Hindawi Publishing Corporation
Advances in Operations Research
Volume 2012, Article ID 768929, 16 pages
doi:10.1155/2012/768929

Research Article

Optimal Policies for a Finite-Horizon Production Inventory Model

Lakdere Benkherouf and Dalal Boushehri

Department of Statistics and Operations Research, College of Science, Kuwait University, P.O. Box 5969, Safat 13060, Kuwait

Correspondence should be addressed to Lakdere Benkherouf, lakdereb@yahoo.com

Received 26 October 2011; Revised 29 February 2012; Accepted 17 March 2012

Academic Editor: Imed Kacem

This paper is concerned with the problem of finding the optimal production schedule for an inventory model with time-varying demand and deteriorating items over a finite planning horizon. This problem is formulated as a mixed-integer nonlinear program with one integer variable. The optimal schedule is shown to exist uniquely under some technical conditions. It is also shown that the objective function of the nonlinear obtained from fixing the integrality constraint is convex as a function of the integer variable. This in turn leads to a simple procedure for finding the optimal production plan.

1. Introduction

This paper is concerned with the optimality of a production schedule for a single-item inventory model with deteriorating items and for a finite planning horizon. The motivation for considering inventory models with time-varying demand and deteriorating items is well documented in the literature. Readers may consult Teng et al. [1], Goyal and Giri [2], and Sana et al. [3] and the references therein.

Earlier models on finding optimal replenishment schedule for a finite planning horizon may be categorized as economic lot size (ELS) models dealing with replenishment only. The model treated in this paper is an extension of the economic production lot size (EPLS) to finite horizon models and time-varying demand. The model is close in spirit to that of [1]. However, in [1], the possibility that products may experience deterioration while in stock was not considered. Deterioration was considered in [3] with the possibility of shortages. Nevertheless, their proposed (EPLS) schedule is not optimal.

Recently, Benkherouf and Gilding [4] suggested a general procedure for finding the optimal inventory policy for finite horizon models. The procedure is based on earlier work by Donaldson [5], Henery [6], and Benkherouf and Mahmoud [7]. This procedure was motivated by applications to (ELS) models. Nevertheless, it turned out that the applicability of the procedure goes beyond its original scope. The procedure has already been successful in finding the optimal inventory policy for an integrated single-vendor single buyer with time-varying demand rate: see Benkherouf and Omar [8]. The current paper presents another extension of the procedure to (EPLS) models. In our treatment, we have opted for a route of simplicity. In that, we selected a model with no shortages and where costs are fixed throughout the planning horizon. Various extensions of the model are discussed in Section 5.

The details of the model of the paper along with the statement of the problem to be discussed are presented in the next section. Section 3 contains some preliminaries on the procedure of Benkherouf and Gilding [4]. The main results are contained in Section 4. Section 5 is concerned with some general remarks and conclusion.

2. Mathematical Model

The model treated in this paper is based on the following assumptions:

(1) the planning horizon is finite;

(2) a single item is considered;

(3) products are assumed to experience deterioration while in stock;

(4) shortages are not permitted;

(5) initial inventory at the beginning of planning horizon is zero, also the inventory depletes to zero at the end of the planning horizon;

(6) the demand function is strictly positive;

We will initially look at a single period i (production cycle), say, starting at time t_{i-1} and ending at time t_i, $i = 1, 2, \ldots$. Some of the notations used in the model are as follows:

H: the total planning horizon,

p: the constant production rate,

$D(t)$: the demand rate at time t, $0 < D(t) < p$,

α: constant deteriorating rate of inventory items with $\alpha > 0$,

t_i^p: the time at which the inventory level reaches it is maximum in the ith production cycle,

K: set up cost for the inventory model,

c_1: the cost of one unit of the item with $c_1 > 0$,

c_h: carrying cost per inventory unit held in the model per unit time,

TC: total system cost during H.

Figure 1 shows the changes of the level of stock for a typical production period.

Let $I(t)$ be the level of stock at time t. The change in period i, the level of inventory, may be described by the following differential equation i, $i = 1, 2, \ldots$,

$$I'(t) = p - D(t) - \alpha I(t), \quad t_{i-1} \leq t < t_i^p, \quad I(t) \downarrow 0 \quad \text{as } t \downarrow t_{i-1}. \tag{2.1}$$

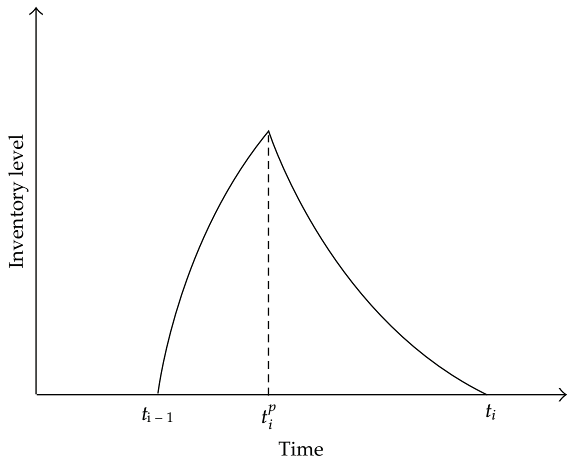

Figure 1: The changes of inventory levels of various components of the model for a typical production batch.

The solution to (2.1) is given by

$$I(t) = e^{-\alpha t} \int_{t_{i-1}}^{t} e^{\alpha u} \{p - D(u)\} du, \quad t_{i-1} \le t < t_i^p, \tag{2.2}$$

$$I'(t) = -D(t) - \alpha I(t), \quad t_i^p \le t < t_i, \quad I(t) \uparrow 0 \quad \text{as } t \uparrow t_i. \tag{2.3}$$

The solution to (2.3) is given by

$$I(t) = e^{-\alpha t} \int_{t}^{t_i} e^{\alpha u} D(u) du, \quad t_i^p \le t < t_i. \tag{2.4}$$

The total costs, (excluding the setup cost) for period i, which consist of holding cost and deterioration cost are given by

$$\begin{aligned} &c_h \int_{t_{i-1}}^{t_i^p} e^{-\alpha t} \left[\int_{t_{i-1}}^{t} e^{\alpha u} \{p - D(u)\} du \right] dt \\ &\quad + c_h \int_{t_i^p}^{t_i} e^{-\alpha t} \left\{ \int_{t}^{t_i} e^{\alpha u} D(u) du \right\} dt, \\ &\quad + c_1 \left\{ \int_{t_{i-1}}^{t_i^p} p du - \int_{t_{i-1}}^{t_i} D(u) du \right\}. \end{aligned} \tag{2.5}$$

We will call models with this cost OHD models. It is possible to consider instead of (2.5) the form

$$c_h \int_{t_{i-1}}^{t_i^p} e^{-\alpha t} \left[\int_{t_{i-1}}^{t} e^{\alpha u} \{p - D(u)\} du \right] dt + c_h \int_{t_i^p}^{t_i} e^{-\alpha t} \left\{ \int_{t}^{t_i} e^{\alpha u} D(u) du \right\} dt + c_1 \int_{t_{i-1}}^{t_i^p} p du, \tag{2.6}$$

which considers only holding and purchasing costs, where the expression $c_1 \int_{t_{i-1}}^{t_i^p} p du$ represents the purchasing cost. We call this OHP models.

Note that since the function I is continuous at t_i^p, we have

$$e^{-\alpha t_i^p} \int_{t_{i-1}}^{t_i^p} e^{\alpha u} \{p - D(u)\} du = e^{-\alpha t_i^p} \int_{t_i^p}^{t_i} e^{\alpha u} D(u) du, \tag{2.7}$$

or

$$\int_{t_{i-1}}^{t_i^p} e^{\alpha u} \{p - D(u)\} du = \int_{t_i^p}^{t_i} e^{\alpha u} D(u) du, \tag{2.8}$$

then

$$p \int_{t_{i-1}}^{t_i^p} e^{\alpha u} du = \int_{t_{i-1}}^{t_i} e^{\alpha u} D(u) du, \tag{2.9}$$

or

$$\frac{p}{\alpha} \left(e^{\alpha t_i^p} - e^{\alpha t_{i-1}} \right) = \frac{p}{\alpha} e^{\alpha t_{i-1}} \left\{ e^{\alpha (t_i^p - t_{i-1})} - 1 \right\} = \int_{t_{i-1}}^{t_i} e^{\alpha u} D(u) du, \tag{2.10}$$

$$e^{\alpha (t_i^p - t_{i-1})} - 1 = \frac{\alpha}{p} e^{-\alpha t_{i-1}} \int_{t_{i-1}}^{t_i} e^{\alpha u} D(u) du. \tag{2.11}$$

Therefore,

$$t_i^p - t_{i-1} = \frac{1}{\alpha} \log \left\{ 1 + \frac{\alpha}{p} e^{-\alpha t_{i-1}} \int_{t_{i-1}}^{t_i} e^{\alpha u} D(u) du \right\}. \tag{2.12}$$

Lemma 2.1. *The expression of the cost in* (2.5) *is equal to*

$$(c_h + \alpha c_1)\left[\frac{p}{\alpha^2}\log\left\{1 + \frac{\alpha}{p}e^{-\alpha t_{i-1}}\int_{t_{i-1}}^{t_i} e^{\alpha u}D(u)du\right\} - \frac{1}{\alpha}\int_{t_{i-1}}^{t_i} D(u)du\right]. \tag{2.13}$$

Proof. Applying integration by parts, we get that (2.5) reduces to

$$\begin{aligned}
&\frac{c_h}{\alpha}\int_{t_{i-1}}^{t_i^p}\left\{1 - e^{\alpha(u-t_i^p)}\right\}\{p - D(u)\}du\\
&\quad + \frac{c_h}{\alpha}\int_{t_i^p}^{t_i}\left\{e^{\alpha(u-t_i^p)} - 1\right\}D(u)du\\
&\quad + c_1\int_{t_{i-1}}^{t_i^p} pdu - c_1\int_{t_{i-1}}^{t_i} D(u)du,
\end{aligned} \tag{2.14}$$

or

$$\begin{aligned}
&\frac{c_h}{\alpha}\left\{p\int_{t_{i-1}}^{t_i^p} du - \int_{t_{i-1}}^{t_i^p} D(u)du - p\int_{t_{i-1}}^{t_i^p} e^{\alpha(u-t_i^p)}du + \int_{t_{i-1}}^{t_i^p} e^{\alpha(u-t_i^p)}D(u)du\right\}\\
&\quad + \frac{c_h}{\alpha}\left\{\int_{t_i^p}^{t_i} e^{\alpha(u-t_i^p)}D(u)du - \int_{t_i^p}^{t_i} D(u)du\right\}\\
&\quad + c_1\int_{t_{i-1}}^{t_i^p} pdu - c_1\int_{t_{i-1}}^{t_i} D(u)du,
\end{aligned} \tag{2.15}$$

$$\begin{aligned}
&= p\left(\frac{c_h}{\alpha} + c_1\right)\left(t_i^p - t_{i-1}\right) - \frac{c_h}{\alpha}e^{-\alpha t_i^p}\int_{t_{i-1}}^{t_i^p} e^{\alpha u}\{p - D(u)\}du\\
&\quad + \frac{c_h}{\alpha}e^{-\alpha t_i^p}\int_{t_i^p}^{t_i} e^{\alpha u}D(u)du - \frac{c_h}{\alpha}\int_{t_{i-1}}^{t_i} D(u)du - c_1\int_{t_{i-1}}^{t_i} D(u)du.
\end{aligned} \tag{2.16}$$

This is equal, using (2.7), to

$$p\left(\frac{c_h}{\alpha} + c_1\right)\left(t_i^p - t_{i-1}\right) - \left(\frac{c_h}{\alpha} + c_1\right)\int_{t_{i-1}}^{t_i} D(u)du. \tag{2.17}$$

Now, the lemma follows from (2.12) and (2.17). □

Note that Lemma 2.1 reduced the dependence of the inventory cost in period i from three variables to two variables. This reduction can be significant for an n-period model. Let

$$R(x,y) := (c_h + \alpha c_1)\left[\frac{p}{\alpha^2}\log\left\{1 + \frac{\alpha}{p}e^{-\alpha x}\int_x^y e^{\alpha t}D(t)dt\right\} - \frac{1}{\alpha}\int_x^y D(t)dt\right]. \tag{2.18}$$

Remark 2.2. Let $\alpha \to 0$, in (2.18), and recall that $\log(1+x)$ may be expanded as

$$x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{1}{5}x^5 + O\left(x^6\right), \tag{2.19}$$

to get that as $\alpha \to 0$, $R(x, y)$ is equivalent to

$$(c_h + \alpha c_1)\left[\int_x^y \left\{\frac{e^{\alpha(t-x)} - 1}{\alpha}\right\} D(t)dt - \frac{1}{2p}\left\{\int_x^y e^{\alpha(t-x)} D(t)dt\right\}^2\right], \tag{2.20}$$

which leads to the expression

$$c_h\left[\int_x^y (t-x)D(t)dt - \frac{1}{2p}\left\{\int_x^y D(t)dt\right\}^2\right]. \tag{2.21}$$

This expression may be found in Hill [9], Omar and Smith [10], and Rau and Ouyang [11]. However, their interest in finding the optimal inventory policy for their model centered around treating special cases for demand rate functions or devising heuristics.

The total inventory costs where n ordered are made may be written as follows:

$$\mathrm{TC}(t_1, \dots, t_n) = nK + \sum_{i=1}^{n} R(t_{i-1}, t_i), \tag{2.22}$$

which is given by (2.18).

The objective now is to find n and $t_1, \dots, t_n$ which minimizes TC subject to $t_0 = 0 < t_1 < \cdots < t_n = H$. The problem becomes a mixed integer programming problem. The approach that we will use to solve it is based on a procedure developed by Benkherouf and Gilding [4]. The next section contains the ingredients of the approach.

3. Technical Preliminaries

This section contains a summary of the work of [4] needed to tackle the problem of this paper. Proofs of the results are omitted. Interested readers may consult [4].

Consider the problem

$$\mathbf{P}: \ \mathrm{TC}(t_1, \dots, t_n; n) = nK + \sum_{i=1}^{n} R_i(t_{i-1}, t_{\mathrm{i}}), \tag{3.1}$$

subject to

$$0 = t_0 < t_1 < \cdots < t_n = H. \tag{3.2}$$

It was shown in [4] that, under some technical conditions, the optimization problem ($\mathbf{P}$) has a unique optimal solution which can be found from solving a system of nonlinear equations derived from the first-order optimality condition. To be precise, let $t_0 = 0$ and $t_n = H$ and ignore the rest of the constraints (3.2).

Write

$$S_n := \sum_{i=1}^{n} R_i(t_{i-1}, t_i). \tag{3.3}$$

Assuming that $R_i's$ are twice differentiable, then, for fixed n, the optimal solution in ($\mathbf{P}$) subject to (3.2) reduces to minimizing S_n.

Use the notation ∇ for the gradient, then setting $\nabla \mathrm{TC}(t_1, \ldots, t_n; n) = 0$ gives

$$(\nabla \mathrm{TC})_i = (\partial R_i)_y (t_{i-1}, t_i) + (\partial R_{i+1})_x (t_i, t_{i+1}) = 0, \quad i = 1, \ldots, n-1. \tag{3.4}$$

Two sets of hypotheses were put forward in [4].

Hypothesis 1. The functions R_i satisfy, for $i = 1, \ldots, n$ and $y > x$,

(1) $R_i(x, y) > 0$,

(2) $R_i(x, x) = 0$,

(3) $(\partial R_i)_x (x, y) < 0 < (\partial R_i)_y (x, y)$,

(4) $(\partial_x \partial_y R)_i (x, y) < 0$.

Hypothesis 2. Define

$$\begin{aligned} \mathcal{L}_x z &= \partial_x^2 z + \partial_x \partial_y z + f(x) \partial_x z, \\ \mathcal{L}_y z &= \partial_y^2 z + \partial_x \partial_y z + f(y) \partial_y z, \end{aligned} \tag{3.5}$$

then there is a continuous function f such that $\mathcal{L}_x R_i \geq 0$, $\mathcal{L}_y R_i \geq 0$ for all $i = 1, \ldots, n$, and $(\partial R_i)_y + (\partial R_{i+1})_x = 0$ on the boundary of the feasible set.

The next theorem shows that under assumptions in Hypotheses 1 and 2, the function S_n has a unique minimum.

Theorem 3.1. *The system* (3.4) *has a unique solution subject to* (3.2)*. Furthermore, this solution is the solution of* (3.1) *subject to* (3.2)*. Recall that a function S_n is convex in n if*

$$S_{n+2} - S_{n+1} \geq S_{n+1} - S_n. \tag{3.6}$$

This is equivalent to

$$\frac{1}{2}(S_n + S_{n+2}) \geq S_{n+1}. \tag{3.7}$$

Theorem 3.2. *If s_n denotes the minimum objective value of* (3.1) *subject to* (3.2) *and $R_i(x, y) = R(x, y)$, then s_n is convex in n.*

Based on the convexity property of s_n, the optimal number of cycles n^* is given by

$$n^* = \min\{n \geq 1 : s_{n+1} - s_n > 0\}. \tag{3.8}$$

Now to solve (3.4) at $i = n-1$,

$$\partial R_{n-1y}(t_{n-2}, t_{n-1}) + (\partial R_n)_x(t_{n-1}, H) = 0. \tag{3.9}$$

Assume that t_{n-1} is known, according to [4], t_{n-2} can be found uniquely as a function of t_{n-1}. Repeating this process for $i = n-2$, down to $i = 1, t_{n-3}, \ldots, t_n$ are a function of t_{n-1}. So, the search for the optimal solution of (3.5) can be conducted using a univariate search method.

4. Optimal Production Plan

This section is concerned with the optimal inventory policy for the production inventory model. The model has been introduced in Section 2. This section will investigate the extent to which the function R given by (2.18) satisfies Hypotheses 1 and 2,

$$R(x,y) = (c_h + \alpha c_1)\left[\frac{p}{\alpha^2}\log\left\{1 + \frac{\alpha}{p}\int_x^y e^{\alpha(t-x)}D(t)dt\right\} - \frac{1}{\alpha}\int_x^y D(t)dt\right]. \tag{4.1}$$

Without loss of generality, we will set $c_h + \alpha c_1$ to 1. As this will have no effect on the solution of the optimization problem where K needs to be replaced by $K/(c_h+\alpha c_1)$, therefore, we set

$$R(x,y) = \frac{p}{\alpha^2}\log\left\{1 + \frac{\alpha}{p}\int_x^y e^{\alpha(t-x)}D(t)dt\right\} - \frac{1}{\alpha}\int_x^y D(\mathrm{t})dt. \tag{4.2}$$

Write

$$G(x,y) = \frac{\alpha}{p}\int_x^y e^{\alpha(t-x)}D(t)dt. \tag{4.3}$$

Direct computations then lead to

$$
\begin{aligned}
\partial_x R &= \frac{p}{\alpha^2}\frac{\partial_x G}{1+G} + \frac{1}{\alpha}D(x),\\
\partial_y R &= \frac{p}{\alpha^2}\frac{\partial_y G}{1+G} - \frac{1}{\alpha}D(y),\\
\partial_x\partial_y R &= \frac{p}{\alpha^2}\frac{\partial_x\partial_y G(1+G) - (\partial_x G)(\partial_y G)}{(1+G)^2},\\
\partial_x^2 R &= \frac{p}{\alpha^2}\frac{\partial_x^2 G(1+G) - (\partial_x G)^2}{(1+G)^2} + \frac{1}{\alpha}D'(x),\\
\partial_y^2 R &= \frac{p}{\alpha^2}\frac{\partial_y^2 G(1+G) - (\partial_x G)^2}{(1+G)^2} - \frac{1}{\alpha}D'(y),\\
\partial_x G &= -\alpha\left\{G + \frac{D(x)}{p}\right\},\\
\partial_y G &= \frac{\alpha}{p}e^{\alpha(y-x)}D(y),\\
\partial_x\partial_y G &= -\alpha\partial_y G,\\
\partial_x^2 G &= -\alpha\left\{-\alpha G - \alpha\frac{D(x)}{p} + \frac{D'(x)}{p}\right\},\\
\partial_y^2 G &= \alpha\partial_y G + \frac{\alpha}{p}e^{\alpha(y-x)}D'(y).
\end{aligned}
\tag{4.4}
$$

The following result indicates that R obtained in (4.2) satisfies Hypothesis 1.

Lemma 4.1. *The function R satisfies Hypothesis 1.*

Proof. It is clear that for any $x \in [0, H]$,

$$R(x, x) = 0. \tag{4.5}$$

Now, direct computations show that

$$\partial_x R(x, y) = \frac{G(x, y)}{\alpha\{1 + G(x, y)\}}\{-p + D(x)\}. \tag{4.6}$$

But $p > D(x)$, therefore $\partial_x R(x, y) < 0$ since $G(x, y) > 0$ for $y > x$. Also, it can be shown that

$$\partial_y R(x, y) = \frac{D(y)}{\alpha}\left\{\frac{e^{\alpha(y-x)}}{1 + G(x, y)} - 1\right\}. \tag{4.7}$$

We claim that $\partial_y R(x, y) > 0$. Indeed, the claim is equivalent to

$$F_x(y) = 1 + G(x, y) - e^{\alpha(y-x)} < 0 \quad \text{for } y > x. \tag{4.8}$$

The function $F_x(y)$ is decreasing since

$$F_x'(y) = -\alpha e^{\alpha(y-x)}\left\{1 - \frac{D(y)}{p}\right\}, \tag{4.9}$$

with $F_x(x) = 0$. Hence, the claim is true. To complete the proof, we need to examine the sign of $\partial_x\partial_y R$. Again, some algebra leads to

$$\partial_x\partial_y R(x,y) = -\alpha\partial_y G(x,y)\left\{1 - \frac{D(y)}{p}\right\}. \tag{4.10}$$

But $\partial_y G(x,y) > 0$, and $p > D(x)$. Therefore, $\partial_x\partial_y R(x,y) < 0$, for $y > x$, and the proof is complete. □

Before we proceed further, we set

$$Z_x(u) = \frac{D'(u)}{D(u)\{1 + 2G(x,u)\}}, \quad \text{with } 0 \le x \le u \le H. \tag{4.11}$$

We assume the following.

(A1) The function Z_x is nonincreasing.

Note that as $\alpha \to 0$, $G(x,u) \to 0$, and consequently $Z_x(u)$ reduces to $D'(u)/D(u)$. In other words, assumption (A1) implies that D is logconcave. This property of the demand rate function may be found in [4, 6], when considering models with infinite production rates. As a matter of fact, this property of D can also be obtained if we let $p \to \infty$.

Example 4.2. Let $D(u) = \alpha e^{\beta u}$, where $\alpha > 0$ and is known and $\beta > 0$ and is known, then $Z_x(u)$ is nonincreasing.

Note that Z_x is nonincreasing which is equivalent to $1/Z_x$ non-decreasing. We have

$$g_x(u) := \frac{1}{Z_x(u)} = \frac{D(u)\{1 + 2G(x,u)\}}{D'(u)} = \frac{1}{\beta}\{1 + 2G(x,u)\}, \tag{4.12}$$

with $g_x'(u) = (2/\beta)\partial_y G(x,y) > 0$, which implies the result.

Example 4.3. Let $D(u) = a + bu$, where $b > 0$, then it is an easy exercise to check that assumption (A1) is satisfied.

Lemma 4.4. *If Z_x satisfies (A1) for all $0 \le x \le H$, then $\mathcal{L}_x R \ge 0$, where $\mathcal{L}_x$ is defined in* (3.5).

Proof. Tedious but direct algebra using the definition of $\mathcal{L}_x R$ leads to

$$\begin{aligned}\mathcal{L}_x R = {} & \frac{\{1-(D(x)/p)\}}{\{1+G(x,y)\}^2}\left\{\alpha\int_x^y e^{\alpha(t-x)}D(t)dt + D(x) - e^{\alpha(y-x)}D(y)\right\} \\ & + \frac{1}{\alpha}\frac{D'(x)G(x,y)}{1+G(x,y)} - \frac{p}{\alpha}\frac{G(x,y)}{1+G(x,y)}\left\{1-\frac{D(x)}{p}\right\}f(x).\end{aligned} \tag{4.13}$$

$\mathcal{L}_x R \geq 0$ is equivalent to

$$\begin{aligned}& \frac{\{1-(D(x)/p)\}}{\{1+G(x,y)\}}\left\{\alpha\int_x^y e^{\alpha(t-x)}D(t)dt + D(x) - e^{\alpha(y-x)}D(y)\right\} \\ & \quad + \frac{1}{\alpha}D'(x)G(x,y) - \frac{p}{\alpha}G(x,y)\left\{1-\frac{D(x)}{p}\right\}f(x) \geq 0.\end{aligned} \tag{4.14}$$

Let

$$f(x) = -\frac{D'(x)}{D'(x)} + \frac{D'(x)}{p\{1-(D(x)/p)\}}. \tag{4.15}$$

It can be shown that (4.14) is true if

$$\frac{\alpha}{p}\frac{e^{\alpha(y-x)}D(y) - D(x) - \alpha\int_x^y e^{\alpha(t-x)}D(t)dt}{G(x,y)\{1+G(x,y)\}} \leq \frac{D'(x)}{D(x)}. \tag{4.16}$$

Define, for $u \geq x$.

$$\begin{aligned}F_x(u) &= e^{\alpha(u-x)}D(u) - \alpha\int_x^u e^{\alpha(t-x)}D(t)dt, \\ H_x(u) &= \frac{\alpha}{p}\int_x^u e^{\alpha(t-x)}D(t)dt\left\{1+\frac{\alpha}{p}\int_x^u e^{\alpha(t-x)}D(t)dt\right\}.\end{aligned} \tag{4.17}$$

The left hand side of (4.16) may be written as

$$\frac{\alpha}{p}\frac{F_x(y) - F_x(x)}{H_x(y) - H_x(x)}. \tag{4.18}$$

This is equal by extended-mean value theorem to $(\alpha/p)((F'_x(\xi))/(H'_x(\xi)))$ for some $x < \xi \leq y$. However,

$$\begin{aligned}F'_x(u) &= \alpha e^{\alpha(u-x)}D'(u), \\ H'_x(u) &= \partial_y G(x,u)\{1+2G(x,u)\}.\end{aligned} \tag{4.19}$$

Therefore,

$$\frac{\alpha}{p}\frac{F_x'(\xi)}{H_x'(\xi)} = \frac{D'(\xi)}{D(\xi)\{1+2G(x,\xi)\}} \leq \frac{D'(x)}{D(x)\{1+2G(x,x)\}} = \frac{D'(x)}{D(x)}. \tag{4.20}$$

The last inequality follows from assumption (A1). This completes the proof. □

Now, set for $0 \leq u \leq y \leq H$,

$$V_y(u) = -\alpha e^{\alpha(y-u)}\left\{e^{\alpha(y-u)}D(y) - D(u) - \alpha\int_u^y e^{\alpha(t-u)}D(t)dt\right\}, \tag{4.21}$$

$$W_y(u) = -\{1+G(u,y)\}\left\{e^{\alpha(y-u)} - 1 - G(u,y)\right\}. \tag{4.22}$$

The next assumption is needed for $\mathcal{L}_y R \geq 0$ of Hypothesis 2 to hold.

(A2) V_y'/W_y' is non-decreasing.

Assumption (A2) is technical and is needed to complete the result of the paper. This assumption may seem complicated but, it is not difficult to check it numerically using MATLAB or Mathematica,say, once the demand rate function is known. Moreover, it can be shown that as $\alpha \to 0$, (A2) reduces to the condition that the function

$$\mathcal{F} = \frac{D'(u)}{1-(D(u)/p)} \tag{4.23}$$

is non-decreasing. This property is satisfied by linear and exponential demand rate functions. In fact, assumption (A1) is also, in this case, satisfied when D is linear or exponential.

Lemma 4.5. *If assumption (A2) is satisfied, then $\mathcal{L}_y R \geq 0$.*

Proof. Recall that

$$\mathcal{L}_y R = \partial_y^2 R + \partial_x\partial_y R + f(y)\partial_y R. \tag{4.24}$$

Direct and tedious computation leads to

$$\begin{aligned}
\mathcal{L}_y R(x,y) = {} & \frac{1}{\alpha}\frac{\partial_y G(x,y)}{\{1+G(x,y)\}^2} \\
& \times \left[\alpha\int_x^y e^{\alpha(t-x)}D(t)dt - \left\{e^{\alpha(y-x)}D(y) - D(x)\right\}\right] \\
& - \frac{1}{\alpha}D'(y) + \frac{1}{\alpha}\frac{e^{\alpha(y-x)}D'(y)}{\{1+G(x,y)\}} \\
& + \frac{1}{\alpha}f(y)D(y)\left\{\frac{e^{\alpha(y-x)}}{1+G(x,y)} - 1\right\}.
\end{aligned} \tag{4.25}$$

Recall the definition of the function f in (4.15). Then $\mathcal{L}_y R \geq 0$ is equivalent to

$$\begin{aligned}
&\frac{\partial_y G(x,y)}{1+G(x,y)}\left\{e^{\alpha(y-x)}D(y)-D(x)-\alpha\int_x^y e^{\alpha(t-x)}D(t)dt\right\}\\
&\quad\leq \frac{D'(y)D(y)}{p\{1-(D(y)/p)\}}\left\{e^{\alpha(y-x)}-1-G(x,y)\right\},
\end{aligned} \tag{4.26}$$

or, by $(\partial_y G)$ and (4.8), we get that the requirement $\mathcal{L}_y R \geq 0$ leads to

$$\alpha\frac{e^{\alpha(y-x)}\left\{e^{\alpha(y-x)}D(y)-D(x)-\alpha\int_x^y e^{\alpha(t-x)}D(t)dt\right\}}{\{1+G(x,y)\}\left\{e^{\alpha(y-x)}-1-G(x,y)\right\}} \leq \frac{D'(y)}{\{1-(D(y)/p)\}}. \tag{4.27}$$

The left hand side of (4.27) is equal to $(V_y(y)-V_y(x))/(W_y(y)-W_y(x))$, where V_y and W_y are given by (4.21), and (4.22) respectively.

Computations show that

$$\begin{aligned}
V_y'(u) &= 2\alpha^2 e^{\alpha(y-u)}\left\{e^{\alpha(y-u)}D(y)-D(u)-\alpha\int_u^y e^{\alpha(t-u)}D(t)dt\right\}\\
&\quad+\alpha e^{\alpha(y-u)}D'(y),\\
W_y'(u) &= -\partial_x G(u,y)\left\{e^{\alpha(y-u)}-1-G(u,y)\right\}\\
&\quad-\{1+G(u,y)\}\left\{-\alpha e^{\alpha(y-u)}-\partial_x G(u,y)\right\}.
\end{aligned} \tag{4.28}$$

Now, the extended-mean value theorem gives that

$$\frac{V_y(y)-V_y(x)}{W_y(y)-W_y(x)} = \frac{V_y'(\xi)}{W_y'(\xi)}, \quad \text{for some } x<\xi<y. \tag{4.29}$$

But assumption (A2) implies that $(V_y'(\xi)/W_y'(\xi)) \leq (V_y'(y)/W_y'(y))$, where the right hand side of the above inequality is equal to

$$\frac{\alpha D'(y)}{-\{-\alpha+\alpha(D(y)/p)\}} = \frac{D'(y)}{\{1-\alpha(D(y)/p)\}}. \tag{4.30}$$

This is the right hand side of (4.27). Hence, $\mathcal{L}_y R \geq 0$. □

As a consequence of Lemmas 4.4 and 4.5 and Theorem 3.1, we have the following result.

Theorem 4.6. *Under the requirements that assumptions (A1) and (A2) hold the function $S_n = \sum_{i=1}^N R(t_{i-1}, t_i)$, with $t_0 = 0 < t_1 < \cdots < t_n$, has a unique minimum, this minimum can be found using the iterative procedure mentioned in [4].*

Let s_n be the minimal value of S_n, then the next theorem follows from Theorem 3.2.

Theorem 4.7. *The function s_n is convex in n.*

As a consequence of Theorem 4.7, the search for the optimal inventory policy can be conducted in two grids: the integer grid and $\mathbb{R}_+^n$. That is, for fixed integer n, the corresponding optimal times are found from the solution of the system of nonlinear equations (3.4) with corresponding objective value s_n. Then, the optimal value of n can be obtained using the following corollary.

Corollary 4.8. *The optimal number of production period n^* is such that*

(1) *if $K > s_1 - s_2$, then $n^* = 1$,*

(2) *if there exists an $N \geq 2$ such that $s_{N-1} - s_N > K > s_N - s_{N+1}$, then $n^* = N$,*

(3) *if there exists an $N \geq 1$ such that $K = s_N - s_{N+1}$, then $n^* = N$ and $n^* = N + 1$.*

5. Conclusion

This paper was concerned with finding the economic-production-lot-size policy for an inventory model with deteriorating items. An optimal inventory policy was proposed for a class of cost functions named OHD models. The proposed optimality approach was based on an earlier work in [4]. The extension to OHD models should not pose any difficulty. Indeed, note that by comparing (2.5) and (2.6), the OHP and OHD models differ in the expression

$$-c_1 \int_{t_{i-1}}^{t_i} D(u)du. \tag{5.1}$$

Now, consider the optimization problem (2.22) with R given by (2.6). It is clear that adding $-c_1 \int_0^H D(u)du = -c_1 \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} D(u)du$ will have no effect on the optimization problem. Consequently, the results obtained for the OHP model apply to the OHD model.

Before we close, we revisit paper [1] and note that the model in [1] allows for the purchasing cost to vary with time, and therefore with fixed unit cost and no deterioration, the model in [1] is a special case of the model of the present paper. The reduction (2.18) in the present paper allows a direct approach as though the problem on hand is an unconstrained optimization problem. The approach adopted in [1] is the standard approach for constrained nonlinear programming problem. The key result in [1] is Theorem 1 (Page 993) which adapted to the model of this paper with $\theta = 0$ requires that $c_h(1 - (D(t)/p)) > 0$ to hold. This is satisfied since $D(t) < p$. Theorem 1 in [1] states, with no conditions imposed on D, that for fixed n the optimal inventory policy is uniquely determined as a solution of the first order condition of the optimization problem on hand. A result similar to Theorem 3.2 related to convexity of the corresponding objective value with respect to n is also presented. The following counterexample shows that Theorem 1 in [1] cannot be entirely correct in its present form. Indeed, for simplicity let $n = 2$, then the problem treated in [1] reduces (equivalently) to minimizing (2.22) with R given by (2.21). The objective function in this case is a function of a single variable. Take $D(t) = 2\sin(10t) + 2\cos(10t) + 4$, $c_h = 1$, and $H \simeq 4.27$, and ignore the setup cost. Figure 2 shows the plot of the objective function. It is clear that multiple critical points can be observed as well as multiple optima. The remark on [1] also applies to part of Balkhi [12].

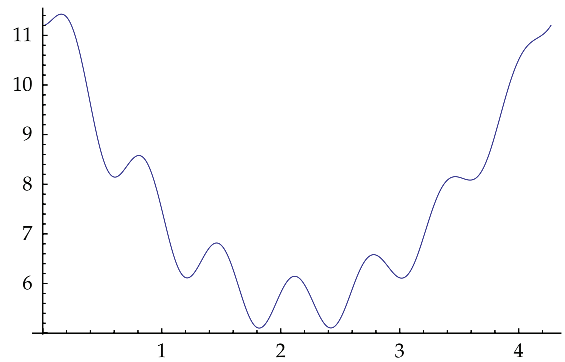

Figure 2: Behaviour of the objective function when $D(t) = 2\sin(10t) + 2\cos(10t) + 4$.

It is worth noting that the keys to success in applying the approach in [4] are the separability of the cost functions between periods and Hypotheses 1 and 2. With this in mind, we believe that the approach of this paper to models with shortages and possibly with costs that are a function of time are possible. The technical requirement needed to generalize the results will be slightly more involved but essentially similar.

Acknowledgments

The authors would like to thank three anonymous referees for helpful comments on an earlier version of the paper.

References

[1] J. T. Teng, L. Y. Ouyang, and C. T. Chang, "Deterministic economic production quantity models with time-varying demand cost," *Applied Mathematical Modelling*, vol. 29, pp. 987–1003, 2005.

[2] S. K. Goyal and B. C. Giri, "The production-inventory problem of a product with time varying demand, production and deterioration rates," *European Journal of Operational Research*, vol. 147, no. 3, pp. 549–557, 2003.

[3] S. Sana, S. K. Goyal, and K. S. Chaudhuri, "A production-inventory model for a deteriorating item with trended demand and shortages," *European Journal of Operational Research*, vol. 157, no. 2, pp. 357–371, 2004.

[4] L. Benkherouf and B. H. Gilding, "On a class of optimization problems for finite time horizon inventory models," *SIAM Journal on Control and Optimization*, vol. 48, no. 2, pp. 993–1030, 2009.

[5] W. A. Donaldson, "Inventory replenishment policy for a linear trend in demand—an analytical solution," *Operational Research Quarterly*, vol. 28, pp. 663–670, 1977.

[6] R. J. Henery, "Inventory replenishment policy for increasing demand," *Operational Research Society Journal*, vol. 30, no. 7, pp. 611–617, 1979.

[7] L. Benkherouf and M. G. Mahmoud, "On an inventory model for deteriorating items with increasing time-varying demand and shortages," *Journal of the Operational Research Society*, vol. 47, pp. 188–200, 1996.

[8] L. Benkherouf and M. Omar, "Optimal integrated policies for a single-vendor single-buyer time-varying demand model," *Computers & Mathematics with Applications*, vol. 60, no. 7, pp. 2066–2077, 2010.

[9] R. J. Hill, "Batching policies for linearly increasing demand with finite input rate," *International Journal of Production Economics*, vol. 43, pp. 149–154, 1996.

[10] M. Omar and D. K. Smith, "An optimal batch size for a production system under linearly increasing-time varying demand process," *Computers and Industrial Engineering*, vol. 42, pp. 35–42, 2002.

[11] H. Rau and B. C. Ou Yang, “A general and optimal approach for three inventory models with a linear trend in demand,” *Computers and Industrial Engineering*, vol. 52, pp. 521–532, 2007.

[12] Z. T. Balkhi, “On a finite horizon production lot size inventory model for deteriorating items: an optimal solution,” *European Journal of Operational Research*, vol. 132, no. 1, pp. 210–223, 2001.