)$$

for every $x, y \in S$, then f is η -invex on S .

Proof. We have

$$\frac{f(\gamma(t_n)) - f(y)}{t_n} \leq f(x) - f(y). \quad (3.9)$$

Since f is differentiable on S , taking the limit as $n \rightarrow \infty$ on both sides, we get

$$df_{\gamma_{x,y}(0)}(\gamma'_{x,y}(0)) \leq f(x) - f(y). \quad (3.10)$$

Therefore,

$$df_y(\eta(x, y)) \leq f(x) - f(y). \quad (3.11)$$

Hence, the result. \square

It is to be noted that the converse of above proposition is not true in general. However, Barani and Pouryayevali [9] proved that a η -invex function on S is geodesic η -preinvex on S if η satisfies the condition C in [9].

It is revealed in the following proposition that like convex functions, η -invex functions are transformed into η -invex functions by a suitable class of monotone functions.

Proposition 3.7. Let $\phi : R \rightarrow R$ be a monotone increasing differentiable convex function. If f is η -invex on geodesic invex set S , then the composite function $\phi(f)$ is η -invex.

Proof. Using the fact that $\phi(x+h) \geq \phi(x) + \phi'(x)h$ for every $x, h \in R$, we have

$$\begin{aligned} \phi(f(x)) &\geq \phi(f(y) + df_y(\eta(x, y))) \\ &\geq \phi(f(y)) + \phi'(f(y))df_y(\eta(x, y)) \\ &= \phi(f(y)) + d(\phi(f))_y \eta(x, y) \\ &\implies \phi(f(x)) - \phi(f(y)) \\ &\geq d(\phi(f))_y \eta(x, y). \end{aligned} \tag{3.12}$$

Hence $\phi(f)$ is η -invex on S . □

4. Properties of Generalized Geodesic η -Preinvex Functions

In [15], Pini introduced the notion of η -pre-pseudo-invex and η -pre-quasi-invex functions on an invex set. We extend these notions to geodesic η -pre-pseudo-invexity and geodesic η -pre-quasi-invexity on a geodesic invex set S by replacing the line segment with the geodesic.

Let f be a function defined on a geodesic invex subset S of a Riemannian manifold M with respect to $\eta : M \times M \rightarrow TM$.

Definition 4.1. Function f is said to be geodesic η -pre-pseudo-invex (p.p.i) on S if there exist a geodesic $\gamma_{x,y}(t)$ and a strictly positive function $b : S \times S \rightarrow R^+$ such that

$$f(x) < f(y) \implies f(\gamma_{x,y}(t)) \leq f(y) + t(t-1)b(x, y), \tag{4.1}$$

for every $t \in (0, 1)$ and $x, y \in S$.

Theorem 4.2. Let M be a Riemannian manifold and S be an open subset of M which is geodesic invex with respect to $\eta : M \times M \rightarrow TM$. If f is geodesic η -preinvex, then f is geodesic η -pre-pseudo-invex for the same geodesic.

Proof. If $f(x) < f(y)$ for every $t \in (0, 1)$ and f is geodesic η -preinvex, then

$$\begin{aligned} f(\gamma_{x,y}(t)) &\leq tf(x) + (1-t)f(y) \\ &= f(y) + t(f(x) - f(y)) \\ &< f(y) + t(f(x) - f(y)) - t^2(f(x) - f(y)) \\ &= f(y) + t(t-1)(f(y) - f(x)) \\ &= f(y) + t(t-1)b(x, y), \end{aligned} \tag{4.2}$$

where $b(x, y) = f(y) - f(x) > 0$. □

Theorem 4.3. Let $f : S \rightarrow R$ be a geodesic η -pre-pseudo-invex function on S and let $g : I \rightarrow R$ be strictly increasing convex function such that $\text{range}(f) \subseteq I$. Then, the composite function $g(f)$ is geodesic η -pre-pseudo-invex on S .

Proof. Since f is geodesic η -pre-pseudo-invex function on S , we have

$$f(x) < f(y) \implies f(\gamma_{x,y}(t)) \leq f(y) + t(t-1)b(x,y), \quad (4.3)$$

for every $t \in (0,1)$ and $x, y \in S$, where $b(x,y)$ is strict positive function.

Since g is strictly increasing convex function, we get

$$\begin{aligned} g(f(x)) < g(f(y)) &\implies g(f(\gamma_{x,y}(t))) \\ &< g[f(y) + t(t-1)b(x,y)] \\ &< g(f(y)) + t(t-1)g(b(x,y)) \end{aligned} \quad (4.4)$$

for every $t \in (0,1)$ and $x, y \in S$, where $g(b(x,y))$ is strict positive function. Which shows that $g(f)$ is a geodesic η -pre-pseudo-invex function on S . \square

Definition 4.4. Function f is said to be geodesic η -pre-quasi-invex (p.q.i) on S if

$$f(\gamma_{x,y}(t)) \leq \max(f(x), f(y)), \quad (4.5)$$

for all $x, y \in S$ and for every $t \in [0,1]$.

Now, we characterize geodesic η -pre-quasi-invex function in terms of its lower level sets.

Theorem 4.5. Let S be a geodesic invex subset of M and $f : S \rightarrow R$. Then, f is geodesic η -pre-quasi-invex on S if and only if its lower level sets are geodesic invex.

Proof. Suppose that f is geodesic η -pre-quasi-invex function on S and $C(\alpha) := \{x : f(x) \leq \alpha\}$ is the subset of S . If $C(\alpha)$ is empty, the result is trivial. If $C(\alpha)$ is neither empty nor the whole set S , take any two points x and y in $C(\alpha)$. We have to show that the geodesic $\gamma_{x,y}(t)$ is contained in $C(\alpha)$. Since f is geodesic η -pre-quasi-invex function, we have

$$f(\gamma_{x,y}(t)) \leq \max(f(x), f(y)) \leq \alpha, \quad (4.6)$$

for all $x, y \in S$ and for every $t \in [0,1]$. Hence $C(\alpha)$ is geodesic invex.

Conversely, suppose that for every real number α the set $C(\alpha)$ is geodesic invex. Take any two points $x, y \in S$ and suppose that $f(x) \leq f(y)$. Consider the lower level set $C(f(y))$. Since $C(\alpha)$ is geodesic invex, the geodesic $\gamma_{x,y}(t)$ is contained in $C(f(y))$. Thus,

$$f(\gamma_{x,y}(t)) \leq f(y) = \max(f(x), f(y)) \quad (4.7)$$

for every $t \in [0,1]$. The proof is complete. \square

Proposition 4.6. *Let f be a geodesic η -pre-quasi-invex function on S . Then,*

- (i) *every strict local minimum of f is also a strict global minimum;*
- (ii) *the set of all strict global minimum points is geodesic invex set.*

Proof. (i) Let y be a strict local minimum which is not global; then there exists a point $x^* \in S$, such that, $f(x^*) < f(y)$. Since, f is geodesic η -pre-quasi-invex, we have $f(\gamma_{x^*,y}(t)) \leq f(y)$, which contradicts the hypothesis that y is a strict local minimum.

(ii) If f has no minimum value in S , then the set of minimum points is empty and hence geodesic invex. If f has the minimum point α on S , then the set of minimum points is $S \cap C(\alpha)$, which is geodesic invex.

The geodesic η -pre-quasi-invexity is preserved under composition with nondecreasing function $\phi : R \rightarrow R$ as can be seen below. \square

Proposition 4.7. *Let f be a geodesic η -pre-quasi-invex function and let $\phi : R \rightarrow R$ be a nondecreasing function. Then, $\phi(f)$ is geodesic η -pre-quasi-invex.*

Proof. Given that f is a geodesic η -pre-quasi-invex function and ϕ is a nondecreasing function. Then, we have

$$\begin{aligned} (\phi(f))(\gamma_{x,y}(t)) &\leq \phi(\max(f(x), f(y))) \\ &= \max(\phi(f(x)), \phi(f(y))), \end{aligned} \quad (4.8)$$

which shows that the composite function $\phi(f)$ is geodesic η -pre-quasi-invex. \square

Proposition 4.8. *If the function f is geodesic η -preinvex on S , then f is geodesic η -pre-quasi-invex on S .*

Proof. Let f be geodesic η -preinvex function on S . Then for every $x, y \in S$ and $t \in [0, 1]$, it follows that

$$\begin{aligned} f(\gamma_{x,y}(t)) &\leq tf(x) + (1-t)f(y) \\ &\leq t \max\{f(x), f(y)\} + (1-t) \max\{f(x), f(y)\} \\ &= \max\{f(x), f(y)\}, \end{aligned} \quad (4.9)$$

which shows that f is geodesic η -pre-quasi-invex on S . \square

Proposition 4.9. *If $f : S \rightarrow R$ is geodesic η -pre-pseudo-invex on S then f is geodesic η -pre-quasi-invex on S .*

Proof. Let $f(x) < f(y)$. Since f is geodesic η -pre-pseudo-invex function on S , for all $x, y \in S$, and for all $t \in (0, 1)$, we have

$$\begin{aligned} f(\gamma_{x,y}(t)) &\leq f(y) + t(t-1)b(x, y) \\ &< f(y) \\ &= \max\{f(x), f(y)\}. \end{aligned} \quad (4.10)$$

Hence, f is geodesic η -pre-quasi-invex on S . \square

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