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Research Article

Dynamic Approaches for Multichoice Solutions

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Based on alternative reduced games, several dynamic approaches are proposed to show how the three extended Shapley values can be reached dynamically from arbitrary efficient payoff vectors on multichoice games.

1. Introduction

A *multichoice transferable-utility (TU) game*, introduced by Hsiao and Raghavan [1], is a generalization of a standard coalition TU game. In a standard coalition TU game, each player is either fully involved or not involved at all in participation with some other agents, while in a multichoice TU game, each player is allowed to participate with many finite different activity levels. Solutions on multichoice TU games could be applied in many fields such as economics, political sciences, management, and so forth. Van den Nouweland et al. [2] referred to several applications of multichoice TU games, such as a large building project with a deadline and a penalty for every day if this deadline is overtime. The date of completion depends on the effort of how all of the people focused on the project: the harder they exert themselves, the sooner the project will be completed. This situation gives rise to a multichoice TU game. The worth of a coalition resulted from the players working in certain levels to a project is defined as the penalty for their delay of the project completion with the same efforts. Another application appears in a large company with many divisions, where the profit-making depends on their performance. This situation also gives rise to a multichoice TU game. The players are the divisions, and the worth of a coalition resulted from the divisions functioning in certain levels is the corresponding profit produced by the company.

Here we apply three solutions for multichoice TU games due to Hsiao and Raghavan [1], Derks and Peters [3], and Peters and Zank [4], respectively. Two main results are as follows.

(1) A solution concept can be given *axiomatic justification*. Oppositely, *dynamic processes* can be described that lead the players to that solution, starting from an arbitrary efficient payoff vector (the foundation of a dynamic theory was laid by Stearns [5]. Related dynamic results may be found in, for example, Billera [6], Maschler and Owen [7], etc.). In Section 3, we firstly define several alternative reductions on multichoice TU games. Further, we adopt these reductions and some axioms introduced by Hsiao and Raghavan [1], Hwang and Liao [8–10], and Klijn et al. [11] to show how the three extended Shapley values can be reached dynamically from arbitrary efficient payoff vectors. In the proofs of Theorems 3.2 and 3.4, we will point out how these axioms would be used in the dynamic approaches.

(2) There are two important factors, the *players* and their *activity levels*, for multichoice games. Inspired by Hart and Mas-Colell [12], Hwang and Liao [8–10] proposed two types of reductions by only reducing the number of the players. In Section 4, we propose two types of *player-action reduced games* by reducing both the number of the players and the activity levels. Based on the *potential*, Hart and Mas-Colell [12] showed that the Shapley value [13] satisfies consistency. Different from Hart and Mas-Colell [12], we show that the three extended Shapley values satisfy related properties of player-action consistency by applying alternative method.

2. Preliminaries

Let U be the universe of players and $N \subseteq U$ be a set of players. Suppose each player i has $m_i \in \mathbb{N}$ levels at which he can actively participate. Let $m = (m_i)_{i\in N}$ be the vector that describes the number of activity levels for each player, at which he can actively participate. For $i \in U$, we set $M^i = \{0, 1, \ldots, m_i\}$ as the action space of player i, where the action 0 means not participating, and $M^i_+ = M^i \setminus \{0\}$. For $N \subseteq$ U, $N \neq \emptyset$, let $M^N = \prod_{i\in N} M^i$ be the product set of the action spaces for players N and $M^N_+ = \prod_{i\in N} M^i_+$. Denote the zero vector in $\mathbb{R}^N$ by 0_N.

A *multichoice TU game* is a triple (N, m, v), where N is a nonempty and finite set of players, m is the vector that describes the number of activity levels for each player, and $v : M^N \to \mathbb{R}$ is a characteristic function which assigns to each action vector $\alpha = (\alpha_i)_{i\in N} \in M^N$ the worth that the players can jointly obtain when each player i plays at activity level $\alpha_i \in M_i$ with $v(0_N) = 0$. If no confusion can arise, a game (N, m, v) will sometimes be denoted by its characteristic function v. Given a multichoice game (N, m, v) and $\alpha \in M^N$, we write (N, α, v) for the *multichoice TU subgame* obtained by restricting v to $\{\beta \in M^N \mid \beta_i \leq \alpha_i \ \forall\, i \in N\}$ only. Denote the class of all multichoice TU games by MC.

Given $(N, m, v) \in MC$, let $L^{N,m} = \{(i, k_i) \mid i \in N,\ k_i \in M^i_+\}$. A *solution* on MC is a map ψ assigning to each $(N, m, v) \in MC$ an element

$$\psi(N, m, v) = \left(\psi_{i,k_i}(N, m, v)\right)_{(i,k_i)\in L^{N,m}} \in \mathbb{R}^{L^{N,m}}. \tag{2.1}$$

Here $\psi_{i,k_i}(N, m, v)$ is the power index or the value of the player i when he takes action k_i to play game v. For convenience, given $(N, m, v) \in MC$ and a solution ψ on MC, we define $\psi_{i,0}(N, m, v) = 0$ for all $i \in N$.

To state the three extended Shapley values, some more notations will be needed. Given $S \subseteq N$, let |S| be the number of elements in S and let $e^S(N)$ be the binary vector in $\mathbb{R}^N$ whose component $e_i^S(N)$ satisfies

$$e_i^S(N) = \begin{cases} 1 & \text{if } i \in \text{S}, \\ 0 & \text{otherwise.} \end{cases} \tag{2.2}$$

Note that if no confusion can arise $e_i^S(N)$ will be denoted by e_i^S.

Given $(N, m, v) \in MC$ and $\alpha \in M^N$, we define $S(\alpha) = \{k \in N \mid \alpha_k \neq 0\}$ and $\|\alpha\| = \sum_{i \in N} \alpha_i$. Let $\alpha, \beta \in \mathbb{R}^N$, we say $\beta \leq \alpha$ if $\beta_i \leq \alpha_i$ for all $i \in N$.

The analogue of unanimity games for multichoice games are *minimal effort games* (N, m, u_N^α), where $\alpha \in M^N$, $\alpha \neq 0_N$, defined by for all $\beta \in M^N$,

$$u_N^\alpha(\beta) = \begin{cases} 1 & \text{if } \beta \geq \alpha; \\ 0 & \text{otherwise.} \end{cases} \tag{2.3}$$

It is known that for $(N, m, v) \in MC$ it holds that $v = \sum_{\alpha \in M^N \setminus \{0_N\}} a^\alpha(v) u_N^\alpha$, where $a^\alpha(v) = \sum_{S \subseteq S(\alpha)} (-1)^{|S|} v(\alpha - e^S)$.

Here we apply three extensions of the Shapley value for multichoice games due to Hsiao and Raghavan [1], Derks and Peters [3], and Peters and Zank [4].

Definition 2.1. *(i) (Hsiao and Raghavan, [1]).*

The H&R *Shapley value* Λ is the solution on MC which associates with each $(N, m, v) \in MC$ and each player $i \in N$ and each $k_i \in M_i^+$ the value (Hsiao and Raghavan [1] provided an alternative formula of the H&R Shapley value. Hwang and Liao [9] defined the H&R Shapley value in terms of the dividends)

$$\Lambda_{i,k_i}(N, m, v) = \sum_{\substack{\alpha \in M^N \\ \alpha_i \leq k_i}} \frac{a^\alpha(v)}{|S(\alpha)|}. \tag{2.4}$$

Note that the so-called dividend $a^\alpha(v)$ is divided equally among the necessary players.

(ii) (Derks and Peters, [3]).

The D&P *Shapley value* Θ is the solution on MC which associates with each $(N, m, v) \in MC$ and each player $i \in N$ and each $k_i \in M_i^+$ the value

$$\Theta_{i,k_i}(N, m, v) = \sum_{\substack{\alpha \in M^N \\ \alpha_i \geq k_i}} \frac{a^\alpha(v)}{\|\alpha\|}. \tag{2.5}$$

Note that the so-called dividend $a^\alpha(v)$ is divided equally among the necessary levels.

(iii) (Peters and Zank, [4]).

The P&Z *Shapley value* Γ is the solution on MC which associates with each $(N, m, v) \in MC$ and each player $i \in N$ and each $k_i \in M_i^+$ the value (Peters and Zank [4] defined the P&Z Shapley value by fixing its values on minimal effort games and imposing linearity. Hwang and Liao [8] defined the P&Z Shapley value based on the dividends)

$$\Gamma_{i,k_i}(N, m, v) = \sum_{\substack{\alpha \in M^N \\ \alpha_i = k_i}} \frac{a^\alpha(v)}{|S(\alpha)|}. \tag{2.6}$$

Clearly, the P&Z Shapley value is a subdivision of the H&R Shapley value. For all $(N, m, v) \in MC$ and for all $(i, \mathrm{k}_i) \in L^{N,m}$,

$$\Lambda_{i,k_i}(N, m, v) = \sum_{\substack{\alpha \in M^N \\ \alpha_i \le k_i}} \frac{a^\alpha(v)}{|S(\alpha)|} = \sum_{t_i=1}^{k_i} \sum_{\substack{\alpha \in M^N \\ \alpha_i = t_i}} \frac{a^\alpha(v)}{|S(\alpha)|} = \sum_{t_i=1}^{k_i} \Gamma_{i,t_i}(N, m, v). \tag{2.7}$$

3. Axioms and Dynamic Approaches

In this section, we propose dynamic processes to illustrate that the three extended Shapley values can be reached by players who start from an arbitrary efficient solution.

In order to provide several dynamic approaches, some more definitions will be needed. Let ψ be a solution on MC. ψ satisfies the following.

(i) *1-efficiency (1EFF)* if for each $(N, m, v) \in MC$, $\sum_{i \in S(m)} \psi_{i,m_i}(N, m, v) = v(m)$.

(ii) *2-efficiency (2EFF)* if for each $(N, m, v) \in MC$, $\sum_{i \in S(m)} \sum_{k_i=1}^{m_i} \psi_{i,k_i}(N, m, v) = v(m)$.

The following axioms are analogues of the balanced contributions property due to Myerson [14]. The solution ψ satisfies the following.

(i) *1-strong balanced contributions (1SBC)* if for each $(N, m, v) \in MC$ and $(i, k_i), (j, k_j) \in L^{N,m}$, $i \neq j$,

$$\begin{aligned} &\psi_{i,k_i}(N, (m_{N \setminus \{j\}}, k_j), v) - \psi_{i,k_i}(N, (m_{N \setminus \{j\}}, 0), v) \\ &\quad = \psi_{j,k_j}(N, (m_{N \setminus \{i\}}, k_i), v) - \psi_{j,k_j}(N, (m_{N \setminus \{i\}}, 0), v). \end{aligned} \tag{3.1}$$

(ii) *2-strong balanced contributions (2SBC)* if for each $(N, m, v) \in MC$ and $(i, k_i), (j, k_j) \in L^{N,m}$, $i \neq j$,

$$\begin{aligned} &\psi_{i,k_i}(N, m, v) - \psi_{i,k_i}(N, (m_{N \setminus \{j\}}, k_j - 1), v) \\ &\quad = \psi_{j,k_j}(N, m, v) - \psi_{j,k_j}(N, (m_{N \setminus \{i\}}, k_i - 1), v). \end{aligned} \tag{3.2}$$

(iii) *3-strong balanced contributions (3SBC)* if for each $(N,m,v)\in MC$ and $(i,k_i),(j,k_j)\in L^{N,m}$, $i\neq j$,

$$\begin{aligned}&\psi_{i,k_i}\left(N,\left(m_{N\setminus\{j\}},k_j\right),v\right)-\psi_{i,k_i}\left(N,\left(m_{N\setminus\{j\}},k_j-1\right),v\right)\\&\quad=\psi_{j,k_j}\left(N,\left(m_{N\setminus\{i\}},k_i\right),v\right)-\psi_{j,k_j}\left(N,\left(m_{N\setminus\{i\}},k_i-1\right),v\right).\end{aligned}\tag{3.3}$$

The following axiom was introduced by Hwang and Liao ([9]). The solution ψ satisfies the following.

(i) *Independence of individual expansions (IIE)* if for each $(N,m,v)\in MC$ and each $(i,k_i)\in L^{N,m}$, $j\neq m_i$,

$$\psi_{i,k_i}\left(N,\left(m_{N\setminus\{i\}},k_i\right),v\right)=\psi_{i,k_i}\left(N,\left(m_{N\setminus\{i\}},k_i+1\right),v\right)=\cdots=\psi_{i,k_i}(N,m,v).\tag{3.4}$$

In the framework of multichoice games, IIE asserts that whenever a player gets available higher activity level, the payoff for all original levels should not be changed under condition that other players are fixed.

The following axiom was introduced by Klijn et al. [11]. The solution ψ satisfies the following.

(i) *Equal loss (EL)* if for each $(N,m,v)\in MC$ and each $(i,k_i)\in L^{N,m}$, $k_i\neq m_i$,

$$\psi_{i,k_i}(N,m,v)-\psi_{i,k_i}\left(N,m-e^{\{i\}},v\right)=\psi_{i,m_i}(N,m,v).\tag{3.5}$$

Klijn et al. [11] provided an interpretation of the equal loss property as follows. EL is also inspired by the balanced contributions property of Myerson [14]. In the framework of multichoice games, EL says that whenever a player gets available higher activity level the payoff for all original levels changes with an amount equal to the payoff for the highest level in the new situation. Note that EL is a vacuous property for standard coalition TU games.

Some considerable weakenings of the previous axioms are as follows. *Weak 1-efficiency (1WEFF)* simply says that for all $(N,m,v)\in MC$ with $|S(m)|=1$, ψ satisfies 1EFF. *Weak 2-efficiency (2WEFF)* simply says that for all $(N,m,v)\in MC$ with $|S(m)|=1$, ψ satisfies 2EFF. *1-upper balanced contributions (1UBC)* only requires that 1SBC holds if $k_i=m_i$ and $k_j=m_j$. *2-upper balanced contributions (2UBC)* only requires that 2SBC or 3SBC holds if $k_i=m_i$ and $k_j=m_j$. *Weak independence of individual expansions (WIIE)* simply says that for all $(N,m,v)\in MC$ with $|S(m)|=1$, ψ satisfies IIE. *Weak equal loss (WEL)* simply says that for all $(N,m,v)\in MC$ with $|S(m)|=1$, ψ satisfies EL.

Subsequently, we recall the reduced games and related consistency properties introduced by Hwang and Liao [8–10]. Let $(N,m,v)\in MC$, $S\subseteq N\setminus\{\emptyset\}$ and ψ be a solution.

(i) The *1-reduced game* $(S,m_S,v^{\psi}_{1,S})$ with respect to ψ and S is defined as, for all $\alpha\in M^S$,

$$v^{\psi}_{1,S}(\alpha)=v\left(\alpha,m_{N\setminus S}\right)-\sum_{i\in N\setminus S}\psi_{i,m_i}\left(N,\left(\alpha,m_{N\setminus S}\right),v\right).\tag{3.6}$$

(ii) The *2-reduced game* $(S, m_S, v^{\psi}_{2,S})$ with respect to ψ and S is defined as, for all $\alpha \in M^S$,

$$v^{\psi}_{2,S}(\alpha) = v(\alpha, m_{N\setminus S}) - \sum_{i \in N\setminus S} \sum_{k_i=1}^{m_i} \psi_{i,k_i}(N, (\alpha, m_{N\setminus S}), v). \tag{3.7}$$

(iii) ψ on MC satisfies *1-consistency (1CON)* if for all $(N, m, v) \in MC$, for all $S \subseteq N$ and for all $(i, k_i) \in L^{S,m_S}$, $\psi_{i,k_i}(N, m, v) = \psi_{i,k_i}(S, m_S, v^{\psi}_{1,S})$.

(iv) ψ on MC satisfies *2-consistency (2CON)* if for all $(N, m, v) \in MC$, for all $S \subseteq N$ and for all $(i, k_i) \in L^{S,m_S}$, $\psi_{i,k_i}(N, m, v) = \psi_{i,k_i}(S, m_S, v^{\psi}_{2,S})$.

Remark 3.1. Hwang and Liao [8–10] characterized the solutions Λ, Γ, and Θ by means of 1CON and 2CON as follows.

(i) The solution Λ is the only solution satisfying 1WEFF (1EFF), WIIE (IIE), 1UBC (1SBC), and 1CON.

(ii) The solution Θ is the only solution satisfying 2WEFF (2EFF), WEL (EL), 2UBC (2SBC), and 2CON.

(iii) The solution Γ is the only solution satisfying 2WEFF (2EFF), WIIE (IIE), 2UBC (3SBC), and 2CON.

Next, we will find dynamic processes that lead the players to solutions, starting from arbitrary efficient payoff vectors.

Let $(N, m, v) \in MC$. A *payoff vector* of (N, m, v) is a vector $(x_{i,k_i})_{(i,k_i)\in L^{N,m}} \in \mathbb{R}^{L^{N,m}}$ where x_{i,k_i} denotes the payoff to player i corresponding to his activity level k_i for all $(i, k_i) \in L^{N,m}$. A payoff vector x of (N, m, v) is *1-efficient (1EFF)* if $\sum_{i\in N} x_{i,m_i} = v(m)$. x is *2-efficient (2EFF)* if $\sum_{i\in N} \sum_{k_i \in M_i^+} x_{i,k_i} = v(m)$. Moreover, the sets of *1-preimputations* and *2-preimputations* of (N, m, v) are denoted by

$$\begin{aligned} X^1(N, m, v) &= \left\{ x \in \mathbb{R}^{L^{N,m}} \mid x \text{ is 1EFF in } (N, m, v) \right\}, \\ X^2(N, m, v) &= \left\{ x \in \mathbb{R}^{L^{N,m}} \mid x \text{ is 2EFF in } (N, m, v) \right\}. \end{aligned} \tag{3.8}$$

In order to exhibit such processes, let us define two alternative reduced games as follows. Let $(N, m, v) \in MC$, $S \subseteq N$, and let ψ be a solution and x a payoff vector.

(i) The *$(1, \psi)$-reduced game* $(S, m_S, v^{x,\psi}_{1,S})$ with respect to S, x, and ψ is defined as, for all $\alpha \in M^S$,

$$v^{x,\psi}_{1,S}(\alpha) = \begin{cases} v(m) - \displaystyle\sum_{i\in N\setminus S} x_{i,m_i}, & \alpha = m_S, \\ v^{\psi}_{1,S}(\alpha), & \text{otherwise}. \end{cases} \tag{3.9}$$

(ii) The $(2,\psi)$-*reduced game* $(S, m_S, v_{2,S}^{x,\psi})$ with respect to S, x, and ψ is defined as, for all $\alpha \in M^S$,

$$v_{2,S}^{x,\psi}(\alpha) = \begin{cases} v(m) - \sum_{i \in N \setminus S} \sum_{k_i \in M^+} x_{i,k_i}, & \alpha = m_S, \\ v_{2,S}^{\psi}(\alpha), & \text{otherwise}. \end{cases} \tag{3.10}$$

Let $(N,m,v) \in MC$, $N \geq 3$ and $(i,k_i) \in L^{N,m}$. Inspired by Maschler and Owen [7], we define $f_{i,k_i} : X^1(N,m,v) \to \mathbb{R}$, $g_{i,k_i} : X^2(N,m,v) \to \mathbb{R}$, $h_{i,k_i} : X^2(N,m,v) \to \mathbb{R}$ to be as follows:

(i) $f_{i,k_i}(x) = x_{i,k_i} + t \cdot \sum_{j \in N \setminus \{i\}} (\Lambda_{i,k_i}(\{i,j\}, m_{\{i,j\}}, v_{1,\{i,j\}}^{x,\Lambda}) - x_{i,k_i})$,

(ii) $g_{i,k_i}(x) = x_{i,k_i} + t \cdot \sum_{j \in N \setminus \{i\}} (\Theta_{i,k_i}(\{i,j\}, m_{\{i,j\}}, v_{2,\{i,j\}}^{x,\Theta}) - x_{i,k_i})$,

(iii) $h_{i,k_i}(x) = x_{i,k_i} + t \cdot \sum_{j \in N \setminus \{i\}} (\Gamma_{i,k_i}(\{i,j\}, m_{\{i,j\}}, v_{2,\{i,j\}}^{x,\Gamma}) - x_{i,k_i})$,

where t is a fixed positive number, which reflects the assumption that player i does not ask for adequate correction (when $t = 1$) but only (usually) a fraction of it. It is easy to check that $(f_{i,k_i}(x))_{(i,k_i) \in L^{N,m}} \in X^1(N,m,v)$ if $x \in X^1(N,m,v)$, $(g_{i,k_i}(x))_{(i,k_i) \in L^{N,m}} \in X^2(N,m,v)$, and $(h_{i,k_i}(x))_{(i,k_i) \in L^{N,m}} \in X^2(N,m,v)$ if $x \in X^2(N,m,v)$.

Inspired by Maschler and Owen [7], we define correction functions $f_{i,k_i}, g_{i,k_i}, h_{i,k_i}$ on multichoice games. In the following, we provided some discussions which are analogues to the discussion of Maschler and Owen [7]. Let $(N,m,v) \in MC$ and x be a 1-efficient payoff vector. By a process of induction we assume that the players have already agreed on the solution Λ for all p-person games, $1 < p < |N|$. In particular, we assume that they agreed on Λ for 1-person games (involving only Pareto optimality) and for 2-person games (which are side-payment games after an appropriate change in the utility scale of one player). Now somebody suggests that x should be the solution for an n-person game (N, m, v), thus suggesting a solution concept Ψ, which should satisfy

$$\Psi(P, m', u) = \begin{cases} \Lambda(P, m', u), & (P, m', u) \in MC, |P| < |N|, \\ x, & (N, m, v) = (P, m', u). \end{cases} \tag{3.11}$$

On the basis of this Ψ, the members of a coalition $S = \{i,j\}$ will examine $v_{1,S}^{x,\Lambda}$ for related 1-consistency. If the solution turns out to be inconsistent, they will modify x "in the direction" which is dictated by $\Lambda_{i,k_i}(S, m_S, v_{1,S}^{x,\Lambda})$ in a manner which will be explained subsequently (see the definition of f_{i,k_i}). These modifications, done simultaneously by all 2-person coalitions, will lead to a new payoff vector x^* and the process will repeat. The hope is that it will converge and, moreover, converge to $\Lambda(N,m,v)$. Similar discussions could be used to g_{i,k_i} and h_{i,k_i}.

Theorem 3.2. *Let* $(N,m,v) \in MC$ *and* $x \in X^1(N,m,v)$. *Define* $x^0 = x$, $x^1 = (f_{i,k_i}(x^0))_{(i,k_i) \in L^{N,m}}$, $\ldots, x^q = (f_{i,k_i}(x^{q-1}))_{(i,k_i) \in L^{N,m}}$ *for all* $q \in \mathbb{N}$.

(1) *If* $0 < t < 4/|N|$, *then for all* $i \in N$ *and for all* $x \in X^1(N,m,v)$, $\{x^q_{i,m_i}\}_{q=1}^{\infty}$ *converges to* $\Lambda_{i,m_i}(N,m,v)$.

(2) *If* $0 < t < 4/|N|$, *then for all* $(i,k_i) \in L^{N,m}$ *and for all* $x \in X^1(N,(m_{N\setminus\{i\}},k_i),v)$, $\{x^q_{i,k_i}\}_{q=1}^{\infty}$ *converges to* $\Lambda_{i,k_i}(N,m,v)$.

Proof. Fix $(N,m,v) \in MC$ and $x \in X^1(N,m,v)$. To prove (1), let $i,j \in S(m)$ and $S = \{i,j\}$. By 1EFF and 1UBC of Λ, and definitions of $v^{\Lambda}_{1,S}$ and $v^{x,\Lambda}_{1,S}$,

$$\Lambda_{i,m_i}\left(S,m_S,v^{x,\Lambda}_{1,S}\right) + \Lambda_{j,m_j}\left(S,m_S,v^{x,\Lambda}_{1,S}\right) = x_{i,m_i} + x_{j,m_j}, \quad (\text{by 1EFF of } \Lambda), \tag{3.12}$$

$$\begin{aligned}
\Lambda_{i,m_i}&\left(S,m_S,v^{x,\Lambda}_{1,S}\right) - \Lambda_{j,m_j}\left(S,m_S,v^{x,\Lambda}_{1,S}\right)\\
&= \Lambda_{i,m_i}\left(S,\left(m_{S\setminus\{j\}},0\right),v^{x,\Lambda}_{1,S}\right) - \Lambda_{j,m_j}\left(S,\left(m_{S\setminus\{i\}},0\right),v^{x,\Lambda}_{1,S}\right) \quad (\text{by 1UBC of } \Lambda)\\
&= \Lambda_{i,m_i}\left(S,\left(m_{S\setminus\{j\}},0\right),v^{\Lambda}_{1,S}\right) - \Lambda_{j,m_j}\left(S,\left(m_{S\setminus\{i\}},0\right),v^{\Lambda}_{1,S}\right) \quad \left(\text{by definition of } v^{\Lambda}_{1,S}\right)\\
&= \Lambda_{i,m_i}\left(S,m_S,v^{\Lambda}_{1,S}\right) - \Lambda_{j,m_j}\left(S,m_S,v^{\Lambda}_{1,S}\right). \quad (\text{by 1UBC of } \Lambda).
\end{aligned} \tag{3.13}$$

Therefore,

$$2\cdot\left[\Lambda_{i,m_i}\left(S,m_S,v^{x,\Lambda}_{1,S}\right) - x_{i,m_i}\right] = \Lambda_{i,m_i}\left(S,m_S,v^{\Lambda}_{1,S}\right) - \Lambda_{j,m_j}\left(S,m_S,v^{\Lambda}_{1,S}\right) - x_{i,m_i} + x_{j,m_j}. \tag{3.14}$$

By definition of f, 1CON and 1EFF of Λ and (3.14),

$$\begin{aligned}
f_{i,m_i}(x) = x_{i,m_i} + \frac{t}{2}\cdot&\left[\sum_{j\in N\setminus\{i\}}\Lambda_{i,m_i}\left(\{i,j\},m_{\{i,j\}},v^{\Lambda}_{1,\{i,j\}}\right) - \sum_{j\in N\setminus\{i\}}x_{i,m_i}\right.\\
&\left.-\sum_{j\in N\setminus\{i\}}\Lambda_{j,m_j}\left(\{i,j\},m_{\{i,j\}},v^{\Lambda}_{1,\{i,j\}}\right) + \sum_{j\in N\setminus\{i\}}x_{j,m_j}\right]\\
&\qquad\left(\text{by definition of } f_{i,m_i} \text{ and equation (3.14)}\right)\\
= x_{i,m_i} + \frac{t}{2}\cdot&\left[\sum_{k\in N\setminus\{i\}}\Lambda_{i,m_i}(N,m,v) - (|N|-1)x_{i,m_i}\right.\\
&\left.-\sum_{j\in N\setminus\{i\}}\Lambda_{j,m_j}(N,m,v) + \left(v(m) - x_{i,m_i}\right)\right]\\
&\qquad(\text{by 1CON of } \Lambda \text{ and 1EFF of } x)
\end{aligned}$$

$$\begin{aligned}
&= x_{i,m_i} + \frac{t}{2} \cdot [(|N|-1)\Lambda_{i,m_i}(N,m,v) - (|N|-1)x_{i,m_i} \\
&\qquad -(v(m) - \Lambda_{i,m_i}(N,m,v)) + (v(m) - x_{i,m_i})] \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{(by 1EFF of } \Lambda) \\
&= x_{i,m_i} + \frac{|N| \cdot t}{2} \cdot [\Lambda_{i,m_i}(N,m,v) - x_{i,m_i}].
\end{aligned} \tag{3.15}$$

Hence, for all $q \in \mathbb{N}$,

$$\begin{aligned}
\left(1 - \frac{|N| \cdot t}{2}\right)^{q+1} \left[\Lambda_{i,m_i}(N,m,v) - x^q_{i,m_i}\right] &= \left[\Lambda_{i,m_i}(N,m,v) - f_{i,m_i}(x^q)\right] \\
&= \left[\Lambda_{i,m_i}(N,m,v) - x^{q+1}_{i,m_i}\right].
\end{aligned} \tag{3.16}$$

If $0 < t < 4/|N|$, then $-1 < (1 - (|N| \cdot t)/2) < 1$ and $\{x^q_{i,m_i}\}_{q=1}^{\infty}$ converges to $\Lambda_{i,m_i}(N,m,v)$.

To prove (2), by 1 of this theorem, if $0 < t < 4/|N|$, then for all $(i,k_i) \in L^{N,m}$ and for all $x \in X^1(N,(m_{N\setminus\{i\}},k_i),v)$, $\{x^q_{i,k_i}\}_{q=1}^{\infty}$ converges to $\Lambda_{i,k_i}(N,(m_{N\setminus\{i\}},k_i),v)$. By IIE of Λ, $\{x^q_{i,k_i}\}_{q=1}^{\infty}$ converges to $\Lambda_{i,k_i}(N,m,v)$. □

Remark 3.3. Huang et al. [15] provided dynamic processes for the P&Z Shapley value as follows. Let $(N,m,v) \in MC$ and $y \in X^2(N,m,\mathrm{v})$. Define $y^0 = y,\ y^1 = (h_{i,k_i}(y^0))_{(i,k_i)\in L^{N,m}}, \ldots, y^q = (h_{i,k_i}(y^{q-1}))_{(i,k_i)\in L^{N,m}}$ for all $q \in \mathbb{N}$.

(1) If $0 < \alpha < 4/|N|$, then for all $i \in N$ and for all $y \in X^2(N,m,v)$, $\{\sum_{k_i=1}^{m_i} y^q_{i,k_i}\}_{q=1}^{\infty}$ converges to $\sum_{k_i=1}^{m_i} \Gamma_{i,k_i}(N,m,v)$.

(2) If $0 < t < 4/|N|$, then for all $(i,k_i) \in L^{N,m}$ and for all $y \in X^2(N,(m_{N\setminus\{i\}},k_i),v)$, $\{\sum_{t_i=1}^{k_i} y^q_{i,t_i}\}_{q=1}^{\infty}$ converges to $\sum_{t_i=1}^{k_i} \Gamma_{i,t_i}(N,m,v)$.

(3) If $0 < \alpha < 4/|N|$, then for all $(i,k_i) \in L^{N,m}$ and for all payoff vectors y with $y_{i,k_i} = v(m_{N\setminus\{i\}},k_i) - v(m_{N\setminus\{i\}},k_i-1)$, $\{y^q_{i,k_i}\}_{q=1}^{\infty}$ converges to $\Gamma_{i,k_i}(N,m,v)$.

In fact, the proofs of (1), (2), and (3) are similar to Theorem 3.2.

Theorem 3.4. *Let $(N,m,v) \in MC$ and $z \in X^2(N,m,v)$. Define $z^0 = z,\ z^1 = (g_{i,k_i}(z^0))_{(i,k_i)\in L^{N,m}}, \ldots, z^q = (g_{i,k_i}(z^{q-1}))_{(i,k_i)\in L^{N,m}}$ for all $q \in \mathbb{N}$.*

(1) *If $0 < \alpha < 4/|N|$, then for all $i \in N$ and for all $z \in X^2(N,m,v)$, $\{\sum_{k_i=1}^{m_i} z^q_{i,k_i}\}_{q=1}^{\infty}$ converges to $\sum_{k_i=1}^{m_i} \Theta_{i,k_i}(N,m,v)$.*

(2) *If $0 < t < 4/|N|$, then for all $(i,k_i) \in L^{N,m}$ and for all $z \in X^2(N,(m_{N\setminus\{i\}},k_i),v)$, $\{\sum_{t_i=1}^{k_i} z^q_{i,t_i}\}_{q=1}^{\infty}$ converges to $\sum_{t_i=1}^{k_i} \Theta_{i,t_i}(N,m,v)$.*

(3) *If $0 < \alpha < 4/|N|$, then for all $(i,k_i) \in L^{N,m}$ and for all payoff vectors z with $z_{i,k_i} = v(m_{N\setminus\{i\}},k_i) - v(m_{N\setminus\{i\}},k_i-1)$, $\{z^q_{i,k_i}\}_{q=1}^{\infty}$ converges to $\Theta_{i,k_i}(N,m,v)$.*

Proof. "EL" instead of "IIE", the proofs of this theorem are immediate analogues Theorem 3.2 and Remark 3.3, hence we omit them. □

4. Player-Action Reduction and Related Consistency

By reducing the number of the players, Hwang and Liao [8–10] proposed 1-reduction and 2-reduction on multichoice games. Here we define two types of player-action reduced games by reducing both the number of the players and the activity levels. Let $(N,m,v) \in MC$, $S \subseteq N \setminus \{\emptyset\}$, ψ be a solution and $\gamma \in M_+^{N\setminus S}$.

(i) The *1-player-action reduced game* $(S, m_S, v_{S,\gamma}^{1,\psi})$ with respect to S, γ and ψ is defined as for all $\alpha \in M^S$,

$$v_{S,\gamma}^{1,\psi}(\alpha) = v(\alpha,\gamma) - \sum_{j\in N\setminus S} \psi_{j,\gamma_j}(N,(\alpha,\gamma),v). \tag{4.1}$$

(ii) The *2-player-action reduced game* $(S, m_S, v_{S,\gamma}^{2,\psi})$ with respect to S, γ and ψ is defined as for all $\alpha \in M^S$,

$$v_{S,\gamma}^{2,\psi}(\alpha) = v(\alpha,\gamma) - \sum_{j\in N\setminus S} \sum_{k_j=1}^{\gamma_j} \psi_{j,k_j}(N,(\alpha,\gamma),v). \tag{4.2}$$

The player-action reduced games are based on the idea that, when renegotiating the payoff distribution within S, the condition $\gamma \in M_+^{N\setminus S}$ means that the members of $N \setminus S$ continue to cooperate with the members of S. All members in $N \setminus S$ take nonzero levels based on the participation vector γ to cooperate. Then in the player-action reduced games, the coalition S with activity level α cooperates with all the members of $N \setminus S$ with activity level γ.

Definition 4.1. Let ψ be a solution on MC.

(i) ψ satisfies *1-player-action consistency (1PACON)* if for all $(N,m,v) \in MC$, for all $S \subseteq N \setminus \{\emptyset\}$, for all $(i,k_i) \in L^{S,m_S}$ and for all $\gamma \in M_+^{N\setminus S}$, $\psi_{i,k_i}(N,(m_S,\gamma),v) = \psi_{i,k_i}(S,m_S,v_{S,\gamma}^{1,\psi})$.

(ii) ψ satisfies *2-player-action consistency (2PACON)* if for all $(N,m,v) \in MC$, for all $S \subseteq N \setminus \{\emptyset\}$, for all $(i,k_i) \in L^{S,m_S}$ and for all $\gamma \in M_+^{N\setminus S}$, $\psi_{i,k_i}(N,(m_S,\gamma),v) = \psi_{i,k_i}(S,m_S,v_{S,\gamma}^{2,\psi})$.

Remark 4.2. Let $(N,m,v) \in MC$, $S \subseteq N \setminus \{\emptyset\}$ and ψ be a solution. Let $\gamma = m_{N\setminus S}$, by definitions of reduced games and player-action reduced games, $v_{S,m_{N\setminus S}}^{1,\psi} = v_{1,S}^{\psi}$ and $v_{S,m_{N\setminus S}}^{2,\psi} = v_{2,S}^{\psi}$. Clearly, if a solution satisfies 1PACON, then it also satisfies 1CON. Similarly, if a solution satisfies 2PACON, then it also satisfies 2CON.

As we knew, each $(N,m,v) \in MC$ can be expressed as a linear combination of minimal effort games and this decomposition exists uniquely. The following lemmas point out the relations of coefficients of expressions among (N,m,v), $(S,m_S,v_{S,\gamma}^{1,\Lambda})$, $(S,m_S,v_{S,\gamma}^{2,\Gamma})$, and $(S,m_S,v_{S,\gamma}^{2,\Theta})$.

Lemma 4.3. *Let $(N,m,v)\in MC$, $(S,m_S,v^{1,\Lambda}_{S,\gamma})$ be a 1-player-action reduced game of v with respect to S, γ, and the solution Λ and let $(S,m_S,v^{2,\Gamma}_{S,\gamma})$ be a 2-player-action reduced game of v with respect to S, γ and the solution Γ. If $v=\sum_{\alpha\in M^N\setminus\{0_N\}} a^{\alpha}(v)\cdot u^{\alpha}_N$, then $v^{1,\Lambda}_{S,\gamma}$ can be expressed as $v^{1,\Lambda}_{S,\gamma}=v^{2,\Gamma}_{S,\gamma}=\sum_{\alpha\in M^S\setminus\{0_S\}} a^{\alpha}(v^{2,\Gamma}_{S,\gamma})\cdot u^{\alpha}_S$, where for all $\alpha\in M^S$,*

$$a^{\alpha}\left(v^{1,\Lambda}_{S,\gamma}\right)=a^{\alpha}\left(v^{2,\Gamma}_{S,\gamma}\right)=\sum_{\lambda\leq\gamma}\frac{|S(\alpha)|}{|S(\alpha)|+|S(\lambda)|}\cdot a^{(\alpha,\lambda)}(v). \tag{4.3}$$

Proof. Let $(N,m,v)\in MC$, $S\subseteq N$ with $S\neq\emptyset$ and $\gamma\in M^{N\setminus S}_{+}$. For all $\alpha\in M^S$,

$$v^{2,\Gamma}_{S,\gamma}(\alpha)=v(\alpha,\gamma)-\sum_{j\in N\setminus S}\sum_{k_j=1}^{\gamma_j}\Gamma_{j,k_j}(N,(\alpha,\gamma),v). \tag{4.4}$$

By 2EFF of Γ, $v^{2,\Gamma}_{S,\gamma}(0_S)=0$. For all $\alpha\in M^S$ with $\alpha\neq 0_S$,

$$\begin{aligned}
(4.2)&=\sum_{j\in S(\alpha)}\sum_{k_j=1}^{\alpha_j}\Gamma_{j,k_j}(N,(\alpha,\gamma),v)\\
&=\sum_{j\in S(\alpha)}\sum_{k_j=1}^{\alpha_j}\sum_{\substack{\beta\leq(\alpha,\gamma)\\ \beta_j=k_j}}\frac{a^{\beta}(v)}{|S(\beta)|}\\
&=\sum_{j\in S(\alpha)}\left[\sum_{\substack{\beta\leq(\alpha,\gamma)\\ \beta_j=1}}\frac{a^{\beta}(v)}{|S(\beta)|}+\cdots+\sum_{\substack{\beta\leq(\alpha,\gamma)\\ \beta_j=\alpha_j}}\frac{a^{\beta}(v)}{|S(\beta)|}\right]\\
&=\sum_{j\in S(\alpha)}\left[\sum_{\substack{\eta\leq\alpha\\ \eta_j=1}}\sum_{\lambda\leq\gamma}\frac{a^{(\eta,\lambda)}(v)}{|S(\eta)|+|S(\lambda)|}+\cdots+\sum_{\substack{\eta\leq\alpha\\ \eta_j=\alpha_j}}\sum_{\lambda\leq\gamma}\frac{a^{(\eta,\lambda)}(v)}{|S(\eta)|+|S(\lambda)|}\right]\\
&=\sum_{\eta\leq\alpha}\sum_{\lambda\leq\gamma}\frac{|S(\eta)|}{|S(\eta)|+|S(\lambda)|}\cdot a^{(\eta,\lambda)}(v).
\end{aligned} \tag{4.5}$$

Set

$$a^{\eta}\left(v^{2,\Gamma}_{S,\gamma}\right)=\sum_{\lambda\leq\gamma}\frac{|S(\eta)|}{|S(\eta)|+|S(\lambda)|}\cdot a^{(\eta,\lambda)}(v). \tag{4.6}$$

By (4.5), for all $\alpha \in M^S$,

$$v_{S,\gamma}^{2,\Gamma}(\alpha) = \sum_{\eta\leq\alpha}\sum_{\lambda\leq\gamma}\frac{|S(\eta)|}{|S(\eta)|+|S(\lambda)|}\cdot a^{(\eta,\lambda)}(v) = \sum_{\eta\leq\alpha}a^{\eta}\left(v_{S,\gamma}^{2,\Gamma}\right). \tag{4.7}$$

Hence $v_{S,\gamma}^{2,\Gamma}$ can be expressed to be $v_{S,\gamma}^{2,\Gamma} = \sum_{\alpha\in M^S\setminus\{0_S\}} a^{\alpha}(v_{S,\gamma}^{1,\Gamma})\cdot u_S^{\alpha}$. By Definition 2.1 and the definitions of $v_{S,\gamma}^{1,\Lambda}$ and $v_{S,\gamma}^{2,\Gamma}$, for all $\alpha \in M^N$,

$$\begin{aligned} v_{S,\gamma}^{2,\Gamma}(\alpha) &= v(\alpha,\gamma) - \sum_{j\in N\setminus S}\sum_{k_j=1}^{\gamma_j}\Gamma_{j,k_j}(N,(\alpha,\gamma),v) \\ &= v(\alpha,\gamma) - \sum_{j\in N\setminus S}\Lambda_{j,\gamma_j}(N,(\alpha,\gamma),v) \\ &= v_{S,\gamma}^{1,\Lambda}(\alpha). \end{aligned} \tag{4.8}$$

Hence, for each $\alpha \in M^S$, $v_{S,\gamma}^{1,\Lambda}(\alpha) = v_{S,\gamma}^{2,\Gamma}(\alpha)$ and $a^{\alpha}(v_{S,\gamma}^{1,\Lambda}) = a^{\alpha}(v_{S,\gamma}^{2,\Gamma})$. □

Lemma 4.4. *Let $(N,m,v) \in MC$ and $(S,m_S,v_{S,\gamma}^{2,\Theta})$ be a 2-player-action reduced game of v with respect to S, γ, and the solution Θ. If $v = \sum_{\alpha\in M^N\setminus\{0_N\}} a^{\alpha}(v)\cdot u_N^{\alpha}$, then $v_{S,\gamma}^{2,\Theta}$ can be expressed to be $v_{S,\gamma}^{2,\Theta} = \sum_{\alpha\in M^S\setminus\{0_S\}} a^{\alpha}(v_{S,\gamma}^{2,\Theta})\cdot u_S^{\alpha}$, where for all $\alpha \in M^S$,*

$$a^{\alpha}\left(v_{S,\gamma}^{2,\Theta}\right) = \sum_{\lambda\leq\gamma}\frac{\|\alpha\|}{\|\alpha\|+\|\lambda\|}\cdot a^{(\alpha,\lambda)}(v). \tag{4.9}$$

Proof. The proof is similar to Lemma 4.3; hence, we omit it. □

By applying Lemmas 4.3 and 4.4, we show that the three extended Shapley values satisfy related properties of player-action consistency.

Proposition 4.5. *The solution Λ satisfies 1PACON. The solutions Γ and Θ satisfy 2PACON.*

Proof. Let $(N,m,v) \in MC$, $S \subseteq N$ with $S \neq \emptyset$ and $\gamma \in M_+^{N\setminus S}$. First, we show that the solution Λ satisfies 1PACON. By Definition 2.1 and Lemma 4.3, for all $(i,k_i) \in L^{S,m_S}$,

$$\begin{aligned} \Lambda_{i,k_i}\left(S,m_S,v_{S,\gamma}^{1,\Lambda}\right) &= \sum_{\substack{\alpha\in M^S\\ \alpha_i\leq k_i}}\frac{a^{\alpha}\left(v_{S,\gamma}^{1,\Lambda}\right)}{|S(\alpha)|} \\ &= \sum_{\substack{\alpha\in M^S\\ \alpha_i\leq k_i}}\frac{1}{|S(\alpha)|}\cdot\sum_{\lambda\leq\gamma}\frac{|S(\alpha)|}{|S(\alpha)|+|S(\lambda)|}\cdot a^{(\alpha,\lambda)}(v) \end{aligned}$$

$$= \sum_{\substack{\beta \leq (m_S,\gamma) \\ \beta_i \leq k_i}} \frac{a^{\beta}(v)}{|S(\beta)|}$$

$$= \Lambda_{i,k_i}(N, (m_S, \gamma), v). \tag{4.10}$$

Hence, the solution Λ satisfies 1PACON. Similarly, by Definition 2.1, Lemmas 4.3, 4.4 and previous proof, we can show that the solutions Θ and Γ satisfy 2PACON. □

Hwang and Liao [8–10] characterized the three extended Shapley values by means of 1CON and 2CON. By Proposition 4.5 and Remarks 3.1 and 4.2, it is easy to check that 1CON and 2CON could be replaced by 1PACON and 2PACON in axiomatizations proposed by Hwang and Liao [8–10].

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