# BOUNDARY VALUE PROBLEMS FOR THE 2ND-ORDER SEIBERG-WITTEN EQUATIONS

CELSO MELCHIADES DORIA

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It is shown that the nonhomogeneous Dirichlet and Neuman problems for the 2nd-order Seiberg-Witten equation on a compact 4-manifold $X$ admit a regular solution once the nonhomogeneous Palais-Smale condition $\mathcal{H}$ is satisfied. The approach consists in applying the elliptic techniques to the variational setting of the Seiberg-Witten equation. The gauge invariance of the functional allows to restrict the problem to the Coulomb subspace $\mathcal{C}_\alpha^{\mathfrak{C}}$ of configuration space. The coercivity of the $\mathcal{SW}_\alpha$-functional, when restricted into the Coulomb subspace, imply the existence of a weak solution. The regularity then follows from the boundedness of $L^\infty$-norms of spinor solutions and the gauge fixing lemma.

## 1. Introduction

Let $X$ be a compact smooth 4-manifold with nonempty boundary. In our context, the Seiberg-Witten equations are the 2nd-order Euler-Lagrange equation of the functional defined in Definition 2.3. When the boundary is empty, their variational aspects were first studied in [3] and the topological ones in [1]. Thus, the main aim here is to obtain the existence of a solution to the nonhomogeneous equations whenever $\partial X \neq \emptyset$. The nonemptiness of the boundary inflicts boundary conditions on the problem. Classically, this sort of problem is classified according to its boundary conditions in *Dirichlet problem* ($\mathcal{D}$) or *Neumann problem* ($\mathcal{N}$).

Originally, the Seiberg-Witten equations were described in [8] as a pair of 1st-order PDE. The solutions of these equations were known as $\mathcal{SW}_\alpha$-monopoles, and their main achievement were to shed light on the understanding of the 4-dimensional differential topology, since new smooth invariants were defined by the topology of their moduli space of solutions (moduli gauge group). In the same article, Witten introduced a variational formulation for the equations and showed that its stable critical points turn out to be exactly the $\mathcal{SW}_\alpha$-monopoles. The variational aspects of the $\mathcal{SW}_\alpha$-equations were first explored in [3], where they proved that the functional satisfies the Palais-Smale condition and the solutions of the Euler-Lagrange (2nd-order) equations share the same important analytical properties as the $\mathcal{SW}_\alpha$-monopoles. Therefore, it is natural to ask if the equations fit into a Morse-Bott-Smale theory, where the lower number of critical points


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is the Betti number of the configuration space. The topology of the configuration space was described in [1]. Besides, if the SW-theory is a Morse theory, another natural question is to argue about the existence of a Morse-Smale-Witten complex, as in [6]. In the last question, the $\mathcal{SW}_\alpha$-equations on manifolds endowed with tubular ends or boundary also demand attention. The analogy of the $\mathcal{SW}_\alpha$-equation's variational formulation, with the variational principle of the Ginzburg-Landau equation in superconductivity, further motivates the present study.

**1.1.** Spin$^c$ **structure.** The space of Spin$^c$ structures on $X$ is identified with

$$\text{Spin}^c(X) = \{\alpha+\beta \in H^2(X,\mathbb{Z}) \oplus H^1(X,\mathbb{Z}_2) \mid w_2(X) = \alpha (\text{mod}\, 2)\}. \tag{1.1}$$

For each $\alpha \in \text{Spin}^c(X)$, there is a representation $\rho_\alpha : \text{SO}_4 \to \mathbb{C}l_4$, induced by a Spin$^c$ representation, and a pair of vector bundles $(\mathcal{S}^+_\alpha, \mathcal{L}_\alpha)$ over $X$ (see [4]). Let $P_{\text{SO}_4}$ be the frame bundle of $X$, so

(i) $\mathcal{S}_\alpha = P_{\text{SO}_4} \times_{\rho_\alpha} V = \mathcal{S}^+_\alpha \oplus \mathcal{S}^-_\alpha$. The bundle $\mathcal{S}^+_\alpha$ is the positive complex spinors bundle (fibers are $\text{Spin}^c_4$-modules isomorphic to $\mathbb{C}^2$),

(ii) $\mathcal{L}_\alpha = P_{\text{SO}_4} \times_{\det(\alpha)} \mathbb{C}$. It is called the *determinant line bundle* associated to the Spin$^c$-structure $\alpha \cdot (c_1(\mathcal{L}_\alpha) = \alpha)$.

Thus, for each $\alpha \in \text{Spin}^c(X)$, we associate a pair of bundles

$$\alpha \in \text{Spin}^c(X) \rightsquigarrow (\mathcal{L}_\alpha, \mathcal{S}^+_\alpha). \tag{1.2}$$

From now on, we considered on $X$ a Riemannian metric $g$ and on $\mathcal{S}_\alpha$ a Hermitian structure $h$.

Let $P_\alpha$ be the $U_1$-principal bundle over $X$ obtained as the frame bundle of $\mathcal{L}_\alpha$ ($c_1(P_\alpha) = \alpha$). Also, we consider the adjoint bundles

$$\text{Ad}\,(U_1) = P_{U_1} \times_{\text{Ad}} U_1, \qquad \text{ad}\,(\mathfrak{u}_1) = P_{U_1} \times_{\text{ad}} \mathfrak{u}_1, \tag{1.3}$$

where $\text{Ad}(U_1)$ is a fiber bundle with fiber $U_1$, and $\text{ad}(\mathfrak{u}_1)$ is a vector bundle with fiber isomorphic to the Lie algebra $\mathfrak{u}_1$.

**1.2. The main theorem.** Let $\mathcal{A}_\alpha$ be (formally) the space of connections (covariant derivative) on $\mathcal{L}_\alpha$, $\Gamma(\mathcal{S}^+_\alpha)$ the space of sections of $\mathcal{S}^+_\alpha$, and $\mathcal{G}_\alpha = \Gamma(\text{Ad}(U_1))$ the gauge group acting on $\mathcal{A}_\alpha \times \Gamma(\mathcal{S}^+_\alpha)$ as follows:

$$g \cdot (A, \phi) = (A + g^{-1}dg, g^{-1}\phi). \tag{1.4}$$

$\mathcal{A}_\alpha$ is an affine space with vector space structure, after fixing an origin, isomorphic to the space $\Omega^1(\text{ad}(\mathfrak{u}_1))$ of $\text{ad}(\mathfrak{u}_1)$-valued 1-forms. Once a connection $\nabla^0 \in \mathcal{A}_\alpha$ is fixed, a bijection $\mathcal{A}_\alpha \leftrightarrow \Omega^1(\text{ad}(\mathfrak{u}_1))$ is exposed by $\nabla^A \leftrightarrow A$, where $\nabla^A = \nabla^0 + A$. $\mathcal{G}_\alpha = \text{Map}(X, U_1)$, since $\text{Ad}(U_1) \simeq X \times U_1$. The curvature of a 1-connection form $A \in \Omega^1(\text{ad}(\mathfrak{u}_1))$ is the 2-form $F_A = dA \in \Omega^2(\text{ad}(\mathfrak{u}_1))$.

*Definition 1.1.* (1) The configuration space of the $\mathcal{D}$-problem is

$$\mathcal{C}_\alpha^{\mathcal{D}} = \{(A,\phi) \in \mathcal{A}_\alpha \times \Gamma(\mathcal{S}_\alpha^+) \,|\, (A,\phi)|_Y \overset{\text{gauge}}{\sim} (A_0,\phi_0)\}, \tag{1.5}$$

(2) the configuration space of the $\mathcal{N}$-problem is

$$\mathcal{C}_\alpha^{\mathcal{N}} = \mathcal{A}_\alpha \times \Gamma(\mathcal{S}_\alpha^+). \tag{1.6}$$

Although each boundary problem requires its own configuration space, the superscripts $\mathcal{D}$ and $\mathcal{N}$ will be used whenever the distinction is necessary, since most arguments work for both sort of problems. The gauge group $\mathcal{G}_\alpha$ action on each of the configuration spaces is given by (1.4).

The Dirichlet ($\mathcal{D}$) and Neumann ($\mathcal{N}$) boundary value problems associated to the $\mathcal{SW}_\alpha$-equations are the following: we consider $(\Theta,\sigma) \in \Omega^1(\text{ad}(\mathfrak{u}_1)) \oplus \Gamma(\mathcal{S}_\alpha^+)$ and $(A_0,\phi_0)$ defined on the manifold $\partial X$ ($A_0$ is a connection on $\mathcal{L}_\alpha|_{\partial X}$, $\phi_0$ is a section of $\Gamma(\mathcal{S}_\alpha^+|_{\partial X})$). In this way, find $(A,\phi) \in \mathcal{C}_\alpha^{\mathcal{D}}$ satisfying $\mathcal{D}$ and $(A,\phi) \in \mathcal{C}_\alpha^{\mathcal{N}}$ satisfying $\mathcal{N}$, where

(1)

$$\mathcal{D} = \begin{cases} d^*F_A + 4\Phi^*(\nabla^A\phi) = \Theta, \\ \Delta_A\phi + \dfrac{(|\phi|^2+k_g)}{4}\phi = \sigma, \\ (A,\phi)|_{\partial X} \overset{\text{gauge}}{\sim} (A_0,\phi_0), \end{cases} \qquad \mathcal{N} = \begin{cases} d^*F_A + 4\Phi^*(\nabla^A\phi) = \Theta, \\ \Delta_A\phi + \dfrac{(|\phi|^2+k_g)}{4}\phi = \sigma, \\ i^*(*F_A) = 0, \quad \nabla_\nu^A\phi = 0, \end{cases} \tag{1.7}$$

(2) the operator $\Phi^* : \Omega^1(\mathcal{S}_\alpha^+) \to \Omega^1(\mathfrak{u}_1)$ is locally given by

$$\Phi^*(\nabla^A\phi) = \frac{1}{2}\nabla^A(|\phi|^2) = \sum_i \langle \nabla_i^A\phi,\phi\rangle \eta_i, \tag{1.8}$$

and $\eta = \{\eta_i\}$ is an orthonormal frame in $\Omega^1(\text{ad}(\mathfrak{u}_1))$,

(3) $i^*(*F_A) = F_4$, where $F_4 = (F_{14},F_{24},F_{34},0)$ is the local representation of the 4th component (normal to $\partial X$) of the 2-form of curvature in the local chart $(x,U)$ of $X$; $x(U) = \{x = (x_1,x_2,x_3,x_4) \in \mathbb{R}^4;\ \|x\| < \epsilon,\ x_4 \geq 0\}$, and $x(U \cap \partial X) \subset \{x \in x(U) \mid x_4 = 0\}$. Let $\{e_1,e_2,e_3,e_4\}$ be the canonical base of $\mathbb{R}^4$, so $\nu = -e_4$ is the normal vector field along $\partial X$.

Theorem 1.2 (main theorem). *If the pair $(\Theta,\sigma) \in L^{k,2} \oplus (L^{k,2} \cap L^\infty)$ satisfies the $\mathcal{H}$-Condition 3.1, then the problems $\mathcal{D}$ and $\mathcal{N}$ admit a $C^r$-regular solution $(A,\phi)$, whenever $2 < k$ and $r < k$.*

## 2. Basic set up

**2.1. Sobolev spaces.** As a vector bundle $E$ over $(X,g)$ is endowed with a metric and a covariant derivative $\nabla$, we define the Sobolev norm of a section $\phi \in \Omega^0(E)$ as

$$\|\phi\|_{L^{k,p}} = \sum_{|i|=0}^{k}\left(\int_X |\nabla^i\phi|^p\right)^{1/p}. \tag{2.1}$$

In this way, the $L^{k,p}$-Sobolev Spaces of sections of $E$ is defined as

$$L^{k,p}(E) = \{\phi \in \Omega^0(E) \mid \|\phi\|_{L^{k,p}} < \infty\}. \tag{2.2}$$

In our context, in which we fixed a connection $\nabla^0$ on $\mathcal{L}_\alpha$, a metric $g$ on $X$, and a Hermitian structure on $\mathcal{S}_\alpha$, the Sobolev spaces on which the basic setting is made are the following:

(i) $\mathcal{A}_\alpha = L^{1,2}(\Omega^1(\mathrm{ad}(\mathfrak{u}_1)))$;
(ii) $\Gamma(\mathcal{S}_\alpha^+) = L^{1,2}(\Omega^0(X,\mathcal{S}_\alpha^+))$;
(iii) $\mathcal{C}_\alpha = \mathcal{A}_\alpha \times \Gamma(\mathcal{S}_\alpha^+)$;
(iv) $\mathcal{G}_\alpha = L^{2,2}(X,U_1) = L^{2,2}(\mathrm{Map}(X,U_1))$. ($\mathcal{G}_\alpha$ is an $\infty$-dimensional Lie group with Lie algebra $\mathfrak{g} = L^{1,2}(X,\mathfrak{u}_1)$).

The above Sobolev spaces induce a Sobolev structure on $\mathcal{C}_\alpha^{\mathcal{D}}$ and on $\mathcal{C}_\alpha^{\mathcal{N}}$. From now on, the configuration spaces will be denoted by $\mathcal{C}_\alpha$ by ignoring the superscripts, unless needed.

The most basic analytical results needed to achieve the main result is the *gauge fixing lemma* (see [7]) and the estimate (2.3), both extended by Marini [5] to manifolds with boundary.

Lemma 2.1 (gauge fixing lemma). *Every connection $\widehat{A} \in \mathcal{A}_\alpha$ is gauge equivalent, by a gauge transformation $g \in \mathcal{G}_\alpha$ named Coulomb ($\mathfrak{C}$) gauge, to a connection $A \in \mathcal{A}_\alpha$ satisfying*

(1) $d_\tau^{*_f} A_\tau = 0$ *on* $\partial X$,
(2) $d^* A = 0$ *on* $X$,
(3) *in the $\mathcal{N}$-problem, the connection $A$ satisfies $A_\nu = 0$ ($\nu \perp \partial X$).*

Corollary 2.2. *Under the hypothesis of Lemma 2.1, there exists a constant $K > 0$ such that the connection $A$, gauge equivalent to $\widehat{A}$ by the Coulomb gauge, satisfies the following estimates:*

$$\|A\|_{L^{1,p}} \leq K \cdot \|F_A\|_{L^p}. \tag{2.3}$$

*Notation.* $*_f$ is the Hodge operator in the flat metric and the index $\tau$ denotes tangential components.

**2.2. Variational formulation.** A global formulation for problems $\mathcal{D}$ and $\mathcal{N}$ is made using the Seiberg-Witten functional.

*Definition 2.3.* Let $\alpha \in \mathrm{Spin}^c(X)$. The Seiberg-Witten functional $\mathcal{SW}_\alpha : \mathcal{C}_\alpha \to \mathbb{R}$ is defined as

$$\mathcal{SW}_\alpha(A,\phi) = \int_X \left\{\frac{1}{4}\left|F_A\right|^2 + \left|\nabla^A\phi\right|^2 + \frac{1}{8}|\phi|^4 + \frac{k_g}{4}|\phi|^2\right\} dv_g + \pi^2\alpha^2, \tag{2.4}$$

where $k_g$ = scalar curvature of $(X,g)$.

*Remark 2.4.* The $\mathcal{G}_\alpha$-action on $\mathcal{C}_\alpha$ has the following properties:

(1) the $\mathcal{SW}_\alpha$-functional is $\mathcal{G}_\alpha$-invariant,

(2) the $\mathcal{G}_\alpha$-action on $\mathcal{C}_\alpha$ induces on $T\mathcal{C}_\alpha$ a $\mathcal{G}_\alpha$-action as follows: let $(\Lambda, V) \in T_{(A,\phi)}\mathcal{C}_\alpha$ and $g \in \mathcal{G}_\alpha$,

$$g \cdot (\Lambda, V) = (\Lambda, g^{-1}V) \in T_{g\cdot(A,\phi)}\mathcal{C}_\alpha. \tag{2.5}$$

Consequently, $d(\mathcal{SW}_\alpha)_{g\cdot(A,\phi)}(g \cdot (\Lambda, V)) = d(\mathcal{SW}_\alpha)_{(A,\phi)}(\Lambda, V)$.

The tangent bundle $T\mathcal{C}_\alpha$ decomposes as

$$T\mathcal{C}_\alpha = \Omega^1(\text{ad}(\mathfrak{u}_1)) \oplus \Gamma(\mathcal{S}_\alpha^+). \tag{2.6}$$

In this way, the 1-form $d\mathcal{SW}_\alpha \in \Omega^1(\mathcal{C}_\alpha)$ admits a decomposition $d\mathcal{SW}_\alpha = d_1\mathcal{SW}_\alpha + d_2\mathcal{SW}_\alpha$, where

$$\begin{aligned} d_1(\mathcal{SW}_\alpha)_{(A,\phi)} &: \Omega^1(\text{ad}(\mathfrak{u}_1)) \longrightarrow \mathbb{R}, \qquad d_1(\mathcal{SW}_\alpha)_{(A,\phi)} \cdot \Lambda = d(\mathcal{SW}_\alpha)_{(A,\phi)} \cdot (\Lambda, 0), \\ d_2(\mathcal{SW}_\alpha)_{(A,\phi)} &: \Gamma(\mathcal{S}_\alpha^+) \longrightarrow \mathbb{R}, \qquad d_2(\mathcal{SW}_\alpha)_{(A,\phi)} \cdot V = d(\mathcal{SW}_\alpha)_{(A,\phi)} \cdot (0, V). \end{aligned} \tag{2.7}$$

By performing the computations, we get

(1) for every $\Lambda \in \mathcal{A}_\alpha$,

$$d_1(\mathcal{SW}_\alpha)_{(A,\phi)} \cdot \Lambda = \frac{1}{4}\int_X \text{Re}\{\langle F_A, d_A\Lambda\rangle + 4\langle \nabla^A(\phi), \Phi(\Lambda)\rangle\} dx, \tag{2.8}$$

where $\Phi : \Omega^1(\mathfrak{u}_1) \to \Omega^1(\mathcal{S}_\alpha^+)$ is the linear operator $\Phi(\Lambda) = \Lambda(\phi)$, with dual defined in (1.8),

(2) for every $V \in \Gamma(\mathcal{S}_\alpha^+)$,

$$d_2(\mathcal{SW}_\alpha)_{(A,\phi)} \cdot V = \int_X \text{Re}\left\{\langle \nabla^A\phi, \nabla^A V\rangle + \left\langle \frac{|\phi|^2 + k_g}{4}\phi, V\right\rangle\right\} dx. \tag{2.9}$$

Therefore, by taking $\text{supp}(\Lambda) \subset \text{int}(X)$ and $\text{supp}(V) \subset \text{int}(X)$, we restrict to the interior of $X$, and so, the gradient of the $\mathcal{SW}_\alpha$-functional at $(A,\phi) \in \mathcal{C}_\alpha$ is

$$\text{grad}(\mathcal{SW}_\alpha)(A,\phi) = \left(d_A^* F_A + 4\Phi^*(\nabla^A\phi), \triangle_A\phi + \frac{|\phi|^2 + k_g}{4}\phi\right). \tag{2.10}$$

It follows from the $\mathcal{G}_\alpha$-action on $T\mathcal{C}_\alpha$ that

$$\text{grad}(\mathcal{SW}_\alpha)(g \cdot (A,\phi)) = \left(d_A^* F_A + 4\Phi^*(\nabla^A\phi), g^{-1} \cdot \left(\triangle_A\phi + \frac{|\phi|^2 + k_g}{4}\phi\right)\right). \tag{2.11}$$

An important analytical aspect of the $\mathcal{SW}_\alpha$-functional is the coercivity lemma proved in [3].

LEMMA 2.5 (coercivity). *For each* $(A,\phi) \in \mathcal{C}_\alpha$, *there exist* $g \in \mathcal{G}_\alpha$ *and a constant* $K_C^{(A,\phi)} > 0$, *where* $K_C^{(A,\phi)}$ *depends on* $(X,g)$ *and* $\mathcal{SW}_\alpha(A,\phi)$, *such that*

$$||g \cdot (A,\phi)||_{L^{1,2}} < K_C^{(A,\phi)}. \tag{2.12}$$

*Proof* (see [3, Lemma 2.3]). The gauge transform is the Coulomb one given in the Lemma 2.1. □

Considering the gauge invariance of the $\mathcal{SW}_\alpha$-theory, and the fact that the gauge group $\mathcal{G}_\alpha$ is an infinite-dimensional Lie group, we cannot hope to handle the problem in general. From now on, we need to restrict the problem to the space, named Coulomb subspace,

$$\mathcal{C}_\alpha^{\mathfrak{C}} = \Big\{(A,\phi) \in \mathcal{C}_\alpha;\ ||(A,\phi)||_{L^{1,2}} < K_{\mathfrak{C}}^{(A,\phi)}\Big\}. \tag{2.13}$$

The superscripts $\mathcal{D}$ and $\mathcal{N}$ have been omitted here for simplicity, although each one should be taken in account according to the problem. These choices of spaces come from the nature of the $\mathcal{G}_\alpha$ action on $\mathcal{C}_\alpha$, they are suggested by the gauge fixing lemma and the coercivity lemma (not shared by an actions in general).

## 3. Existence of a solution

**3.1. Nonhomogeneous Palais-Smale condition —$\mathcal{H}$.** In the variational formulation, the problems $\mathcal{D}$ and $\mathcal{N}$ (1.7) are written as

$$(\mathcal{D}) = \begin{cases} \operatorname{grad}(\mathcal{SW}_\alpha)(A,\phi) = (\Theta,\sigma), \\ (A,\phi)|_{\partial X} \overset{\text{gauge}}{\sim} (A_0,\phi_0), \end{cases}$$
$$(\mathcal{N}) = \begin{cases} \operatorname{grad}(\mathcal{SW}_\alpha)(A,\phi) = (\Theta,\sigma), \\ i^*(*F_A) = 0, \quad \nabla_n^A\phi = 0. \end{cases} \tag{3.1}$$

The equations in (1.7) may not admit a solution for any pair $(\Theta,\sigma) \in \Omega^1(\operatorname{ad}(\mathfrak{u}_1)) \oplus \Gamma(\mathcal{S}_\alpha^+)$. In finite dimension, if we consider a function $f : X \to \mathbb{R}$, the analogous question would be to find a point $p \in X$ such that, for a fixed vector $u$, $\operatorname{grad}(f)(p) = u$. This question is more subtle if $f$ is invariant under a Lie group action on $X$. Therefore, we need a hypothesis about the pair $(\Theta,\sigma) \in \Omega^1(\operatorname{ad}(\mathfrak{u}_1)) \oplus \Gamma(\mathcal{S}_\alpha^+)$.

*Condition 3.1* ($\mathcal{H}$). Let $(\Theta,\sigma) \in L^{1,2}(\Omega^1(\operatorname{ad}(\mathfrak{u}_1))) \oplus (L^{1,2}(\Gamma(\mathcal{S}_\alpha^+)) \cap L^\infty(\Gamma(\mathcal{S}_\alpha^+)))$ be a pair such that there exists a sequence $\{(A_n,\phi_n)\}_{n\in\mathbb{Z}} \subset \mathcal{C}_\alpha^{\mathfrak{C}}$ (2.13) with the following properties:

(1) $\{(A_n,\phi_n)\}_{n\in\mathbb{Z}} \subset L^{1,2}(\mathcal{A}_\alpha) \times (L^{1,2}(\Gamma(\mathcal{S}_\alpha^+)) \cup L^\infty(\Gamma(\mathcal{S}_\alpha^+)))$ and there exists a constant $c_\infty > 0$ such that, for all $n \in \mathbb{Z}$, $\|\phi_n\|_\infty < c_\infty$,

(2) there exists $c \in \mathbb{R}$ such that, for all $n \in \mathbb{Z}$, $\mathcal{SW}_\alpha(A_n,\phi_n) < c$,

(3) the sequence $\{d(\mathcal{SW}_\alpha)_{(A_n,\phi_n)}\}_{n\in\mathbb{Z}} \subset (L^{1,2}(\Omega^1(\operatorname{ad}(\mathfrak{u}_1))) \oplus L^{1,2}(\Gamma(\mathcal{S}_\alpha^+)))^*$, of linear functionals, converges weakly to

$$L_\Theta + L_\sigma : T\mathcal{C}_\alpha \longrightarrow \mathbb{R}, \tag{3.2}$$

where

$$L_\Theta(\Lambda) = \int_X \langle \Theta, \Lambda \rangle, \qquad L_\sigma(V) = \int_X \langle \sigma, V \rangle. \tag{3.3}$$

**3.2. Strong convergence of $\{(A_n, \phi_n)\}_{n\in\mathbb{Z}}$ in $L^{1,2}$.** As a consequence of Lemma 2.5, the sequence $\{(A_n, \phi_n)\}_{n\in\mathbb{Z}}$ given by the $\mathcal{H}$-condition converges to a pair $(A, \phi)$;

(1) weakly in $\mathcal{C}_\alpha$,
(2) weakly in $L^4(\mathcal{A}_\alpha \times \Gamma(\mathcal{S}_\alpha^+))$,
(3) strongly in $L^p(\mathcal{A}_\alpha \times \Gamma(\mathcal{S}_\alpha^+))$, for every $p < 4$.

*Remark 3.2.* Let $\{A_n\}_{n\in\mathbb{N}} \subset L^2$ be a converging sequence in $L^2$ satisfying $d^* A_n = 0$, for all $n \in \mathbb{N}$, and let $A = \lim_{n\to\infty} A_n \in L^2$. So, $d^* A = 0$, once

$$|\langle d^* A, \rho \rangle| \leq |A - A_n|_{L^2} \cdot |d\rho|_{L^2}, \tag{3.4}$$

for all $\rho \in \Omega^0(\mathrm{ad}(\mathfrak{u}_1))$.

Theorem 3.3. *The limit $(A, \phi) \in L^2(\mathcal{A}_\alpha \times \Gamma(\mathcal{S}_\alpha^+))$, obtained as a limit of the sequence $\{(A_n, \phi_n)\}_{n\in\mathbb{Z}}$, is a weak solution of (1.7).*

*Proof.* The proof goes along the same lines as in the 2nd step in the proof of the compactness theorem in [3].

(1) For every $\Lambda \in \mathcal{A}_\alpha$,

$$\begin{aligned} d_1(\mathcal{SW}_\alpha)_{(A_n,\phi_n)} \cdot \Lambda = \frac{1}{4}\int_X &\mathrm{Re}\{\langle F_{A_n}, d_{A_n}\Lambda\rangle + 4\langle \nabla^{A_n}(\phi_n), \Phi(\Lambda)\rangle\} dx \\ &+ \int_{\partial X} \mathrm{Re}\{\Lambda \wedge *F_{A_n}\}, \end{aligned} \tag{3.5}$$

where

(a) $\Phi : \Omega^1(\mathfrak{u}_1) \to \Omega^1(\mathcal{S}_\alpha^+)$ is the linear operator $\Phi(\Lambda) = \Lambda(\phi)$; its dual is defined in (1.8). Assuming $\phi \in L^\infty$ (Lemma 3.4), it follows that

$$\lim_{n\to\infty} d_1(\mathcal{SW}_\alpha)_{(A_n,\phi_n)} \cdot \Lambda = d_1(\mathcal{SW}_\alpha)_{(A,\phi)} \cdot \Lambda. \tag{3.6}$$

Therefore, $d_1(\mathcal{SW}_\alpha)_{(A,\phi)} \cdot \Lambda = \int_X \langle \Theta, \Lambda \rangle$,

(b) $\Lambda \wedge *F_A = -\langle \Lambda, F_4 \rangle dx_1 \wedge dx_2 \wedge dx_3$. Since the above equation is true for all $\Lambda$, let $\mathrm{supp}(\Lambda) \subset \partial X$, so $F_4 = 0$ ($\Rightarrow i^*(*F_A) = 0$).

(2) For every $V \in \Gamma(\mathcal{S}_\alpha^+)$,

$$\begin{aligned} d_2(\mathcal{SW}_\alpha)_{(A_n,\phi_n)} \cdot V = \int_X &\mathrm{Re}\left\{\langle \nabla^{A_n}\phi_n, \nabla^{A_n} V\rangle + \left\langle \frac{|\phi_n|^2 + k_g}{4}\phi_n, V \right\rangle\right\} dx \\ &+ \int_{\partial X} \mathrm{Re}\{\langle \nabla_\nu^{A_n}\phi_n, V\rangle\}. \end{aligned} \tag{3.7}$$

Analogously, it follows that $(A,\phi)$ is a weak solution of the equation

$$d_2(\mathcal{SW}_\alpha)_{(A,\phi)} \cdot V = \int_X \langle \sigma, V\rangle. \tag{3.8}$$

So, in the $\mathcal{N}$-problem, $\nabla_\nu^A \phi = 0$. □

In order to pursue the strong $L^{1,2}$-convergence for the sequence $\{(A_n,\phi_n)\}_{n\in\mathbb{Z}}$, we obtain in the following an upper bound for $\|\phi\|_{L^\infty}$, whenever $(A,\phi)$ is a weak solution.

Lemma 3.4. *Let $(A,\phi)$ be a solution of either $\mathfrak{D}$ or $\mathcal{N}$ in (1.7), so the following hold.*

(1) *If $\sigma = 0$, then there exists a constant $k_{X,g}$, depending on the Riemannian metric on $X$, such that*

$$\|\phi\|_\infty < k_{X,g}\,\mathrm{vol}(X). \tag{3.9}$$

(2) *If $\sigma \neq 0$, then there exist constant $c_1 = c_1(X,g)$ and $c_2 = c_2(X,g)$ such that*

$$\|\phi\|_{L^p} < c_1 + c_2\|\sigma\|^3_{L^{3p}}. \tag{3.10}$$

*In particular, if $\sigma \in L^\infty$, then $\phi \in L^\infty$.*

*Proof.* Fix $r \in \mathbb{R}$ and suppose that there is a ball $B_{r^{-1}}(x_0)$, around the point $x_0 \in X$, such that

$$|\phi(x)| > r, \quad \forall x \in B_{r^{-1}}(x_0). \tag{3.11}$$

Define

$$\eta = \begin{cases} \left(1 - \dfrac{r}{|\phi|}\right)\phi & \text{if } x \in B_{r^{-1}}(x_0), \\ 0 & \text{if } x \in X - B_{r^{-1}}(x_0). \end{cases} \tag{3.12}$$

So,

$$\begin{aligned} |\eta| &\le |\phi|, \\ \nabla\eta &= r\frac{\langle\phi,\nabla\phi\rangle}{|\phi|^3}\phi + \left(1 - \frac{r}{|\phi|}\right)\nabla\phi \\ \Longrightarrow |\nabla\eta|^2 &= r^2\frac{\langle\phi,\nabla\phi\rangle^2}{|\phi|^4} + 2r\left(1 - \frac{r}{|\phi|}\right)\frac{\langle\phi,\nabla\phi\rangle^2}{|\phi|^3} + \left(1 - \frac{r}{|\phi|}\right)^2|\nabla\phi|^2 \\ \Longrightarrow |\nabla\eta|^2 &< r^2\frac{|\nabla\phi|^2}{|\phi|^2} + 2r\left(1 - \frac{r}{|\phi|}\right)\frac{|\nabla\phi|^2}{|\phi|} + \left(1 - \frac{r}{|\phi|}\right)^2|\nabla\phi|^2. \end{aligned} \tag{3.13}$$

Since $r < |\phi|$,

$$|\nabla \eta|^2 < 4|\nabla \phi|^2. \tag{3.14}$$

Hence, by (3.13) and (3.14), $\eta \in L^{1,2}$. The directional derivative of $\mathcal{SW}_\alpha$ in direction $\eta$ is given by

$$d(\mathcal{SW}_\alpha)_{(A,\phi)}(0,\eta) = \int_X \Big[ \langle \nabla^A \phi, \nabla^A \eta \rangle + \frac{|\phi|^2 + k_g}{4} |\phi| (|\phi| - r) \Big]. \tag{3.15}$$

By (2.9),

$$\int_X \Big[ \langle \nabla^A \phi, \nabla^A \eta \rangle + \frac{|\phi|^2 + k_g}{4} |\phi| (|\phi| - r) \Big] = \int_X \Big\langle \sigma, \Big(1 - \frac{r}{|\phi|}\Big)\phi \Big\rangle. \tag{3.16}$$

However,

$$\int_X \langle \nabla^A \phi, \nabla^A \eta \rangle = \int_X \Big[ r \frac{\langle \phi, \nabla^A \phi \rangle^2}{|\phi|^3} + \Big(1 - \frac{r}{|\phi|}\Big) |\nabla \phi|^2 \Big] > 0. \tag{3.17}$$

So,

$$\int_X \frac{|\phi|^2 + k_g}{4} |\phi| (|\phi| - r) < \int_X \Big\langle \sigma, \Big(1 - \frac{r}{|\phi|}\Big)\phi \Big\rangle < \int_X |\sigma| (|\phi| - r). \tag{3.18}$$

Hence,

$$\int_X (|\phi| - r) \Big( \frac{|\phi|^2 + k_g}{4} |\phi| - |\sigma| \Big) < 0. \tag{3.19}$$

Since $r < |\phi(x)|$, whenever $x \in B_{r^{-1}}(x_0)$, it follows that

$$(|\phi|^2 + k_g)|\phi| < 4|\sigma|, \quad \text{a.e. in } B_{r^{-1}}(x_0). \tag{3.20}$$

There are two cases to be analysed independently.

(1) $\sigma = 0$. In this case, we get

$$(|\phi|^2 + k_g)|\phi| < 0, \quad \text{a.e.} \tag{3.21}$$

The scalar curvature plays a central role here: if $k_g \geq 0$, then $\phi = 0$; otherwise,

$$|\phi| \leq \max\{0, (-k_g)^{1/2}\}. \tag{3.22}$$

Since $X$ is compact, we let $k_{X,g} = \max_{x\in X}\{0,[-k_g(x)]^{1/2}\}$, and so,

$$\|\phi\|_\infty < k_{X,g}\,\mathrm{vol}(X). \tag{3.23}$$

(2) Let $\sigma \neq 0$. The inequality (3.20) implies that

$$|\phi|^3 + k_g|\phi| - 4|\sigma| < 0, \quad \text{a.e.} \tag{3.24}$$

Consider the polynomial

$$Q_{\sigma(x)}(w) = w^3 + k_g w - 4\,|\,\sigma(x)\,|. \tag{3.25}$$

An estimate for $|\phi|$ is obtained by estimating the largest real number $w$ satisfying $Q_{\sigma(x)}(w) < 0$. $Q_{\sigma(x)}$ being monic implies that $\lim_{w\to\infty} Q_{\sigma(x)}(w) = +\infty$. So, either $Q_{\sigma(x)} > 0$, whenever $w > 0$, or there exists a root $\rho \in (0,\infty)$. The first case would imply that

$$Q_{\sigma(x)}(\,|\,\phi(x)\,|\,) > 0, \quad \text{a.e.,} \tag{3.26}$$

contradicting (3.20). By the same argument, there exists a root $\rho \in (0,\infty)$ such that $Q_{\sigma(x)}(w)$ changes its sign in a neighborhood of $\rho$. Let $\rho$ be the largest root in $(0,\infty)$ with this property. By the Corollary A.2, there exist constants $c_1 = c_1(X,g)$ and $c_2$ such that

$$|\rho| < c_1 + c_2\,|\,\sigma(x)\,|^3. \tag{3.27}$$

Consequently,

$$|\,\phi(x)\,| < c_1 + c_2\,|\,\sigma(x)\,|^3, \quad \text{a.e. in } B_{r^{-1}}(x_0) \tag{3.28}$$

and

$$\|\phi\|_{L^p} < C_1 + C_2\|\sigma\|^3_{L^{3p}} \quad \text{restricted to } B_{r^{-1}}(x_0), \tag{3.29}$$

where $C_1$, $C_2$ are constants depending on $\mathrm{vol}(B_{r^{-1}}(x_0))$. The inequality (3.29) can be extended over $X$ by using a $C^\infty$ partition of unity. Moreover, if $\sigma \in L^\infty$, then

$$\|\phi\|_\infty < C_1 + C_2\|\sigma\|^3_\infty, \tag{3.30}$$

where $C_1$, $C_2$ are constants depending on $\mathrm{vol}(X)$. □

A sort of concentration lemma, proved in [3], can be extended as follows.

Lemma 3.5. *Let $\{(A_n,\phi_n)\}_{n\in\mathbb{Z}}$ be the sequence given by the $\mathcal{H}$-Condition 3.1. Then,*

$$\lim_{n\to\infty}\int_X \langle \Phi^*(\nabla^{A_n}\phi_n), A_n - A\rangle = 0. \tag{3.31}$$

*Proof.* By (1.8),

$$\lim_{n\to\infty}\int_X \langle \Phi^*(\nabla^{A_n}\phi_n), A_n - A\rangle = \lim_{n\to\infty}\int_X \langle \nabla_i^{A_n}\phi_n, \phi_n\rangle \cdot \langle \eta_i, A_n - A\rangle,$$

$$\begin{aligned}
\lim_{n\to\infty}\int_X &\langle \nabla_i^{A_n}\phi_n, \phi_n\rangle \cdot \langle \eta_i, A_n - A\rangle \\
&\le \lim_{n\to\infty}\int_X |\langle \nabla_i^{A_n}\phi_n, \phi_n\rangle|^2 \cdot \int_X |\langle \eta_i, A_n - A\rangle|^2 \\
&\le \lim_{n\to\infty}\left[\int_X |\nabla_i^{A_n}\phi_n|^2 \cdot |\phi_n|^2\right] \cdot \int_X |A_n - A|^2 \\
&\le \lim_{n\to\infty} c_\infty \cdot \left[\int_X |\nabla_i^{A_n}\phi_n|^2\right] \cdot ||A_n - A||_{L^2}^2 \\
&\le \lim_{n\to\infty} c_\infty \cdot ||\phi_n||_{L^{1,2}}^2 \cdot ||A_n - A||_{L^2}^2 = 0.
\end{aligned} \tag{3.32}$$

□

THEOREM 3.6. *Let* $(\Theta, \sigma)$ *be a pair satisfying the* $\mathcal{H}$*-Condition 3.1. Then, the sequence* $\{(A_n, \phi_n)\}_{n\in\mathbb{Z}}$*, given by Condition 3.1, converges strongly to* $(A, \phi) \in \mathcal{C}_\alpha$.

*Proof.* From Theorem 3.3, $\{(A_n, \phi_n)\}_{n\in\mathbb{Z}}$ converges weakly in $L^{1,2}$ to $(A, \phi) \in \mathcal{C}_\alpha$. The proof is splitted into 2 parts.

(1) $\lim_{n\to\infty} \|A_n - A\|_{L^{1,2}} = 0$. Let $d^* : \Omega^1(\mathrm{ad}(\mathfrak{u}_1)) \to \Omega^0(\mathrm{ad}(\mathfrak{u}_1))$. The operator $d : \ker(d^*) \to \Omega^2(\mathrm{ad}(\mathfrak{u}_1))$ being elliptic implies, by the fundamental elliptic estimate, that

$$||A_n - A||_{L^{1,2}} \le c||d(A_n - A)||_{L^2} + ||A_n - A||_{L^2}. \tag{3.33}$$

The first term in the right-hand side is controlled as follows:

$$\begin{aligned}
||dA_n - dA||_{L^2}^2 &= \int_X \langle d(A_n - A), d(A_n - A)\rangle \\
&= \int_X \langle dA_n, d(A_n - A)\rangle - \int_X \langle dA, d(A_n - A)\rangle \\
&= \int_X \langle d^* F_{A_n}, A_n - A\rangle - \int_X \langle d^* F_A, A_n - A\rangle \\
&= d(\mathcal{SW}_\alpha)_{(A_n,\phi_n)}(A_n - A) - 4\int_X \langle \Phi^*(\nabla^{A_n}\phi_n), A_n - A\rangle \\
&\quad - d(\mathcal{SW}_\alpha)_{(A,\phi)}(A_n - A) - 4\int_X \langle \Phi^*(\nabla^{A}\phi), A_n - A\rangle + o(1) \\
&= -4\left\{\int_X \langle \Phi^*(\nabla^{A_n}\phi_n), A_n - A\rangle + \int_X \langle \Phi^*(\nabla^{A}\phi), A_n - A\rangle\right\} \\
&\quad + o(1), \quad \lim_{n\to\infty} o(1) = 0.
\end{aligned} \tag{3.34}$$

Thus, it follows from Lemma 3.5 that $\lim_{n\to\infty} \|A_n - A\|_{L^{1,2}} = 0$, and consequently, $A_n \to A$ strongly in $L^4$.

(2) $\lim_{n\to\infty} \|\phi_n - \phi\|_{L^{1,2}} = 0$.

$$\|\nabla^0\phi_n - \nabla^0\phi\|_{L^2}^2 = \overbrace{\int_X \langle \nabla^0\phi_n, \nabla^0(\phi_n - \phi)\rangle}^{(1)} - \overbrace{\int_X \langle \nabla^0\phi, \nabla^0(\phi_n - \phi)\rangle}^{(2)}. \tag{3.35}$$

The term (1) leads to

$$\begin{aligned}
&\int_X \langle \nabla^0\phi_n, \nabla^0(\phi_n - \phi)\rangle \\
&\quad= \int_X \langle (\nabla^{A_n} - A_n)\phi_n, (\nabla^{A_n} - A_n)(\phi_n - \phi)\rangle \\
&\quad= \int_X \langle \nabla^{A_n}\phi_n, \nabla^{A_n}(\phi_n - \phi)\rangle - \int_X \langle \nabla^{A_n}\phi_n, A_n(\phi_n - \phi)\rangle \\
&\qquad - \int_X \langle A_n\phi_n, \nabla^{A_n}(\phi_n - \phi)\rangle + \int_X \langle A_n\phi_n, A_n(\phi_n - \phi)\rangle \\
&\quad= \overbrace{d(\mathcal{SW}_\alpha)_{(A_n,\phi_n)}(\phi_n - \phi) - \int_X \frac{|\phi_n|^2 + k_g}{4}\langle \phi_n, \phi_n - \phi\rangle}^{(11)} \\
&\qquad - \overbrace{\int_X \langle \nabla^{A_n}\phi_n, A_n(\phi_n - \phi)\rangle}^{(12)} - \overbrace{\int_X \langle A_n\phi_n, \nabla^{A_n}(\phi_n - \phi)\rangle}^{(13)} \\
&\qquad + \overbrace{\int_X \langle A_n\phi_n, A_n(\phi_n - \phi)\rangle}^{(14)}.
\end{aligned} \tag{3.36}$$

The term (2) in (3.35) leads to similar terms named (21), (22), (23), and (24). We analyze each one of the above-obtained overbraced terms.

(a) Terms (11) and (21):

$$\begin{aligned}
&d(\mathcal{SW}_\alpha)_{(A_n,\phi_n)}(\phi_n - \phi) - \int_X \frac{|\phi_n|^2 + k_g}{4}\langle \phi_n, \phi_n - \phi\rangle + o(1) \\
&\quad= \langle \sigma, \phi_n - \phi\rangle - \int_X \frac{|\phi_n|^2 + k_g}{4}|\phi_n - \phi|^2 - \int_X \frac{|\phi_n|^2 + k_g}{4}\langle \phi, \phi_n - \phi\rangle + o(1) \\
&\quad\le \langle \sigma, \phi_n - \phi\rangle - \int_X \frac{|\phi_n|^2 + k_g}{4}\langle \phi, \phi_n - \phi\rangle + o(1) \\
&\quad\le \|\sigma\|_{L^2}^2 \cdot \|\phi_n - \phi\|_{L^2}^2 + \left\|\frac{|\phi_n|^2 + k_g}{4}\right\|_{L^2}^2 \cdot \|\phi\|_\infty \cdot \|\phi_n - \phi\|_{L^2}^2 + o(1),
\end{aligned} \tag{3.37}$$

where $\lim_{n\to\infty} o(1) = 0$. By the similarity between (11) and (21), we conclude the boundedness of term (22).

(b) Terms (12) and (22):

(i) term (12):

$$\begin{aligned}
&\int_X \langle \nabla^{A_n}\phi_n, A_n(\phi_n-\phi)\rangle \\
&\quad = \int_X \langle \nabla^{A_n}\phi_n, (A_n-A)(\phi_n-\phi)\rangle + \int_X \langle \nabla^{A_n}\phi_n, A(\phi_n-\phi)\rangle \\
&\quad \le \int_X |\nabla^{A_n}\phi_n|^2 \cdot \int_X |A_n-A|^4 \cdot \int_X |\phi_n-\phi|^4 \\
&\qquad + \int |\nabla^{A_n}\phi_n|^2 \cdot \int_X |A(\phi_n-\phi)|^2,
\end{aligned} \tag{3.38}$$

(ii) term (22)

$$\int_X \langle \nabla^A\phi, A(\phi_n-\phi)\rangle \le \int_X |\nabla^A\phi|^2 \cdot \int_X |A(\phi_n-\phi)|^2. \tag{3.39}$$

The term $\int_X |\nabla^A\phi|^2$ is bounded by Proposition 4.1 and $A \in C^0$ by Theorem 4.4.

(c) Term {(13)-(23)}:

$$\begin{aligned}
&\int_X \langle A_n\phi_n, \nabla^{A_n}(\phi_n-\phi)\rangle - \int_X \langle A\phi, \nabla^A(\phi_n-\phi)\rangle \\
&\quad = \int_X \langle (A_n-A)\phi_n, \nabla^{A_n}(\phi_n-\phi)\rangle + \overbrace{\int_X \langle A\phi_n, \nabla^{A_n}(\phi_n-\phi)\rangle}^{\text{(i)}} \\
&\qquad - \int_X \langle (A_n-A)\phi, \nabla^A(\phi_n-\phi)\rangle - \overbrace{\int_X \langle A_n\phi, \nabla^A(\phi_n-\phi)\rangle}^{\text{(ii)}}.
\end{aligned} \tag{3.40}$$

In each of the last two lines above, the first terms are bounded by $\|A_n - A\|_{L^4}$, while the term {(i)-(ii)} can be written as

$$\begin{aligned}
&\int_X \langle (A-A_n)\phi_n, \nabla^{A_n}(\phi_n-\phi)\rangle + \int_X \langle A_n(\phi_n-\phi), \nabla^{A_n}(\phi_n-\phi)\rangle \\
&\quad + \int_X \langle A_n\phi, (\overbrace{\nabla^{A_n} - \nabla^A}^{(A_n-A)})(\phi_n-\phi)\rangle.
\end{aligned} \tag{3.41}$$

So, it is also bounded by $\|A_n - A\|_{L^4}$.

(d) Term {(14)-(24)}:

$$\begin{aligned}
&\int_X \langle A_n\phi_n, A_n(\phi_n-\phi)\rangle - \int_X \langle A\phi, A(\phi_n-\phi)\rangle \\
&\quad = \int_X \langle A_n\phi_n, (A_n-A)(\phi_n-\phi)\rangle + \int_X \langle (A_n-A)\phi_n, A(\phi_n-\phi)\rangle \\
&\qquad + \int |A(\phi_n-\phi)|^2.
\end{aligned} \tag{3.42}$$

Since $A \in C^0$, it follows that $\lim_{n\to\infty} \|A(\phi_n - \phi)\|^2 = 0$. □

**4. Regularity of the solution** $(A,\phi)$

Let $\beta = \{e_i;\ 1 \le i \le 4\}$ be an orthonormal frame fixed on $TX$ with the following properties; for all $i,j \in \{1,2,3,4\}$:

(1) $[e_i,e_j] = 0$,

(2) $\nabla_{e_i} e_j = 0$ ($\nabla$ = Levi-Civita connection on $X$).

Let $\beta^* = \{dx_1,\dots,dx_n\}$ be the dual frame induced on $\mathcal{S}_\alpha^*$. From the 2nd property of the frame $\beta$, it follows that $\nabla_{e_i} dx^j = 0$ for all $i,j \in \{1,2,3,4\}$. For the sake of simplicity, let $\nabla^A_{e_i} = \nabla^A_i$. Therefore, $\nabla^A : \Omega^0(\mathrm{ad}(\mathrm{u}_1)) \to \Omega^1(\mathrm{ad}(\mathrm{u}_1))$ is given by

$$\begin{gathered}\nabla^A\phi = \sum_l (\nabla^A_l\phi)dx_l \Longrightarrow |\nabla^A\phi|^2 = \sum_l |\nabla^A_l\phi|^2,\\ (\nabla^A)^2 = \sum_{k,l}(\nabla^A_k\nabla^A_l\phi)dx_l\wedge dx_k \Longrightarrow |(\nabla^A)^2|^2 = \sum_{k,l}|\nabla^A_k\nabla^A_l\phi|^2.\end{gathered} \tag{4.1}$$

In this setting, the 2 form of curvature of the connection $A$ is given by

$$(F_A)_{kl} = F_{kl} = \nabla^A_l\nabla^A_k - \nabla^A_k\nabla^A_l. \tag{4.2}$$

In order to compute the operator $\Delta_A = (\nabla^A)^*\nabla^A : \Omega^0(\mathcal{S}^+_\alpha) \to \Omega^0(\mathcal{S}^+_\alpha)$, let $* : \Omega^i(\mathcal{S}_\alpha) \to \Omega^{4-i}(\mathcal{S}_\alpha)$ be the Hodge operator and consider the identity

$$(\nabla^A)^* = -*\nabla^A* : \Omega^1(\mathcal{S}^+_\alpha) \longrightarrow \Omega^0(\mathcal{S}^+_\alpha). \tag{4.3}$$

Hence,

$$\Delta_A\phi = -\sum_k \nabla^A_k\nabla^A_k\phi. \tag{4.4}$$

In this way,

$$\begin{aligned}|\Delta_A\phi|^2 &= \sum_{k,l}\langle\nabla^A_k\nabla^A_k\phi,\nabla^A_l\nabla^A_l\phi\rangle\\ &= \sum_{k,l}\big[\nabla^A_k(\langle\nabla^A_k\phi,\nabla^A_l\nabla^A_l\phi\rangle) - \langle\nabla^A_k\phi,\nabla^A_k\nabla^A_l\nabla^A_l\phi\rangle\big]\\ &= \sum_{k,l}\big[\nabla^A_k(\langle\nabla^A_k\phi,\nabla^A_l\nabla^A_l\phi\rangle) - \langle\nabla^A_k\phi,\nabla^A_l\nabla^A_k\nabla^A_l\phi\rangle - \langle\nabla^A_k\phi,F_{lk}\nabla^A_l\phi\rangle\big]\\ &= \sum_{k,l}\big[\nabla^A_k(\langle\nabla^A_k\phi,\nabla^A_l\nabla^A_l\phi\rangle) - \nabla^A_l(\langle\nabla^A_k\phi,\nabla^A_k\nabla^A_l\phi\rangle)\big]\\ &\quad+\sum_{k,l}\big[\langle\nabla^A_l\nabla^A_k\phi,\nabla^A_k\nabla^A_l\phi\rangle + \langle\nabla^A_k\phi,F_{lk}\nabla^A_l\phi\rangle\big]\\ &= \sum_{k,l}\big[\nabla^A_k(\langle\nabla^A_k\phi,\nabla^A_l\nabla^A_l\phi\rangle) - \nabla^A_l(\langle\nabla^A_k\phi,\nabla^A_k\nabla^A_l\phi\rangle)\big] + \sum_{k,l}|\nabla^A_k\nabla^A_l\phi|^2\\ &\quad+\sum_{k,l}\big[\langle F_{kl}\phi,\nabla^A_k\nabla^A_l\phi\rangle + \langle\nabla^A_k\phi,F_{kl}\nabla^A_l\phi\rangle\big]\end{aligned} \tag{4.5}$$

and so,

$$\begin{aligned}|(\nabla^A)^2\phi|^2 \le |\Delta_A\phi|^2 &+ \sum_{k,l}\{|\nabla_k^A(\langle\nabla_k^A\phi,\nabla_l^A\nabla_l^A\phi\rangle)|\} + \sum_{k,l}\{|\nabla_l^A(\langle\nabla_k^A\phi,\nabla_k^A\nabla_l^A\phi\rangle)|\} \\ &+ \sum_{k,l}\{|\langle F_{kl}\phi,\nabla_k^A\phi\nabla_l^A\phi\rangle|\} + \sum_{k,l}\{|\langle\nabla_k^A\phi,F_{kl}\nabla_l^A\phi\rangle|\}.\end{aligned} \tag{4.6}$$

Now, by applying the inequalities

$$\Big(\sum_i a_i\Big)^r \le K_r\cdot\sum_i|a_i|^r, \qquad \sqrt{\sum_{i=1}^n a_i} \le \sum_{i=1}^n\sqrt{a_i} \tag{4.7}$$

to (4.6), we get

$$\begin{aligned}|(\nabla^A)^2\phi|^p \le K_p\cdot|\Delta_A\phi|^p &+ K_p\cdot\sum_{k,l}\{|\nabla_k^A(\langle\nabla_k^A\phi,\nabla_l^A\nabla_l^A\phi\rangle)|^{p/2}\} \\ &+ K_p\sum_{k,l}\{|\nabla_l^A(\langle\nabla_k^A\phi,\nabla_k^A\nabla_l^A\phi\rangle)|^{p/2}\} \\ &+ \sum_{k,l}\{|\langle F_{kl}\phi,\nabla_k^A\phi\nabla_l^A\phi\rangle|^{p/2}\} + \sum_{k,l}\{|\langle\nabla_k^A\phi,F_{kl}\nabla_l^A\phi\rangle|^{p/2}\}.\end{aligned} \tag{4.8}$$

After integrating, it follows that

$$\begin{aligned}k_1\cdot\|(\nabla^A)^2\phi\|_{L^p}^p \le \|\Delta_A\phi\|_{L^p}^p &+ k_2\cdot\|\nabla^A\phi\|_{L^p}^p + k_3\cdot\|F_A(\phi)\|_{L^p}^p \\ &+ k_4\cdot\|F_A(\nabla^A\phi)\|_{L^p}^p + k_5\cdot\sum_{k,l}\int_x\{|\nabla_k^A(\langle\nabla_k^A\phi,\nabla_l^A\nabla_l^A\phi\rangle)|^{p/2}\} \\ &+ k_6\sum_{k,l}\int_X\{|\nabla_l^A(\langle\nabla_k^A\phi,\nabla_k^A\nabla_l^A\phi\rangle)|^{p/2}\}.\end{aligned} \tag{4.9}$$

The boundedness of the right-hand side of (4.9) results from the analysis of each term.

Proposition 4.1. *Let $(A,\phi)\in\mathscr{C}_\alpha$ be a solution of equations in (1.7). If $\sigma\in L^\infty$, then*

(1) $\nabla^A\phi\in L^2$,
(2) $\Delta_A\phi\in L^2$.

*Proof.* (1) $\nabla^A\phi\in L^2$:

$$\begin{aligned}&\langle\Delta_A\phi,\phi\rangle + \Big(\frac{|\phi|^2+k_g}{4}\Big)|\phi|^2 = \langle\sigma,\phi\rangle \\ &\Longrightarrow |\nabla^A\phi|^2 + \Big(\frac{|\phi|^2+k_g}{4}\Big)|\phi|^2 = \langle\sigma,\phi\rangle \le \frac{1}{\epsilon^2}|\sigma|^2+\epsilon^2|\phi|^2.\end{aligned} \tag{4.10}$$

Therefore,

$$|\nabla^A\phi|^2 < \frac{1}{\epsilon^2}|\sigma|^2 + \Big(\epsilon^2-\frac{k_g}{4}\Big)|\phi|^2 - \frac{|\phi|^4}{4}. \tag{4.11}$$

From Lemma 3.4, there exists a polynomial $p$, with coefficients depending on $(X,g)$ and $\epsilon$, such that

$$||\nabla^A\phi||^2_{L^2} < p(\|\sigma\|_\infty). \tag{4.12}$$

So, $\nabla^A\phi \in L^2$.

(2) $\Delta_A\phi \in L^2$:

$$\langle \Delta_A\phi, \Delta_A\phi\rangle + \frac{|\phi|^2 + k_g}{4}\langle \phi, \Delta_A\phi\rangle = \langle \sigma, \Delta_A\phi\rangle; \tag{4.13}$$

let $0 < \epsilon < 1$,

$$\begin{gathered} |\Delta_A\phi|^2 + \frac{|\phi|^2 + k_g}{4}|\nabla^A\phi|^2 = \langle \sigma, \Delta_A\phi\rangle < \frac{1}{\epsilon^2}|\sigma|^2 + \epsilon^2|\Delta_A\phi|^2, \\ (1-\epsilon^2)|\Delta_A\phi|^2 + \frac{|\phi|^2 + k_g}{4}|\nabla^A\phi|^2 < \frac{1}{\epsilon^2}|\sigma|^2. \end{gathered} \tag{4.14}$$

By the boundedness of the term

$$\int_X |\phi|^2 \cdot |\nabla^A\phi|^2 < \|\phi\|^2_\infty \cdot ||\nabla^A\phi||^2_{L^2}, \tag{4.15}$$

one deduces the existence of a polynomial $q$, with coefficients depending on $\epsilon$ and $(X,g)$, such that

$$||\Delta_A\phi||_{L^2} < q(\|\sigma\|_\infty). \tag{4.16}$$

□

Proposition 4.2. *Let $(A,\phi)$ be solutions of the $\mathcal{SW}_\alpha$-equations, where $(\Theta,\sigma) \in L^{1,2} \times (L^{1,2} \cap L^\infty)$, then $F_A \in L^q$, for all $q < \infty$.*

*Proof.* By (1.8), $\Phi^*(\nabla^A\phi) = (1/2)\nabla^A(|\phi|^2)$, and so,

$$d^*F_A + 4\Phi^*(\nabla^A\phi) = \Theta \Longrightarrow ||d^*F_A||^2_{L^2} \le \|\phi\|^2_{L^{1,2}} + \|\Theta\|_{L^2}. \tag{4.17}$$

There are two cases to be analysed.

(1) $F_A$ is harmonic. Since the Laplacian defined on $\mathfrak{u}_1$-forms is an elliptic operator, the fundamental inequality for elliptic operators asserts that there exists a constant $C_k$ such that

$$||F_A||_{L^{k+2,2}} \le ||\Delta F_A||_{L^{k,2}} + C_k||F_A||_{L^2}. \tag{4.18}$$

Consequently, $F_A$ being harmonic implies, for all $k \in \mathbb{N}$, that

$$||F_A||_{L^{k,2}} \le C_k||F_A||_{L^2} \Longrightarrow F_A \in C^\infty. \tag{4.19}$$

(2) $F_A$ is not harmonic. In this case, since $\Theta \in L^{1,2}$, $\phi \in L^\infty$ and

$$\Delta_A F_A = d(\langle \phi, \nabla^A\phi\rangle) + d\Theta = \langle \phi, F_A(\phi)\rangle + d\Theta, \tag{4.20}$$

it follows that $F_A \in L^{2,2}$. Therefore, by the Sobolev embedding theorem, $F_A \in L^q$, for all $q < \infty$. □

Proposition 4.3. *Let $(A,\phi)$ be solutions of the $\mathcal{SW}_\alpha$-equations, where $(\Theta,\sigma) \in L^{1,2} \times (L^{1,2} \cap L^\infty)$, then $(\nabla^A)^2\phi \in L^p$, for all $1 < p < 2$.*

*Proof.* In (4.9), we must take care of the last terms.

(1) $F(\nabla^A\phi) \in L^p$, for all $1 < p < 2$. By Young's inequality,

$$||F(\nabla^A\phi)||_{L^p} \leq ||F_A||_{L^{2p/(2-p)}} \cdot ||\nabla^A\phi||_{L^2}. \tag{4.21}$$

(2) There is no contribution from the divergent terms, since

$$\int_x \{|\nabla_k^A(\langle\nabla_k^A\phi, \nabla_l^A\nabla_l^A\phi\rangle)|^{p/2}\} \leq [\mathrm{vol}(X)]^{(2-p)/p}\int_x \{|\nabla_k^A(\langle\nabla_k^A\phi, \nabla_l^A\nabla_l^A\phi\rangle)|\}. \tag{4.22}$$

In the same way,

$$\begin{aligned} &\sum_{k,l}\int_x \{|\nabla_k^A(\langle\nabla_k^A\phi, \nabla_l^A\nabla_l^A\phi\rangle)|^{p/2}\} = 0, \\ &\sum_{k,l}\int_X \{|\nabla_l^A(\langle\nabla_k^A\phi, \nabla_k^A\nabla_l^A\phi\rangle)|^{p/2}\} = 0. \end{aligned} \tag{4.23}$$

The estimates above applied to (4.9) implies that

$$\begin{aligned} ||(\nabla^A)^2\phi||_{L^p} \leq{}& k_1||\Delta_A\phi||_{L^p}^p + k_2||\nabla^A\phi||_{L^p}^p + k_3||\nabla^A\phi||_{L^p}^p \\ &+ k_4||F_A(\phi)||_{L^p}^p + k_5||F_A||_{L^{p/(2-p)}} \cdot ||\nabla^A\phi||_{L^p}^p. \end{aligned} \tag{4.24}$$

□

Thus, $\phi \in L^{2,p}$, for all $1 < p < 2$. Considering that $\sigma \in L^{1,2}$, the bootstrap argument applied on (1.7) implies that $\phi \in L^{3,p}$, for every $k \geq 2$ and $1 < p < 2$. Hence, by Sobolev embedding theorem, $\phi \in C^0$.

Theorem 4.4. *Let $(A,\phi)$ be a solution of the $\mathcal{SW}_\alpha$-equations, where $(\Theta,\sigma) \in L^{k,2}(\Omega^1(\mathrm{ad}(\mathfrak{u}_1))) \oplus (L^{k,2}(\Gamma(\mathcal{S}_\alpha^+)) \cap L^\infty(\Gamma(\mathcal{S}_\alpha^+)))$, then $(A,\phi) \in L^{k+2,p} \times (L^{k+2,2} \cap L^\infty)$, for all $1 < p < 2$. Moreover, if $k > 2$, then $(A,\phi) \in C^r \times C^r$, for all $r < k$.*

*Proof.* (1) If $\Theta \in L^{k,2}$, then by Proposition 4.2 $F_A \in L^{k+1,2}$. Consequently, by Corollary 2.2, $A \in L^{k+2,2}$.

(2) The Sobolev class of $\phi$ is obtained by the bootstrap argument. □

## Appendix

### Estimates for solutions of 3rd-degree equation

Let $p, q \in \mathbb{R}$ and consider the equation

$$x^3 + px + q = 0. \tag{A.1}$$

Proposition A.1. *The solutions of (A.1) are given in* [2] *by*

$$x_1 = z_1 + z_2, \qquad x_2 = z_1 + \lambda z_2, \qquad y_3 = z_1 + \lambda^2 z_2, \tag{A.2}$$

*where*

$$z_1 = \sqrt[3]{-\frac{q}{2} + \sqrt[2]{D}}, \quad z_2 = \sqrt[3]{-\frac{q}{2} - \sqrt[2]{D}}, \quad D = \frac{p^3}{27} + \frac{q^2}{4}, \tag{A.3}$$

*and* $\lambda \in \mathbb{C}$ *satisfies* $\lambda^3 = 1$.

Corollary A.2. *Let $p$ and $q$ be negative real numbers. So, the solutions of (A.1) are estimated according to the following cases:*

(1) $D \geq 0$:

$$|x_i| \leq \frac{8}{3} + \frac{1}{3}|q| + \frac{1}{12}q^2 + \frac{1}{81}p^3, \tag{A.4}$$

(2) $D < 0$:

$$|x_i| \leq 3 + \frac{1}{6}q^2 + \frac{1}{81}|p|^3. \tag{A.5}$$

*Proof.* Since

$$|x_i| \leq |z_1| + |z_2|, \tag{A.6}$$

it is enough to estimate $|z_1|$ and $|z_2|$. The basics identities needed are the following: suppose $x \geq 0$, whence

$$\sqrt[2]{x} \leq 1 + \frac{1}{2}x, \qquad \sqrt[3]{x} \leq 1 + \frac{1}{3}x. \tag{A.7}$$

(1) $D \geq 0$. In this case, $z_1, z_2 \in \mathbb{R}$ and

$$|z_1| = \sqrt[3]{\left| -\frac{q}{2} + \sqrt[2]{D} \right|} \leq 1 + \frac{1}{3}\left| -\frac{q}{2} + \sqrt[2]{D} \right| \leq \frac{4}{3} + \frac{1}{6}|q| + \frac{1}{6}D. \tag{A.8}$$

Thus,

$$|z_1| \leq \frac{4}{3} + \frac{1}{6}|q| + \frac{1}{24}q^2 + \frac{1}{162}p^3. \tag{A.9}$$

The same estimate can be obtained for $|z_2|$. Hence,

$$|x_i| \leq \frac{8}{3} + \frac{1}{3}|q| + \frac{1}{12}q^2 + \frac{1}{81}p^3. \tag{A.10}$$

(2) $D \leq 0$. In this case, $z_1, z_2 \in \mathbb{C} - \mathbb{R}$. Since $D \in \mathbb{R}$, we can write $\sqrt[2]{D} = i\sqrt[2]{|D|}$ and

$$z_1 = \sqrt[3]{-\frac{1}{2}q + i\sqrt[2]{D}}, \qquad z_2 = \sqrt[3]{-\frac{1}{2}q - i\sqrt[2]{D}}. \tag{A.11}$$

Therefore,

$$\begin{gathered} |z_i|^2 = \sqrt[3]{\frac{q^2}{4} + |D|} < 1 + \frac{1}{12}q^2 + \frac{1}{3}|D| \leq 1 + \frac{1}{6}q^2 + \frac{1}{81}|p|^3, \\ |z_i| < \frac{3}{2} + \frac{1}{12}q^2 + \frac{1}{162}|p|^3. \end{gathered} \tag{A.12}$$

Hence,

$$|x_i| < 3 + \frac{1}{6}q^2 + \frac{1}{81}|p|^3. \tag{A.13}$$

□

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Celso Melchiades Doria: Departamento de Matemática, Universidade Federal de Santa Catarina, Campus Universitario, Trindade, 88040900 Florianópolis - SC, Brazil

*E-mail address*: cmdoria@mtm.ufsc.br