ON BOUNDARY VALUE PROBLEMS FOR SECOND-ORDER DISCRETE INCLUSIONS

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Received 8 October 2004

We prove some existence theorems regarding solutions to boundary value problems for systems of second-order discrete inclusions. For a certain class of right-hand sides, we present some lemmas showing that all solutions to discrete second-order inclusions satisfy an *a priori* bound. Then we apply these *a priori* bounds, in conjunction with an appropriate fixed point theorem for inclusions, to obtain the existence of solutions. The theory is highlighted with several examples.

1. Introduction

The theory of differential inclusions has received much attention due to its versatility and generality. For example, differential inclusions can accurately model discontinuous processes, such as systems with dry friction; the work of an electric oscillator; and autopilot (and other) control systems [8]. When considering these (or other) situations in discrete time, the modeling process gives rise to a discrete (or difference) inclusion, rather than a differential inclusion. In many cases, considering the model in discrete time gives a more precise or realistic description [1].

Let X and Y be two normed spaces. A *set-valued map* $G: X \to Y$ is a map that associates with any $x \in X$ a set $G(x) \subset Y$. By $CK(E)$, we denote the set of nonempty, convex, and closed subsets of a Banach space E. We say that $G: \mathbb{R}^n \to CK(\mathbb{R}^n)$ is *upper semicontinuous* if for all sequences $\{u_i\} \subseteq \mathbb{R}^n$, $\{v_i\} \subseteq \mathbb{R}^n$, where $i \in \mathbb{N}$, the conditions $u_i \to u_0$, $v_i \to v_0$, and $v_i \in G(u_i)$ imply that $v_0 \in G(u_0)$. Since the upper semicontinuity plays an essential role in this paper, we illustrate this notion by the simple example [5, Example 4.1.1].

Example 1.1. The set-valued map $f_1: \mathbb{R} \to \mathbb{R}$ defined by

$$f_1(t) = \begin{cases} \{0\} & \text{for } t = 0, \\ [0,1] & \text{for } t \in \mathbb{R} \setminus \{0\}, \end{cases} \tag{1.1}$$

is not upper semicontinuous. On the other hand, the set-valued map $f_2: \mathbb{R} \to \mathbb{R}$ defined

Boundary Value Problems 2005:2 (2005) 153–163
DOI: 10.1155/BVP.2005.153

by

$$f_2(t) = \begin{cases} [0,1] & \text{for } t = 0, \\ \{0\} & \text{for } t \in \mathbb{R} \setminus \{0\}, \end{cases} \tag{1.2}$$

is upper semicontinuous.

For more information about set-valued maps and differential inclusions, see Aubin and Cellina [3], Smirnov [8], or Erbe, Ma and Tisdell [6].

We are interested in the following boundary value problem (BVP) for second-order discrete inclusions:

$$\Delta^2 y(k-1) \in F(k, y(k), \Delta y(k)), \quad k = 1, \dots, T, \qquad y(0) = A, \qquad y(T+1) = B, \tag{1.3}$$

where $A, B \in \mathbb{R}^d$ are constants and $F : \{1,\dots,T\} \times \mathbb{R}^d \times \mathbb{R}^d \to CK(\mathbb{R}^d)$ is a set-valued map. A solution $\bar{y} = \{y(k)\}_{k=0}^{T+1} \in \mathbb{R}^{(T+2)d}$ to (1.3) is a vector $\bar{y} = \{y(0), \dots, y(T+1)\}$ such that each element $y(k) \in \mathbb{R}^d$ satisfies the discrete inclusion for $k = 1, \dots, T$ and the boundary conditions for $k = 0$ and $k = T+1$.

In Section 2, we show that under certain conditions on the right-hand side F, all solutions of (1.3) are bounded. The inequalities employed rely on growth conditions on F and on appropriate discrete maximum principles.

Section 3 contains the appropriate operator formulations for (1.3) to be considered as a fixed point problem.

In Section 4, we apply the results of Sections 2 and 3 to prove the existence of solutions to (1.3), in conjunction with the following fixed-point theorem [2, Theorem 1.2].

Theorem 1.2. *Let E be a Banach space, U an open subset of E, and $0 \in U$. Suppose that $P : \overline{U} \to CK(E)$ is an upper semicontinuous and compact map. Then either*

(A1) *P has a fixed point in $\overline{U}$, or*

(A2) *there exist $u \in \partial U$ and $\lambda \in (0,1)$ with $u \in \lambda P(u)$.*

To prove the compactness of the image of an upper semicontinuous map, we will use a criterion which can be found in Berge [4, théorème VI.3].

Theorem 1.3. *Let $P : X \to Y$ be an upper semicontinuous map. If K is a compact set in X, then $P(K)$ is a compact set in Y.*

In [2], Agarwal et al. gave conditions under which the following BVP has at least one solution:

$$\Delta^2 y(k-1) \in F(k, y(k)), \quad k = 1, \dots, T, \qquad y(0) = 0, \qquad y(T+1) = 0, \tag{1.4}$$

where $F : \{1,\dots,T\} \times \mathbb{R} \to CK(\mathbb{R})$. In comparison with the results and conditions in [2], we introduce new inequalities unrelated to those in [2] and we also extend some of the results in [2].

2. A priori bound

In this section, we prove two different *a priori* bound results for the following system of BVPs for second-order discrete inclusions:

$$\Delta^2 y(k-1) \in \lambda F(k, y(k), \Delta y(k)), \quad k = 1, \dots, T, \qquad y(0) = \lambda A, \qquad y(T+1) = \lambda B, \tag{2.1}$$

where $\lambda \in [0,1]$.

The study of the above family of BVPs is motivated by the family of inclusions in Theorem 1.2, $u \in \lambda P(u)$.

We denote $\langle \cdot, \cdot \rangle$ as the Euclidean inner product and by $\| \cdot \|$ the Euclidean norm on $\mathbb{R}^n$.

Lemma 2.1. *Let $F : \{1, \dots, T\} \times \mathbb{R}^d \times \mathbb{R}^d \to CK(\mathbb{R}^d)$ be a set-valued map. If there exist constants $\alpha \geq 0$ and $K \geq 0$ such that*

$$\|\phi\| \leq \alpha(2\langle p, \phi \rangle + \|q\|^2) + K, \quad \textit{for } k = 1, \dots, T \textit{ and all } (p,q) \in \mathbb{R}^{2d},\ \phi \in F(k,p,q), \tag{2.2}$$

then all solutions $\bar{y}$ of BVP for the system of discrete inclusions (2.1) satisfy

$$||y(k)|| < R, \quad k = 0, \dots, T+1, \tag{2.3}$$

for $\lambda \in [0,1]$, and R is defined by

$$R := \alpha\beta^2 + \beta + K\frac{(T+1)^2}{8} + 1, \quad \beta := \max\{\|A\|, \|B\|\}. \tag{2.4}$$

Proof. We suppose that $\bar{y}$ is a solution of (2.1). Since we work on a discrete topology, every solution of (2.1) is a solution of a system of discrete BVPs,

$$\Delta^2 y(k-1) = \lambda \hat{f}(k, y(k), \Delta y(k)), \quad k = 1, \dots, T, \qquad y(0) = \lambda A, \qquad y(T+1) = \lambda B, \tag{2.5}$$

where $\hat{f} : \{1, \dots, T\} \times \mathbb{R}^d \times \mathbb{R}^d \to CK(\mathbb{R}^d)$ is a single-valued function such that $\hat{f}(k,p,q) \in F(k,p,q)$ for every $k = 1, \dots, T$ and $(p,q) \in \mathbb{R}^{2d}$. In the theory of the set-valued map, $\hat{f}$ is called a *selector* of F.

Then $\bar{y}$ solves the summation equation

$$y(k) = \lambda \Phi(k) + \lambda \sum_{l=1}^{T} G(k,l) \hat{f}(l, y(l), \Delta y(l)), \quad k = 0, \dots, T+1, \tag{2.6}$$

where

$$\lambda \Phi(k) = \lambda \frac{A(T+1) + (B-A)k}{T+1}, \tag{2.7}$$

and $G:\{0,\dots,T+1\}\times\{1,\dots,T\}\to\mathbb{R}^d$ defined by

$$G(k,l)=\begin{cases}-\dfrac{1}{T+1}l(T+1-k), & l=1,\dots,k-1,\\ -\dfrac{1}{T+1}k(T+1-l), & l=k,\dots,T,\end{cases} \tag{2.8}$$

is Green's function for the BVP,

$$\Delta^2 y(k-1)=0,\quad k=1,\dots,T,\qquad y(0)=0,\qquad y(T+1)=0. \tag{2.9}$$

Since $\|\lambda\Phi(k)\|\le\beta$ for each $k=0,\dots,T+1$, we obtain that

$$||y(k)||\le\beta+\sum_{l=1}^{T}|G(k,l)|\lambda\left|\left|\hat{f}(l,y(l),\Delta y(l))\right|\right|. \tag{2.10}$$

Using (2.2), we have that

$$\begin{aligned}||y(k)||&\le\beta+\sum_{l=1}^{T}|G(k,l)|\lambda\left\{\alpha\left[2\langle y(l),\hat{f}(l,y(l),\Delta y(l))\rangle+||\Delta y(l)||^2\right]+K\right\}\\&\le\beta+\sum_{l=1}^{T}|G(k,l)|\left\{\alpha\left[2\langle y(l),\lambda\hat{f}(l,y(l),\Delta y(l))\rangle+||\Delta y(l)||^2\right]+K\right\}.\end{aligned} \tag{2.11}$$

Define

$$r(k):=||y(k)||^2,\quad k=0,\dots,T+1, \tag{2.12}$$

and use the discrete product rule to calculate the second difference of r at the point $k-1$ to obtain

$$\Delta^2 r(k-1)=||\Delta y(k)||^2+2\langle y(k),\Delta^2 y(k-1)\rangle+||\Delta y(k-1)||^2. \tag{2.13}$$

By using the first equation in (2.5), we deduce that

$$\Delta^2 r(k-1)\ge||\Delta y(k)||^2+2\langle y(k),\lambda\hat{f}(k,y(k),\Delta y(k))\rangle. \tag{2.14}$$

We install this into (2.11) to obtain that

$$||y(k)||\le\beta+\alpha\sum_{l=1}^{T}|G(k,l)|\Delta^2 r(l-1)+\sum_{l=1}^{T}|G(k,l)|K. \tag{2.15}$$

Using (2.8) we make the following computations:

$$\begin{aligned}
\sum_{l=1}^{T}|G(k,l)|\Delta^2 r(l-1) &= \frac{T+1-k}{T+1}\sum_{l=1}^{k-1} l\Delta^2 r(l-1) + \frac{k}{T+1}\sum_{l=k}^{T}(T+1-l)\Delta^2 r(l-1)\\
&= \frac{T+1-k}{T+1}\left(\left[l\Delta r(l-1)\right]_1^k - \sum_{l=1}^{k-1}\Delta r(l)\right)\\
&\quad + \frac{k}{T+1}\left(\left[(T+1-l)\Delta r(l-1)\right]_k^{T+1} + \sum_{l=k}^{T}\Delta r(l)\right)\\
&= \frac{T+1-k}{T+1}\left(k\Delta r(k-1) - \Delta r(0) - r(k) + r(1)\right)\\
&\quad + \frac{k}{T+1}\left(-(T+1-k)\Delta r(k-1) + r(T+1) - r(k)\right)\\
&= \frac{T+1-k}{T+1}r(0) + \frac{k}{T+1}r(T+1) - r(k) \le \beta^2.
\end{aligned} \tag{2.16}$$

Finally, if we consider this estimation and (see, e.g., [7, Exercise 6.20])

$$\max_{k\in\{0,\dots,T+1\}}\sum_{l=1}^{T}|G(k,l)| \le \frac{(T+1)^2}{8}, \tag{2.17}$$

we rewrite (2.15) as

$$||y(k)|| \le \beta + \alpha\beta^2 + K\frac{(T+1)^2}{8} < R, \quad k = 0,\dots,T+1, \tag{2.18}$$

and this concludes the proof. □

Definition 2.2. Let $R>0$ be a constant. Define the set $D_R \subset \{1,\dots,T\}\times\mathbb{R}^d\times\mathbb{R}^d$ by the set containing all triplets (k,p,q) such that

$$k = 1,\dots,T, \qquad (p,q)\in\mathbb{R}^{2d} : \|p\| \ge R, \qquad 2\langle p,q\rangle + \|q\|^2 \le 0. \tag{2.19}$$

By using the same selector technique as above, but employing an unrelated inequality, we now prove the second *a priori* bound result.

Lemma 2.3. *Let $R>0$ be a constant such that*

$$\max\{\|A\|,\|B\|\} < R, \tag{2.20}$$

and let $F:\{1,\dots,T\}\times\mathbb{R}^d\times\mathbb{R}^d \to CK(\mathbb{R}^d)$ be a set-valued map. If

$$2\langle p,\phi\rangle + \|q\|^2 > 0, \quad \textit{for every } (k,p,q)\in D_R \textit{ and all } \phi\in F(k,p,q), \tag{2.21}$$

then all solutions $\bar{y}$ for the system of discrete inclusions (2.1) satisfy

$$||y(k)|| < R, \quad k = 0,\dots,T+1, \tag{2.22}$$

for $\lambda\in[0,1]$.

Proof. Suppose that $\bar{y}$ is a solution of (2.1). As in the previous proof, we can deduce that $\bar{y}$ is a solution of (2.5) for some selector $\widehat{f}(k,p,q) \in F(k,p,q)$.

Assume that the conclusion is not true. Then the function $r(k) := \|y(k)\|^2 - R^2$ must have a nonnegative maximum in $\{0,\ldots,T+1\}$. From the assumption (2.20), this maximum must be achieved in $\{1,\ldots,T\}$. Choose $c \in \{1,\ldots,T\}$ such that $r(c) = \max\{r(k);\, k \in \{1,\ldots,T\}\}$ and suppose that there is no $k < c$ for which $r(k) = r(c)$. This choice of c implies that the conditions

$$\Delta r(c) \leq 0, \tag{2.23}$$

$$\Delta^2 r(c-1) \leq 0 \tag{2.24}$$

must be satisfied simultaneously. Since

$$\begin{aligned} \Delta r(c) &= \langle y(c) + y(c+1), \Delta y(c) \rangle = \langle 2y(c) + \Delta y(c), \Delta y(c) \rangle \\ &= 2\langle y(c), \Delta y(c) \rangle + \|\Delta y(c)\|^2 \end{aligned} \tag{2.25}$$

holds, we rewrite (2.23) as

$$2\langle y(c), \Delta y(c) \rangle + \|\Delta y(c)\|^2 \leq 0. \tag{2.26}$$

Similarly as in the proof of Lemma 2.1, we obtain with the help of the product rule

$$\Delta^2 r(c-1) \geq \|\Delta y(c)\|^2 + 2\langle y(c), \widehat{f}(c, y(c), \Delta y(c)) \rangle > 0, \tag{2.27}$$

which contradicts (2.24). Hence, $\|y(k)\| < R$ for $k = 0,\ldots,T+1$. □

Lemma 2.3 is a natural extension to the lower and upper solution methods used in [2] for the case $d = 1$. If the right-hand side in (1.3) is a single-valued map, then we have the following corollary to Lemma 2.3 for systems of discrete BVPs.

Corollary 2.4. *Let $R > 0$ be a constant. If*

$$2\langle p, F(k,p,q) \rangle + \|q\|^2 > 0, \quad \forall (k,p,q) \in D_R, \tag{2.28}$$

$$\max\{\|A\|, \|B\|\} < R, \tag{2.29}$$

then all solutions $\bar{y}$ of (2.5) satisfy (2.22) for $\lambda \in [0,1]$.

Remark 2.5. Note that if $F(k, y(k), \Delta y(k)) = F(k, y(k))$, then in place of (2.28), we would require only

$$\langle p, F(k,p) \rangle > 0, \quad \forall k = 1,\ldots,T \text{ and all } p \in \mathbb{R}^d : \|p\| \geq R \tag{2.30}$$

to be satisfied.

The advantage of (2.19) rather than assuming (2.28) for, say, all $q \in \mathbb{R}^d$ is highlighted in the following example.

Example 2.6. Consider the single, scalar-valued function $f(k,p,q) = p^3 - q$. For all $q \in \mathbb{R}$, we have that

$$2pf(k,p,q) + q^2 = 2p^4 - 2pq + q^2, \tag{2.31}$$

and for all $|p| \geq R$, we can find $q \in \mathbb{R}$ such that (2.31) is negative. But on the reduced set D_R, the assumption (2.19) implies that $2pq + q^2 \leq 0$ ($-2pq \geq q^2$ equivalently), and thus

$$2pf(k,p,q) + q^2 = 2p^4 - 2pq + q^2 \geq 2p^4 + q^2 + q^2 > 0, \tag{2.32}$$

for all $(p,q) \in \mathbb{R}^2$, $|p| \geq R$, such that (2.19) holds for any $R > 0$. Hence we can use Corollary 2.4 to prove the *a priori* bound for the discrete BVP,

$$\Delta^2 y(k-1) = \lambda y^3(k) - \lambda \Delta y(k), \quad k = 1,\dots,T, \qquad y(0) = \lambda A, \qquad y(T+1) = \lambda B, \tag{2.33}$$

where $\lambda \in [0,1]$.

3. Operator formulation

In this section, we formulate the necessary operators to apply Theorem 1.2.

Solving (2.1) is equivalent to finding a vector $\bar{y} = \{y(k)\}_{k=0}^{T+1} \in \mathbb{R}^{(T+2)d}$ which satisfies

$$y(k) \in \lambda\Phi(k) + \lambda \sum_{l=1}^{T} G(k,l) F\big(l, y(l), \Delta y(l)\big), \quad k = 0,\dots,T+1, \tag{3.1}$$

where $\lambda\Phi$ is the function defined by (2.7) and $G : \{0,\dots,T+1\} \times \{1,\dots,T\} \to \mathbb{R}^d$, defined by (2.8), is the Green's function for the BVP

$$\Delta^2 y(k-1) = 0, \quad k = 0,\dots,T, \qquad y(0) = 0, \qquad y(T+1) = 0. \tag{3.2}$$

We suppose that $F : \{1,\dots,T\} \times \mathbb{R}^d \times \mathbb{R}^d \to CK(\mathbb{R}^d)$ then we can define the operator $\mathcal{F} : \mathbb{R}^{(T+2)d} \to CK(\mathbb{R}^{Td})$ by

$$\mathcal{F}(\overline{u}) := \big\{\overline{v} \in \mathbb{R}^{Td} : v(k) \in F\big(k, u(k), \Delta u(k)\big),\ k = 1,\dots,T\big\}, \tag{3.3}$$

and the operator $\mathcal{T} : \mathbb{R}^{Td} \to \mathbb{R}^{(T+2)d}$ by

$$\mathcal{T} y(k) := \sum_{l=1}^{T} G(k,l) y(l), \quad k = 1,\dots,T. \tag{3.4}$$

The discreteness of topology implies that $\mathcal{T}$ is continuous and linear, and thus, we have $\mathcal{T} \circ \mathcal{F} : \mathbb{R}^{(T+2)d} \to CK(\mathbb{R}^{(T+2)d})$. We can rewrite (3.1) as

$$\bar{y} \in \widehat{\mathcal{S}}(\bar{y}), \tag{3.5}$$

where $\widehat{\mathcal{S}} : \mathbb{R}^{(T+2)d} \to CK(\mathbb{R}^{(T+2)d})$ is defined by

$$\widehat{\mathcal{S}}(\bar{y}) := \lambda\mathcal{T} \circ \mathcal{F}(\bar{y}) + \lambda\Phi. \tag{3.6}$$

In order to use Theorem 1.2, we need to define a suitable open subset U of the Banach space $\mathbb{R}^{(T+2)d}$ and the operator $\mathcal{S} : \overline{U} \to CK(\mathbb{R}^{(T+2)d})$. Introduce $U \subset \mathbb{R}^{(T+2)d}$ by

$$U := \{\overline{u} \in \mathbb{R}^{(T+2)d} : ||u(k)|| < R,\ k = 0,\dots,T+1\}, \tag{3.7}$$

where R is the constant defined either in (2.4) or in Lemma 2.3. Next, we define the operator $\mathcal{S} : \overline{U} \to CK(\mathbb{R}^{(T+2)d})$ by

$$\mathcal{S} = \widehat{\mathcal{S}}|_{\overline{U}}. \tag{3.8}$$

To satisfy the remaining assumptions of Theorem 1.2 on the operator $\mathcal{S}$, we need to prove that it is upper semicontinuous and compact.

Lemma 3.1. *If F satisfies*

(US) *$F(k,p,q)$ is upper semicontinuous for all $(p,q) \in \mathbb{R}^{2d}$, for $k = 1,\dots,T$ and the assumptions either of Lemma 2.1 or Lemma 2.3 hold,*

then the operator $\mathcal{S}$ is upper semicontinuous and compact.

Proof. Consider the sequences $\{u_i\}_{i=1}^{\infty}$ and $\{w_i\}_{i=1}^{\infty}$ such that $w_i \in \mathcal{S}(u_i)$ and $u_i \to u_0$ and $w_i \to w_0$ as $i \to \infty$ in $\mathbb{R}^{(T+2)d}$. To prove our assertion, we must show that $w_0 \in \mathcal{S}(u_0)$. For each $i \in \mathbb{N}$, there exists $v_i \in \mathbb{R}^{Td}$ such that $w_i = \lambda\mathcal{T}v_i + \lambda\Phi$ and $v_i \in \mathcal{F}(u_i)$. Since condition (US) holds and $\overline{U}$ is a compact set, we can deduce from Theorem 1.3 that $\mathcal{F}(\overline{U})$ is a compact set. This implies that there exists at least a subsequence $\{v_{i_n}\}_{n=1}^{\infty}$ of $\{v_i\}_{i=1}^{\infty}$ such that $v_{i_n} \to v_0 \in \mathcal{F}(u_0)$. Since $\mathcal{T}$ is linear and continuous, we have that

$$w_i = \lambda\mathcal{T}v_i + \lambda\Phi \longrightarrow \lambda\mathcal{T}v_0 + \lambda\Phi, \tag{3.9}$$

and noting that $w_i \to w_0$ in $\mathbb{R}^{(T+2)d}$, we can conclude that

$$w_0 \in \mathcal{S}(u_0). \tag{3.10}$$

This proves that $\mathcal{S}$ is upper semicontinuous and we can use Theorem 1.3 to obtain the compactness of $\mathcal{S}$. □

4. Existence results and examples

In this section, we combine the theory of Sections 2 and 3 to formulate existence results.

Theorem 4.1. *If the set-valued map $F : \{1,\dots,T\} \times \mathbb{R}^d \times \mathbb{R}^d \to CK(\mathbb{R}^d)$ satisfies (US) and the assumptions of Lemma 2.1 hold, then the system of discrete inclusion boundary value problems (1.3) has a solution.*

Proof. We showed in the previous section that the problem of existence of a solution of (1.3) (where F satisfies the assumptions of Lemma 2.1) is equivalent to the problem of existence of a fixed point of $\mathcal{S}(\bar{y})$, where $\mathcal{S} : \overline{U} \to CK(\mathbb{R}^{(T+2)d})$ is defined in (3.8) and U is defined in (3.7). Since $\mathcal{S}$ is compact and upper semicontinuous (cf. Lemma 3.1), we are ready to use Theorem 1.2, and thanks to the conclusion of Lemma 2.1, we can exclude the possibility (A2) there. Therefore the operator $\mathcal{S}$ has a fixed point and the problem (1.3) has a solution. □

We illustrate the above result on the following example with $n = 1$.

Example 4.2. Let $\widetilde{F} : \{1,\dots,T\} \times \mathbb{R} \to CK(\mathbb{R})$ be a set-valued map defined by

$$\widetilde{F}(k,p) := \bigcup_{\epsilon\in[-1;1]} (k \pm \epsilon)p^5. \tag{4.1}$$

Consider the BVP

$$\Delta^2 y(k-1) \in \widetilde{F}(k,y(k)), \quad k = 1,\dots,T, \qquad y(0) = A, \qquad y(T+1) = B, \tag{4.2}$$

where $A,B \in \mathbb{R}$. Since $k \in \{1,\dots,T\}$ and $|\epsilon| \le 1$, the inequality $(k\pm\epsilon)p^5 \le (k\pm\epsilon)(p^6 + 1)$ holds and thus for every selector $\widetilde{f}(k,p) \in \widetilde{F}(k,p)$, we have that

$$\begin{aligned} \left|\widetilde{f}(k,p)\right| &\le (k\pm\epsilon)(p^6+1) = p(k\pm\epsilon)p^5 + t \pm \epsilon \\ &\le p\widetilde{f}(k,p) + T + 1, \end{aligned} \tag{4.3}$$

and the inequality (2.2) is satisfied with $\alpha = 1/2$ and $K = T + 1$. The set-valued map $\widetilde{F}$ satisfies (US) and thus we can use Theorem 4.1 to prove that the problem (4.2) has a solution.

As a natural corollary to Theorem 4.1, we have the following result.

Corollary 4.3. *Let $K \ge 0$ be a constant. If the set-valued map $F : \{1,\dots,T\} \times \mathbb{R}^d \times \mathbb{R}^d \to CK(\mathbb{R}^d)$ satisfies (US) and*

$$\|\phi\| \le K, \quad \text{for } k = 1,\dots,T \text{ and all } (p,q) \in \mathbb{R}^{2d},\ \phi \in F(k,p,q), \tag{4.4}$$

then the system of discrete inclusion boundary value problems (1.3) has a solution.

Proof. See that this is a special case of Theorem 4.1 with $\alpha = 0$. □

We now prove an existence result for the conditions from Lemma 2.3.

Theorem 4.4. *If the set-valued map $F : \{1,\dots,T\} \times \mathbb{R}^d \times \mathbb{R}^d \to CK(\mathbb{R}^d)$ satisfies (US) and the assumptions of Lemma 2.3 hold, then the system of discrete inclusion boundary value problems (1.3) has a solution.*

Proof. The proof is identical to that of Theorem 4.1. The only difference consists of the different definition of the constant R when defining the set $\overline{U}$, see Lemma 2.3. □

We illustrate the above result with an example in two dimensions.

Example 4.5. Let $\widehat{F} : \{1,\dots,T\} \times \mathbb{R}^2 \times \mathbb{R}^2 \to CK(\mathbb{R}^2)$ be a set-valued map defined by

$$\widehat{F}(k,p,q) := \bigcup_{\epsilon\in(0;1]} \begin{pmatrix} \epsilon k p_1 - q_1 \\ p_2^3(1+q_1^2) - p_2 - q_2 \end{pmatrix}, \tag{4.5}$$

where $p = \binom{p_1}{p_2}$ and $q = \binom{q_1}{q_2}$. Consider the BVP

$$\Delta^2 y(k-1) \in \widehat{F}(k, y(k), \Delta y(k)), \quad k = 1,\dots,T, \qquad y(0) = A, \qquad y(T+1) = B, \tag{4.6}$$

where $A,B \in \mathbb{R}^2$. Let $\widehat{f}(k,p,q) \in \widehat{F}(k,p,q)$ be an arbitrary selector for all $k = 1,\dots,T$, $q,p \in \mathbb{R}^d$, $\|p\| \geq R$, where $R > 1$ and p, q are such that $2\langle p,q\rangle + \|q\|^2 \leq 0$, for example,

$$2p_1q_1 + 2p_2q_2 + q_1^2 + q_2^2 \leq 0. \tag{4.7}$$

We make the following calculation:

$$\begin{aligned} 2\langle p, \widehat{f}(k,p,q)\rangle + \|q\|^2 &= 2\left\langle \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}, \begin{pmatrix} kp_1 - q_1 \\ p_2^3(1+q_1^2) - p_2 - q_2 \end{pmatrix} \right\rangle + q_1^2 + q_2^2 \\ &= 2\epsilon k p_1^2 + 2p_2^4(1+q_1^2) - 2p_2^2 - 2p_1q_1 - 2p_2q_2 + q_1^2 + q_2^2. \end{aligned} \tag{4.8}$$

Using (4.7), we can provide the estimation

$$2\langle p, \widehat{f}(k,p,q)\rangle + \|q\|^2 \geq 2\epsilon k p_1^2 + 2p_2^4(1+q_1^2) - p_2^2 + 2q_1^2 + 2q_2^2 > 0, \tag{4.9}$$

since either $\epsilon p_1^2 > 0$ or $p_2^4(1+q_1^2) - p_2^2 > 0$. The set-valued map $\widehat{F}$ satisfies also the condition (US), and thus we can use Theorem 4.4 to prove that the problem (4.6) has a solution.

Acknowledgments

The research of the first author was supported by the Grant Agency of Czech Republic, no. 201/03/0671. The research of the second author was supported by the Australian Research Council's Discovery Projects (DP0450752). The research of both authors was conducted at the University of New South Wales (UNSW), Sydney, Australia.

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