EIGENVALUE PROBLEMS FOR DEGENERATE NONLINEAR ELLIPTIC EQUATIONS IN ANISOTROPIC MEDIA

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We study nonlinear eigenvalue problems of the type $-\operatorname{div}(a(x)\nabla u) = g(\lambda, x, u)$ in $\mathbb{R}^N$, where $a(x)$ is a degenerate nonnegative weight. We establish the existence of solutions and we obtain information on qualitative properties as multiplicity and location of solutions. Our approach is based on the critical point theory in Sobolev weighted spaces combined with a Caffarelli-Kohn-Nirenberg-type inequality. A specific minimax method is developed without making use of Palais-Smale condition.

1. Introduction

We are concerned in this paper with the existence of critical points to Euler-Lagrange energy functionals generated by nonlinear equations involving degenerate differential operators. Precisely, we study the existence of nontrivial weak solutions to degenerate elliptic equations of the type

$$-\operatorname{div}\left(a(x)\nabla u\right) = g(\lambda, x, u), \quad x \in \Omega, \tag{1.1}$$

where λ is a real parameter, Ω is a (bounded or unbounded) domain in $\mathbb{R}^N$ $(N \geq 2)$, and a is an nonnegative measurable weight function that is allowed to have "essential" zeroes at some points. Problems like this have a long history (see the pioneering papers [3, 16, 17, 18, 22]) and come from the consideration of standing waves in anisotropic Schrödinger equations (see, e.g., [23]). Such problems in anisotropic media can be regarded as equilibrium solutions of the evolution equations

$$u_t = \mathcal{F}(\lambda, u, \nabla u) \quad \text{in } \Omega \times (0, T), \tag{1.2}$$

where $u = u(x,t)$ is the state of a certain system. For instance, in describing the behavior of a bacteria culture, the state variable u represents the number of mass of the bacteria.

It is worth to stress that the study of nontrivial solutions of the problem $\mathcal{F}(\lambda, u, \nabla u) = 0$ in Ω is motivated by important phenomena. For example, consider a fluid which flows irrotationally along a flat-bottomed canal. Then the flow can be modelled by an equation of the form $\mathcal{F}(\lambda, u, \nabla u) = 0$, with $\mathcal{F}(\lambda, 0, 0) = 0$. One possible motion is a uniform stream

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(corresponding to the trivial solution $u = 0$), but it is of course the nontrivial solutions which are of physical interest. Other problems of this type are encountered in various reaction-diffusion processes (cf. [1, 2]).

A model equation that we consider in this paper is

$$-\operatorname{div}(a(x)\nabla u) = f(x,u)u - \lambda u, \quad x \in \Omega \subset \mathbb{R}^N, \tag{1.3}$$

where a is a nonnegative weight and λ is a real parameter. The behavior of solutions to the above equation depends heavily on the sign of λ. Here we focus on the attractive case $\lambda > 0$ which, from an analytical point of view, seems to be the richest one. The main interest of these equations is due to the presence of the singular potential $a(x)$ in the divergence operator. Problems of this kind arise as models for several physical phenomena related to equilibrium of continuous media which may somewhere be "perfect insulators" (cf. [13, page 79]). These equations can be often reduced to elliptic equations with Hardy singular potential (see [23]). For further results and extensions we refer to [5, 7, 12, 14, 24, 25, 26].

In this paper we first establish the existence of solutions to the above problem involving the singular potential $a(x)$ under verifiable conditions for the nonlinear term f when $\lambda > 0$ is sufficiently small. Then we investigate a related nonlinear eigenvalue problem obtaining an existence result which contains information about the location and multiplicity of eigensolutions.

The proofs of our main results rely on an adequate variational approach where, in view of the presence of a singular potential and a (possibly) unbounded domain, the usual methods fail to apply. Namely, on suitable Sobolev weighted spaces, we apply the mountain-pass theorem and a special version of it involving a suitable hyperplane. Among other things, we employ an inequality due to Caldiroli and Musina [11] (see also [10] for the case $a(x) = |x|^\alpha$) which extends the inequalities of Hardy [15] and Caffarelli et al. [9]. Our results are different from the ones in [11]. In particular, they are not related to the first eigenvalue of the linear part, but exploit the behavior of the nonlinear term at infinity. Another specific feature of our variational approach is that due to the lack of compactness, we do not make use of the Palais-Smale condition.

The rest of the paper is organized as follows. Section 2 presents our main results which are Theorems 2.6 and 2.9. In Section 3 we prove some auxiliary results. The proofs of Theorems 2.6 and 2.9 are given in Sections 4 and 5, respectively.

2. Abstract framework and main results

Let Ω be a (bounded or unbounded) domain in $\mathbb{R}^N$, with $N \geq 2$, and let $a : \Omega \to [0,+\infty)$ be a weight function satisfying $a \in L^1_{\text{loc}}(\Omega)$. Suppose that a fulfills the following condition:

(h_α) $\liminf_{x\to z} |x-z|^{-\alpha}a(x) > 0$, $\forall z \in \overline{\Omega}$, with a real number $\alpha \geq 0$.

If Ω is unbounded we impose the additional assumption

(h_α^∞) $\liminf_{|x|\to\infty} |x|^{-\alpha}a(x) > 0$.

A model example is $a(x) = |x|^\alpha$. The case $\alpha = 0$ covers the "isotropic" case corresponding to the Laplace operator.

Under assumptions (h_α) and (h_α^∞), Caldiroli and Musina have proved in [11] that there exists a finite set $Z = \{z_1, \dots, z_k\} \subset \overline{\Omega}$ and numbers $r, \delta > 0$ such that the balls $B_i = B_r(z_i)$ ($i = 1, \dots, k$) are mutually disjoint and

$$\begin{gathered} a(x) \geq \delta \left| x - z_i \right|^\alpha \quad \forall x \in B_i \cap \Omega,\ i = 1, \dots, k, \\ a(x) \geq \delta \quad \forall x \in \Omega \setminus \bigcup_{i=1}^{k} B_i. \end{gathered} \tag{2.1}$$

In addition, if Ω is unbounded, there exists $R > 0$ such that

$$\overline{B_i} \subset B_R(0) \quad (i = 1, \dots, k), \qquad a(x) \geq \delta |x|^\alpha, \quad \forall x \in \Omega,\ |x| > R. \tag{2.2}$$

For any $u \in C_c^\infty(\Omega)$, we set

$$\begin{aligned} \|u\|_a^2 :&= \int_\Omega a(x) |\nabla u|^2 dx, \\ \|u\|_{H,a}^2 :&= \int_\Omega a(x) |\nabla u|^2 dx + \int_\Omega u^2 dx. \end{aligned} \tag{2.3}$$

Let $\mathfrak{D}_a^{1,2}(\Omega)$ and $H_0^1(\Omega, a)$ be the closures of $C_c^\infty(\Omega)$ with respect to $\|\cdot\|_a$ and $\|\cdot\|_{H,a}$, respectively. It is obvious that $H_0^1(\Omega, a) \hookrightarrow \mathfrak{D}_a^{1,2}(\Omega)$ with continuous embedding. For any $\alpha \in (0, 2)$, denote

$$2_\alpha^* := \frac{2N}{N - 2 + \alpha}. \tag{2.4}$$

The following generalization of the Caffarelli-Kohn-Nirenberg inequality is given in [11] (see also [10] for the case $a(x) = |x|^\alpha$).

Lemma 2.1 (Caldiroli and Musina [11]). *Assume that the function $a \in L_{\text{loc}}^1(\Omega)$ satisfies conditions (h_α) and (h_α^∞), for some $\alpha \in (0, 2)$. Then there exists a positive constant C such that*

$$\left(\int_\Omega |u|^{2_\alpha^*} dx \right)^{2/2_\alpha^*} \leq C \|u\|_a^2, \tag{2.5}$$

for any $u \in C_c^\infty(\Omega)$.

Using the above inequality combined with variational methods, Caldiroli and Musina have studied in [11] the boundary value problem

$$\begin{gathered} -\operatorname{div}(a(x) \nabla u) = \lambda u + g(x, u), \quad \text{in } \Omega, \\ u = 0, \quad \text{on } \partial\Omega, \end{gathered} \tag{2.6}$$

where $\lambda \in \mathbb{R}$ and $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function with superlinear growth. Their existence result is related to the first eigenvalue of the degenerate differential elliptic operator $Lu := -\operatorname{div}(a(x) \nabla u)$. Namely, problem (2.6) has a solution for any $\lambda < \lambda_1(a)$, where

$$\lambda_1(a) := \inf \left\{ \int_\Omega a(x) |\nabla \varphi|^2 dx;\ \varphi \in H_0^1(\Omega, a),\ \int_\Omega \varphi^2 dx = 1 \right\}. \tag{2.7}$$

In the statement of our Theorem 2.6, the existence of a solution does not depend on $\lambda_1(a)$, but on the behavior of the nonlinearity at infinity.

Hypotheses (h_α) and (h_α^∞) ensure that the potential $a(x)$ behaves like $|x|^\alpha$ around the degenerate points z_i ($i = 1,\dots,k$). For this reason, in order to simplify the arguments we admit throughout the paper that $a(x) = |x|^\alpha$, for some $\alpha \in (0,2)$, and that $\lambda > 0$. Since we are interested in the case of lack of compactness, we suppose $\Omega = \mathbb{R}^N$.

Consider the model problem

$$-\operatorname{div}\left(|x|^\alpha \nabla u\right) + \lambda u = f(x,u)u, \quad \text{in } \mathbb{R}^N. \tag{2.8}$$

We assume that the nonlinearity $f = f(x,t) : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ in (2.8) is continuous and satisfies the following hypotheses:

(c1) $f(x,t) \geq 0$ for all $t \geq 0$, $\lim_{t\to 0^+}(f(x,t)/t^\tau) = 0$ uniformly in $x \in \mathbb{R}^N$, with some constant $\tau > 0$, $f(x,t) \equiv 0$ for all $t < 0$, $x \in \mathbb{R}^N$, that the mapping $(x,t) \mapsto tf(x,t)$ is of class C^1, and there exists the limit $\lim_{t\to+\infty}(d/dt)f(x,t)$ for all $x \in \mathbb{R}^N$;

(c2) $\lim_{t\to+\infty} f(x,t) = \ell > 0$ uniformly in $x \in \mathbb{R}^N$;

(c3) for any $M > 0$ there exists $\theta > 0$ such that $(2+\theta)F(x,t) \leq f(x,t)t^2$, for all $t \in (0,M)$, where

$$F(x,u) := \int_0^u sf(x,s)\,ds; \tag{2.9}$$

(c4) there exists $\eta > 0$ such that

$$\lim_{t\to+\infty} \frac{f(x,t)t^2 - 2F(x,t)}{t^r} = q(x) \geq \eta > 0 \quad \text{uniformly in } x \in \mathbb{R}^N, \tag{2.10}$$

with some $r \in (2N/(N+2-\alpha), 2)$;

(c5) the function $f(\cdot,t)$ is bounded from above uniformly with respect to t belonging to any bounded subset of $\mathbb{R}_+$.

Remark 2.2. A useful consequence of assumption (c1) is that the derivative with respect to t of the mapping $(x,t) \mapsto tf(x,t)$ vanishes at $t = 0$ uniformly in $x \in \mathbb{R}^N$. Indeed, condition (c1) insures that

$$\frac{d}{dt}\left(tf(x,t)\right)(0) = \lim_{t\to 0^+} \frac{tf(x,t)}{t} = \lim_{t\to 0^+} f(x,t) = 0 \tag{2.11}$$

uniformly in $x \in \mathbb{R}^N$. We also point out that the condition imposed in assumption (c1) of having $\lim_{t\to 0^+}(f(x,t)/t^\tau) = 0$ uniformly in $x \in \mathbb{R}^N$, for some constant $\tau > 0$, is stronger than having $\lim_{t\to 0^+} f(x,t) = 0$ uniformly in $x \in \mathbb{R}^N$. For instance, the function $f(t) = -1/\ln(t)$ for $t > 0$ near 0 verifies $\lim_{t\to 0^+} f(t) = 0$, but $\lim_{t\to 0^+}(f(t)/t^\tau) = +\infty$ whenever $\tau > 0$ (in addition, f is increasing). Moreover, without loss of generality, we may suppose that $0 < \tau < 2_\alpha^* - 2$.

Remark 2.3. It is worth noting that assumption (c3) ensures

$$F(x,t) \leq \frac{1}{2} f(x,t)t^2, \quad \forall x \in \mathbb{R}^N,\ t \geq 0. \tag{2.12}$$

This follows readily from (c3) because $M > 0$ is arbitrary and $\theta F(x,t) \geq 0$ for all $x \in \mathbb{R}^N$ and $t \geq 0$.

Remark 2.4. A significant case where assumption (c5) applies is when the function $t \mapsto f(x,t)$ is nondecreasing for all $x \in \mathbb{R}^N$. It is so because then (c2) implies (c5).

Let $\mathcal{D}_\alpha^{1,2}(\mathbb{R}^N)$ denote the space obtained as the completion of $C_c^\infty(\mathbb{R}^N)$ with respect to the inner product

$$\langle u, v\rangle_\alpha := \int_{\mathbb{R}^N} |x|^\alpha \nabla u \cdot \nabla v \, dx. \tag{2.13}$$

We are seeking solutions of problem (2.8) belonging to the space $\mathcal{D}_\alpha^{1,2}(\mathbb{R}^N)$ in the sense below.

Definition 2.5. We say that $u \in \mathcal{D}_\alpha^{1,2}(\mathbb{R}^N)$ is a *weak solution* of problem (2.8) if

$$\int_{\mathbb{R}^N} \left(|x|^\alpha \nabla u \cdot \nabla v + \lambda uv\right) dx - \int_{\mathbb{R}^N} f(x,u) uv \, dx = 0, \tag{2.14}$$

for all $v \in C_c^\infty(\mathbb{R}^N \setminus \{0\})$.

We are working with $C_c^\infty(\mathbb{R}^N \setminus \{0\})$ instead of $C_c^\infty(\mathbb{R}^N)$ because in our approach it is essential to keep the support of the test functions away from 0 exploiting that every bounded sequence in the space $\mathcal{D}_\alpha^{1,2}(\mathbb{R}^N)$ contains a strongly convergent subsequence in $L_{\text{loc}}^{2_\alpha^*}(\mathbb{R}^N \setminus \{0\})$.

Our main result in solving problem (2.8) is the following.

Theorem 2.6. *Assume that conditions (c1), (c2), (c3), (c4), and (c5) are fulfilled. Then problem (2.8) has a nontrivial weak solution for every $\lambda \in (0, \ell)$, where $\ell > 0$ is the constant in (c2).*

The proof of Theorem 2.6 is given in Section 4. We now provide an example verifying all the assumptions (c1), (c2), (c3), (c4), and (c5) of Theorem 2.6.

Example 2.7. Fix $Q \in C^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, $Q > 0$. Set

$$f(x,t) = \frac{Q(x)t^{2-r}}{1 + Q(x)t^{2-r}} \quad \text{for } t \geq 0,\ x \in \mathbb{R}^N, \tag{2.15}$$

where r is as in (c4), and $f(x,t) = 0$ for $t < 0$ and $x \in \mathbb{R}^N$. It is easy to verify that f satisfies (c1) (with $\tau \in (0, 2-r)$), (c2), and (c5). Since

$$\frac{df}{dt}(x,t) = \frac{(2-r)Q(x)t^{1-r}}{\left(1 + Q(x)t^{2-r}\right)^2} \geq 0 \quad \forall t > 0, \tag{2.16}$$

we deduce that for any $M > 0$,

$$\inf_{x \in \mathbb{R}^N} \min_{t \in [0,M]} \frac{t(df/dt)(x,t)}{f(x,t)} = \inf_{x \in \mathbb{R}^N} \min_{t \in [0,M]} \frac{2-r}{1 + Q(x)t^{2-r}} \geq \frac{2-r}{1 + \|Q\|_{L^\infty(\mathbb{R}^N)} M^{2-r}} > 0. \tag{2.17}$$

Choosing

$$\theta = \frac{2-r}{1+\|Q\|_{L^\infty(\mathbb{R}^N)}M^{2-r}}, \tag{2.18}$$

we have

$$s\frac{df}{ds}(x,s) \geq \theta f(x,s) \quad \forall x \in \mathbb{R}^N,\ s \in [0,M]. \tag{2.19}$$

Multiplying the above relation by $s > 0$, then integrating over $[0,t]$ with $t \in [0,M]$ and taking into account the definition of the function $F(x,t)$, we obtain

$$F(x,t) \leq \frac{1}{2+\theta} f(x,t)t^2 \quad \forall t \in [0,M]. \tag{2.20}$$

It follows that f satisfies (c3). Finally, we note that

$$\lim_{t\to+\infty}\left[t^{3-r}\frac{df}{dt}(x,t)\right] = \lim_{t\to+\infty}\frac{(2-r)Q(x)t^{4-2r}}{\left(1+Q(x)t^{2-r}\right)^2} = \frac{2-r}{Q(x)} \geq \frac{2-r}{\|Q\|_{L^\infty(\mathbb{R}^N)}} > 0. \tag{2.21}$$

Thus there exists $S > 0$ such that

$$s^{3-r}\frac{df}{ds}(x,s) \geq \frac{\eta}{2} > 0 \quad \forall s \geq S, \tag{2.22}$$

where $\eta = (2-r)/\|Q\|_{L^\infty(\mathbb{R}^N)}$. Since

$$\begin{aligned} f(x,t)t^2 - 2F(x,t) &= f(x,t)t^2 - 2\int_0^t sf(x,s)ds \\ &= \int_0^t s^2\frac{df}{ds}(x,s)ds = \int_0^t s^{r-1}\left(s^{3-r}\frac{df}{ds}(x,s)\right)ds, \end{aligned} \tag{2.23}$$

the above estimate yields

$$\begin{aligned} \lim_{t\to+\infty}\left[f(x,t)t^2 - 2F(x,t)\right] &= \lim_{t\to+\infty}\int_0^t s^{r-1}\left(s^{3-r}\frac{df}{ds}(x,s)\right)ds \\ &\geq \frac{\eta}{2}\lim_{t\to+\infty}\int_S^t s^{r-1}ds = +\infty. \end{aligned} \tag{2.24}$$

Hence

$$\lim_{t\to+\infty}\frac{f(x,t)t^2 - 2F(x,t)}{t^r} = \frac{1}{r}\lim_{t\to+\infty}\frac{t^2(df/dt)(x,t)}{t^{r-1}} = \frac{2-r}{rQ(x)} \geq \frac{\eta}{r} > 0. \tag{2.25}$$

The last relation shows that condition (c4) holds true. Therefore all the assumptions (c1), (c2), (c3), (c4), and (c5) are satisfied for the function $f(x,t)$ and Theorem 2.6 can be applied for the corresponding (2.8).

In order to present our second main result, we precisely give the functional setting. Let E be the space defined as the completion of $C_c^\infty(\mathbb{R}^N \setminus \{0\})$ with respect to the norm

$$\|u\|^2 := \int_{\mathbb{R}^N} (|x|^\alpha |\nabla u|^2 + \lambda u^2)\, dx. \tag{2.26}$$

The corresponding inner product is denoted by $\langle \cdot, \cdot \rangle_E$. The notation $\langle \cdot, \cdot \rangle$ will stand for the duality pairing between E and E^*.

Remark 2.8. We have $E \hookrightarrow \mathcal{D}_\alpha^{1,2}(\mathbb{R}^N)$ with continuous embedding.

Next, we state a nonlinear eigenvalue problem corresponding to the degenerate potential $|x|^\alpha$ with $\alpha \in (0,2)$.

Fix a positive number $\nu > 0$. Let $J : E \to \mathbb{R}$ be a C^1 function satisfying

$$J(0) \geq 0, \qquad J'(0) \neq 0, \tag{2.27}$$

$$J(u) \leq a_1 + a_2 \|u\|^p \quad \forall u \in E, \tag{2.28}$$

with constants $a_1 \geq 0$, $a_2 \geq 0$, $p \geq 2$,

$$\frac{1}{\gamma}\langle J'(u), u\rangle - J(u) \geq -b_1 - b_2 \|u\|^2 \quad \forall u \in E, \tag{2.29}$$

with constants $\gamma > 2$, $b_1 \geq 0$, $b_2 \in [0, \nu(1/2 - 1/\gamma))$, and

$$J'(v_n) \rightharpoonup J'(v) \text{ in } E^* \quad \text{whenever } v_n \rightharpoonup v \text{ in } E. \tag{2.30}$$

The notation $\rightharpoonup$ in (2.30) means the weak convergence.

We formulate a nonlinear eigenvalue problem with fixed constants $\alpha > 0$ and $\lambda > 0$ as follows: find $u \in E \setminus \{0\}$ and $\mu > 0$ such that

$$\int_{\mathbb{R}^N} (|x|^\alpha \nabla u \cdot \nabla \varphi + \lambda u \varphi)\, dx = \mu \langle J'(u), \varphi \rangle \quad \forall \varphi \in C_c^\infty(\mathbb{R}^N \setminus \{0\}). \tag{2.31}$$

The concept of solution in (2.31) is clearly compatible with Definition 2.5. Thanks to the assumption $J'(0) \neq 0$ in (2.27), a solution $u \in E$ of (2.31) is necessarily nontrivial, that is, $u \in E \setminus \{0\}$. Assume further that

$$\frac{1}{\nu} \text{ is not an eigenvalue of (2.31)}, \tag{2.32}$$

that is, problem (2.31) is not solvable for $\mu = 1/\nu$.

Our main result in studying problem (2.31) is now stated.

Theorem 2.9. *Assume (2.27),(2.28),(2.29), and (2.30) and (2.32) with a given number $\nu > 0$ hold. Then, for every number $\rho \geq \sqrt{pa_2}$, there exists an eigensolution $(u, \mu) \in (E \setminus \{0\}) \times (0, +\infty)$ of problem (2.31) such that*

$$0 < \mu < \frac{1}{\nu + \rho^2 \|u\|^{p-2}}. \tag{2.33}$$

If $p = 2$ in (2.28), then for all $\rho \geq \sqrt{2a_2}$ and $r > \rho$ there exists an eigensolution $(u,\mu) \in (E \setminus \{0\}) \times (0,+\infty)$ of problem (2.31) such that

$$\frac{1}{\nu + r^2} < \mu < \frac{1}{\nu + \rho^2}. \tag{2.34}$$

3. Auxiliary results

Consider the energy functional $I : E \to \mathbb{R}$ given by

$$I(u) := \frac{1}{2} \int_{\mathbb{R}^N} \left(|x|^\alpha |\nabla u|^2 + \lambda u^2\right) dx - \int_{\mathbb{R}^N} F(x,u)dx \quad \forall u \in E, \tag{3.1}$$

where the function F has been introduced in Section 2. A straightforward argument based on Lemma 2.1, Remark 2.8, and assumptions (c1) and (c2) shows that $I \in C^1(E,\mathbb{R})$ with

$$\langle I'(u), v\rangle = \int_{\mathbb{R}^N} \left(|x|^\alpha \nabla u \cdot \nabla v + \lambda uv\right) dx - \int_{\mathbb{R}^N} f(x,u)uv\, dx \tag{3.2}$$

for all $u, v \in E$.

Using Definition 2.5, we observe that the weak solutions of problem (2.8) correspond to the critical points of the functional I. Moreover, we indicate a method for achieving a solution of (2.8).

Lemma 3.1. *Assume (c1) and (c2). Let $\{u_n\} \subset E$ be a sequence such that, for some $c \in \mathbb{R}$, one has $I(u_n) \to c$ and $I'(u_n) \to 0$ as $n \to \infty$. If $\{u_n\}$ converges weakly to some u_0 in E, then $I'(u_n)$ converges weakly to $I'(u_0) = 0$, so u_0 is a weak solution of problem (2.8).*

Proof. In view of Remark 2.8 we may assume that $\{u_n\}$ converges strongly to the same u_0 in $L^{2^*_\alpha}(\omega)$, for all bounded domains ω in $\mathbb{R}^N$ with $0 \notin \overline{\omega}$. Consider an arbitrary bounded domain ω in $\mathbb{R}^N$ with $0 \notin \overline{\omega}$ and an arbitrary function $\varphi \in C_c^\infty(\mathbb{R}^N \setminus \{0\})$ satisfying $\operatorname{supp}(\varphi) \subset \omega$. The convergence $I'(u_n) \to 0$ in E^* implies $\langle I'(u_n), \varphi\rangle \to 0$ as $n \to \infty$, that is,

$$\lim_{n\to\infty} \left(\int_\omega \left(|x|^\alpha \nabla u_n \cdot \nabla\varphi + \lambda u_n \varphi\right) dx - \int_\omega f(x,u_n)u_n\varphi\, dx \right) = 0. \tag{3.3}$$

Since $u_n \rightharpoonup u_0$ in E, it follows that

$$\lim_{n\to\infty} \int_\omega \left(|x|^\alpha \nabla u_n \cdot \nabla\varphi + \lambda u_n\varphi\right) dx = \int_\omega \left(|x|^\alpha \nabla u_0 \nabla\varphi + \lambda u_0\varphi\right) dx. \tag{3.4}$$

We show that

$$\lim_{n\to\infty} \int_\omega f(x,u_n)u_n\varphi\, dx = \int_\omega f(x,u_0)u_0\varphi\, dx. \tag{3.5}$$

Because $u_n \to u_0$ in $L^{2^*_\alpha}(\omega)$, we have

$$u_n \longrightarrow u_0 \text{ in } L^i(\omega) \quad \forall i \in [1, 2^*_\alpha]. \tag{3.6}$$

By Remark 2.2, we obtain that

$$\lim_{t\to 0}\frac{d}{dt}(tf(x,t)) = 0 \tag{3.7}$$

and, by (c2),

$$0 = \lim_{t\to+\infty}\frac{f(x,t)}{t} = \lim_{t\to+\infty}\frac{tf(x,t)}{t^2} = \lim_{t\to+\infty}\frac{(d/dt)(tf(x,t))}{2t} \tag{3.8}$$

uniformly in $x \in \mathbb{R}^N$, where the last limit exists owing to the final part of (c1) in conjunction with (c2). Then for every $\epsilon > 0$ there exists a positive constant C_ϵ such that

$$\left|\frac{d}{dt}(tf(x,t))\right| \leq \epsilon + C_\epsilon t \quad \forall t \geq 0,\ \forall x \in \omega. \tag{3.9}$$

Here we essentially used that the derivative $(d/dt)(f(x,t)t)$ is continuous, so bounded on any compact set in $\mathbb{R}^N \times \mathbb{R}$.

As known from (3.6), the sequence $\{u_n\}$ converges strongly to u_0 in $L^2(\omega)$. Then, along a relabelled subsequence, there is a function $h \in L^2(\omega)$ such that $|u_n| \leq h$ a.e. in ω. Using (3.9) and the Cauchy-Schwarz inequality, we get the estimate

$$\begin{aligned}
&\left|\int_\omega (f(x,u_n)u_n\varphi - f(x,u_0)u_0\varphi)\,dx\right| \\
&\quad\leq \|\varphi\|_{L^\infty(\omega)}\int_\omega (\epsilon + C_\epsilon h(x))\,|u_n(x) - u_0(x)|\,dx \\
&\quad\leq \|\varphi\|_{L^\infty(\omega)}\Big[\epsilon\|u_n - u_0\|_{L^1(\omega)} + C_\epsilon\|h\|_{L^2(\omega)}\|u_n - u_0\|_{L^2(\omega)}\Big]
\end{aligned} \tag{3.10}$$

for all $n \in \mathbb{N}$. This leads to (3.5).

Finally, from (3.3), (3.4), and (3.5), we deduce

$$\int_\omega (|x|^\alpha \nabla u_0 \cdot \nabla\varphi + \lambda u_0\varphi)\,dx - \int_\omega f(x,u_0)u_0\varphi\,dx = 0. \tag{3.11}$$

The density of $C_c^\infty(\mathbb{R}^N \setminus \{0\})$ in E ensures that $I'(u_0) = 0$. The proof is thus complete. □

Remark 3.2. Lemma 3.1 holds assuming in (c1), (c2) that the convergences are uniform only on the bounded subsets of $\mathbb{R}^N$.

Towards the application of a mountain-pass argument, we need the result below.

Lemma 3.3. *Assume that the conditions (c1), (c2), and (c5) hold. Then there exist constants $\rho > 0$ and $a > 0$ such that for all $u \in E$ with $\|u\| = \rho$, one has $I(u) \geq a$.*

Proof. By (c1), (c2) it follows that for any $\sigma > 0$, uniformly with respect to $x \in \mathbb{R}^N$, it is true that

$$\lim_{t\to 0} f(x,t) = 0, \qquad \lim_{t\to+\infty}\frac{f(x,t)}{t^\sigma} = 0. \tag{3.12}$$

In particular, we have

$$\lim_{t\to 0}\frac{F(x,t)}{t^2}=\lim_{t\to 0}\frac{tf(x,t)}{2t}=\frac{1}{2}\lim_{t\to 0}f(x,t)=0 \tag{3.13}$$

and, for any $\sigma'>2$,

$$\lim_{t\to+\infty}\frac{F(x,t)}{t^{\sigma'}}=\lim_{t\to+\infty}\frac{tf(x,t)}{\sigma' t^{\sigma'-1}}=\frac{1}{\sigma'}\lim_{t\to+\infty}\frac{f(x,t)}{t^{\sigma'-2}}=0. \tag{3.14}$$

Taking $\sigma'=2_\alpha^*$ implies

$$\lim_{t\to+\infty}\frac{F(x,t)}{t^{2_\alpha^*}}=0. \tag{3.15}$$

Then for every $\epsilon>0$ there exist constants $0<\delta_1<\delta_2$ such that, uniformly with respect to $x\in\mathbb{R}^N$, the following estimates hold

$$\begin{aligned}&0\le F(x,t)<\epsilon t^2 \quad \forall t \text{ with } |t|<\delta_1,\\ &0\le F(x,t)<\epsilon t^{2_\alpha^*} \quad \forall t \text{ with } |t|>\delta_2.\end{aligned} \tag{3.16}$$

Assumption (c5) guarantees that F is bounded on $\mathbb{R}^N\times[\delta_1,\delta_2]$. We deduce that there exists a positive constant C_ϵ such that

$$0\le F(x,t)\le\epsilon t^2+C_\epsilon t^{2_\alpha^*}. \tag{3.17}$$

Then (3.17) and Lemma 2.1 show that

$$\begin{aligned}I(u)&=\frac{1}{2}\|u\|^2-\int_{\mathbb{R}^N}F(x,u)dx\\ &\ge\frac{1}{4}\|u\|^2+\left(\frac{\lambda}{4}-\epsilon\right)\|u\|^2_{L^2(\mathbb{R}^N)}-C_\epsilon\int_{\mathbb{R}^N}|u|^{2_\alpha^*}dx\\ &\ge\frac{1}{4}\|u\|^2+\left(\frac{\lambda}{4}-\epsilon\right)\|u\|^2_{L^2(\mathbb{R}^N)}-C_\epsilon C^{2_\alpha^*/2}\|u\|^{2_\alpha^*}.\end{aligned} \tag{3.18}$$

Finally, choosing $\epsilon\in(0,\lambda/4)$ and since $2_\alpha^*>2$, we find $\rho>0$ and $a>0$ as required. □

Remark 3.4. Using the same techniques as in the proof of relation (3.17), we may conclude, on the basis of (c1), (c2), and (c5), that for any $\epsilon>0$ there exists a positive constant D_ϵ such that

$$|f(x,t)|\le\epsilon+D_\epsilon|t|^\sigma, \tag{3.19}$$

where $\sigma=r((N+2-\alpha)/2N)-1>0$.

Now we construct an important element of the space E. Denote

$$(d(N))^2:=\int_{\mathbb{R}^N}e^{-2|x|^2}dx, \tag{3.20}$$

and, for an arbitrary number $a > 0$,

$$w_a(x) := \left(d(N)\right)^{-1} a^{N/4} e^{-a|x|^2} \quad \forall x \in \mathbb{R}^N. \tag{3.21}$$

Proposition 3.5. *The function w_a satisfies $w_a \in E$ whenever $a > 0$.*

Proof. Fix $a > 0$. Defining

$$h(x) = e^{-a|x|^2} \quad \forall x \in \mathbb{R}^N, \tag{3.22}$$

it is enough to show that $h \in E$. We have to establish that for any $\epsilon > 0$ there is $\psi \in C_c^\infty(\mathbb{R}^N \setminus \{0\})$ such that $\|h - \psi\| < \epsilon$. First we prove that, for every $\varepsilon > 0$, there exists a function $\psi_1 \in C_c^\infty(\mathbb{R}^N)$ such that

$$||h - \psi_1|| < \epsilon. \tag{3.23}$$

Given any number $z > 0$, we have $\lim_{t\to+\infty} t^{z+1}e^{-t} = 0$. We derive that a positive constant $C = C(z)$ can be found with the property that $|t^{z+1}e^{-t}| \le C$, for all $t \in [1,\infty)$, so

$$\left| t^{z-1}e^{-t} \right| \le \frac{C}{t^2} \quad \forall t \in [1,\infty). \tag{3.24}$$

We check that there exists some constant $\delta > 0$ such that

$$\frac{1}{4a^2} \int_{\mathbb{R}^N \setminus B(0,\delta)} |x|^\alpha \left| \nabla h(x) \right|^2 dx < \frac{\epsilon^2}{4(4a^2+2)}. \tag{3.25}$$

To this end we note that the below equality holds

$$\frac{1}{4a^2} \int_{\mathbb{R}^N \setminus B(0,\delta)} |x|^\alpha \left| \nabla h(x) \right|^2 dx = \omega_N \int_\delta^{+\infty} r^{\alpha+N+1} e^{-2ar^2} dr, \tag{3.26}$$

where ω_N is the surface measure of the unit sphere in $\mathbb{R}^N$. Then, using (3.24), for $\delta^2 \ge 1/(2a)$ we have

$$\frac{1}{4a^2} \int_{\mathbb{R}^N \setminus B(0,\delta)} |x|^\alpha \left| \nabla h(x) \right|^2 dx \le \frac{C_1}{\delta^2}, \tag{3.27}$$

with a positive constant C_1. To obtain (3.25) it is enough to choose

$$\delta^2 > \max\left\{ \frac{1}{2a}, \frac{4(4a^2+2)C_1}{\epsilon^2} \right\}. \tag{3.28}$$

Let $\varphi \in C_c^\infty(\mathbb{R}^N)$ satisfy $\varphi = 1$ on $B(0,\delta)$ and $0 \le \varphi \le 1$ on $\mathbb{R}^N$. Using (3.25) and taking δ sufficiently large, we find positive constants A_1 and A_2 satisfying

$$\begin{gathered} \|h - \varphi h\|_\alpha^2 < \frac{a^2\epsilon^2}{4a^2+2} + \int_{\mathbb{R}^N \setminus B(0,\delta)} |x|^\alpha \left| \nabla(1-\varphi)(x) \right|^2 e^{-2a|x|^2} dx < A_1\epsilon^2, \\ \lambda\|h - \varphi h\|_{L^2(\mathbb{R}^N)}^2 = \lambda \int_{\mathbb{R}^N \setminus B(0,\delta)} \left((1-\varphi)(x)\right)^2 e^{-2a|x|^2} dx < A_2\epsilon^2. \end{gathered} \tag{3.29}$$

Setting $\psi_1 := \varphi h \in C_c^\infty(\mathbb{R}^N)$, it follows that

$$||h-\psi_1||^2 < (A_1+A_2)\epsilon^2. \tag{3.30}$$

Since $A_1 + A_2$ is independent of ε, it turns out that property (3.23) is true.

Fix now a function $\psi \in C_0^\infty(\mathbb{R}^N \setminus \{0\})$ such that $\|\psi_1 - \psi\| < \epsilon$. Then, combining with (3.23), we arrive at the conclusion of Proposition 3.5 □

The next result sets forth that the functional I fits with the geometry of mountain-pass theorem.

Lemma 3.6. *If the conditions (c1), (c2), (c5) hold and $\lambda \in (0,\ell)$ with the number ℓ in (c2), then for the positive number ρ given in Lemma 3.3 there exists $e \in E$ such that $\|e\| > \rho$ and $I(e) < 0$.*

Proof. Fix the element $w_a \in E$ in Proposition 3.5 for some $a > 0$. We have $\|w_a\|_{L^2(\mathbb{R}^N)} = 1$. Using the notation $d(N)$ entering the formula of w_a, we introduce

$$D(N) := 4\big(d(N)\big)^{-2}\int_{\mathbb{R}^N} e^{-2|x|^2}|x|^{2+\alpha}dx. \tag{3.31}$$

We find

$$\begin{aligned} ||w_a||_\alpha^2 &= \int_{\mathbb{R}^N} |x|^\alpha \left|\nabla w_a(x)\right|^2 dx \\ &= 4a^{(N+4)/2}\big(d(N)\big)^{-2}\omega_N\int_0^{+\infty} r^{N+\alpha+1}e^{-2ar^2}dr \\ &= a^{1-\alpha/2}D(N). \end{aligned} \tag{3.32}$$

Recalling that $0 < \alpha < 2$ and making use of the assumption $\lambda \in (0,\ell)$, we choose

$$a \in \left(0, \left(\frac{\ell-\lambda}{D(N)}\right)^{2/(2-\alpha)}\right). \tag{3.33}$$

One obtains

$$||w_a||_\alpha^2 < \ell - \lambda. \tag{3.34}$$

Since $tw_a(x) \to +\infty$ as $t \to +\infty$ and, by (c2),

$$\lim_{u\to+\infty}\frac{F(x,u)}{u^2} = \frac{\ell}{2}, \tag{3.35}$$

it follows that

$$\lim_{t\to+\infty}\frac{F(x,tw_a(x))}{t^2} = \frac{\ell}{2}w_a^2(x) \tag{3.36}$$

for a.e. $x \in \mathbb{R}^N$. Using Fatou's lemma (see, e.g., [4]), we get

$$\begin{aligned}\limsup_{t\to+\infty}\frac{I(tw_a)}{t^2} &= \frac{1}{2}||w_a||_\alpha^2+\frac{\lambda}{2}||w_a||_{L^2(\mathbb{R}^N)}^2-\liminf_{t\to+\infty}\int_{\mathbb{R}^N}\frac{F(x,tw_a(x))}{t^2}dx\\ &\le \frac{1}{2}||w_a||_\alpha^2+\frac{\lambda}{2}-\int_{\mathbb{R}^N}\lim_{t\to+\infty}\frac{F(x,tw_a(x))}{t^2}dx\\ &= \frac{1}{2}||w_a||_\alpha^2-\frac{\ell-\lambda}{2}<0.\end{aligned}\tag{3.37}$$

In particular, we obtain $I(tw_a)\to-\infty$ as $t\to+\infty$. If $t_0>0$ is large enough and $e=t_0w_a$, then we achieve the conclusion of Lemma 3.6 with $e=t_0w_a$. □

4. Proof of Theorem 2.6

Arguing on the space E described in Section 2, we set

$$\Gamma := \{\gamma\in C([0,1],E);\ \gamma(0)=0,\ \gamma(1)=e\},\tag{4.1}$$

where $e\in E$ is determined by Lemma 3.6, and

$$c:=\inf_{\gamma\in\Gamma}\max_{t\in[0,1]}I(\gamma(t)).\tag{4.2}$$

According to Lemma 3.6 we know that $\|e\|>\rho$, so every path $\gamma\in\Gamma$ intersects the sphere $\|x\|=\rho$. Then Lemma 3.3 implies

$$c\ge\inf_{\|u\|=\rho}I(u)\ge a,\tag{4.3}$$

with the constant $a>0$ in Lemma 3.3, thus $c>0$.

By the mountain-pass theorem (see, e.g., [6]), we obtain a sequence $\{u_n\}\subset E$ such that

$$I(u_n)\longrightarrow c,\qquad I'(u_n)\longrightarrow 0.\tag{4.4}$$

We claim that $\{u_n\}$ is bounded in E. Indeed, from the first convergence in (4.4) we have

$$I(u_n)=\frac{1}{2}||u_n||^2-\int_{\mathbb{R}^N}F(x,u_n)dx=c+o(1).\tag{4.5}$$

We note that

$$||u_n||^2-\int_{\mathbb{R}^N}f(x,u_n)u_n^2dx=\langle I'(u_n),u_n\rangle.\tag{4.6}$$

By (c4) there exist constants $C>0$ and $M>0$ such that

$$f(x,t)t^2-2F(x,t)\ge Ct^r\quad \forall t\ge M,\ x\in\mathbb{R}^N.\tag{4.7}$$

Corresponding to the number $M>0$ in (4.7), by (c3) there exists some constant $\theta>0$ such that

$$F(x,t)\le\frac{1}{2+\theta}f(x,t)t^2\quad\forall t\in(0,M).\tag{4.8}$$

Using (4.5) and (4.6), we find

$$\left(\frac{1}{2}-\frac{1}{2+\theta}\right)||u_n||^2-\int_{\mathbb{R}^N}\left[F(x,u_n)-\frac{1}{2+\theta}f(x,u_n)u_n^2\right]dx=c-\frac{1}{2+\theta}\langle I'(u_n),u_n\rangle+o(1). \tag{4.9}$$

This estimate, together with (4.8) and Remark 2.3, yields

$$\begin{aligned}\frac{\theta}{2(2+\theta)}||u_n||^2&=c+\left\{\int_{\{x;|u_n|\geq M\}}+\int_{\{x;|u_n|<M\}}\right\}\left[F(x,u_n)-\frac{1}{2+\theta}f(x,u_n)u_n^2\right]dx\\&\quad-\frac{1}{2+\theta}\langle I'(u_n),u_n\rangle+o(1)\\&\leq c+\frac{\theta}{2(2+\theta)}\int_{\{x;|u_n|\geq M\}}f(x,u_n)u_n^2dx+\frac{1}{2+\theta}|\langle I'(u_n),u_n\rangle|+o(1).\end{aligned} \tag{4.10}$$

On the other hand, it follows from (4.5) and (4.6) that

$$\int_{\mathbb{R}^N}[f(x,u_n)u_n^2-2F(x,u_n)]dx=2c-\langle I'(u_n),u_n\rangle+o(1). \tag{4.11}$$

Then, from (4.7) and Remark 2.3, we have

$$\begin{aligned}&||I'(u_n)||||u_n||+2c+o(1)\\&\quad\geq\int_{\{x;|u_n|\geq M\}}[f(x,u_n)u_n^2-2F(x,u_n)]dx\geq C\int_{\{x;|u_n|\geq M\}}|u_n|^r dx.\end{aligned} \tag{4.12}$$

Thus, for a constant $C_0>0$, we infer that

$$\int_{\{x;|u_n|\geq M\}}|u_n|^r dx\leq C_0[1+||I'(u_n)||\,||u_n||]. \tag{4.13}$$

Relations (3.19) and (4.10) ensure

$$\begin{aligned}&\frac{\theta}{2(2+\theta)}||u_n||^2\\&\quad\leq c+\frac{\theta}{2(2+\theta)}\int_{\{x;|u_n|\geq M\}}\left(\epsilon|u_n|^2+D_\epsilon|u_n|^{2+\sigma}\right)dx+\frac{1}{2+\theta}||I'(u_n)||\,||u_n||+o(1)\\&\quad\leq c+\epsilon\int_{\mathbb{R}^N}|u_n|^2dx+D_\epsilon\int_{\{x;|u_n|\geq M\}}|u_n|^{1+\sigma}|u_n|dx+\frac{1}{2+\theta}||I'(u_n)||||u_n||+o(1).\end{aligned} \tag{4.14}$$

Notice from the expressions of 2_α^* and σ in (3.19) that

$$\frac{1+\sigma}{r}=1-\frac{1}{2_\alpha^*}<1. \tag{4.15}$$

Then, using the Hölder inequality, Lemma 2.1, and (4.13), we obtain that there exist a constant $D'_\epsilon > 0$ depending on ϵ and a constant $C > 0$ such that

$$\begin{aligned}\frac{\theta}{2(2+\theta)}\|u_n\|^2 &\le c+\epsilon\|u_n\|^2_{L^2(\mathbb{R}^N)}+D_\epsilon\left(\int_{\{x;|u_n|\ge M\}}|u_n|^r dx\right)^{(1+\sigma)/r}\left(\int_{\mathbb{R}^N}|u_n|^{2^*_\alpha}dx\right)^{1/2^*_\alpha}\\ &\quad+\frac{1}{2+\theta}\|I'(u_n)\|\|u_n\|+o(1)\\ &\le c+\epsilon\|u_n\|^2_{L^2(\mathbb{R}^N)}+D'_\epsilon\left[1+\|I'(u_n)\|\|u_n\|\right]^{(1+\sigma)/r}\|u_n\|\\ &\quad+\frac{1}{2+\theta}\|I'(u_n)\|\|u_n\|+o(1)\\ &\le c+\epsilon C\|u_n\|^2+D'_\epsilon\left[1+\|I'(u_n)\|\|u_n\|\right]\|u_n\|+\frac{1}{2+\theta}\|I'(u_n)\|\|u_n\|+o(1).\end{aligned} \tag{4.16}$$

Fix

$$\epsilon\in\left(0,\frac{\theta}{2C(2+\theta)}\right). \tag{4.17}$$

Recalling that $\|I'(u_n)\|\to 0$ (cf. the second relation in (4.4)), the above inequality shows that $\{u_n\}$ is bounded. Thus there exist $u_0\in E$ and a subsequence of $\{u_n\}$ converging weakly to u_0 in E. Consequently, Lemma 3.1 and (4.4) imply that u_0 is a weak solution of problem (2.8).

To complete the proof of Theorem 2.6 it remains to show that u_0 is nontrivial. By (4.4) and (4.6) we see that

$$c=\lim_{n\to\infty}I(u_n)=\lim_{n\to\infty}\left[\frac{1}{2}\langle I'(u_n),u_n\rangle_E+\int_{\mathbb{R}^N}\left(\frac{1}{2}f(x,u_n)u_n^2-F(x,u_n)\right)dx\right]. \tag{4.18}$$

Then using the boundedness of the sequence $\{u_n\}$ and the second relation in (4.4), we find

$$c=\lim_{n\to\infty}\int_{\mathbb{R}^N}\left(\frac{1}{2}f(x,u_n)u_n^2-F(x,u_n)\right)dx. \tag{4.19}$$

We justify that here Fatou's lemma can be applied. To this end, we notice that assumptions (c1), (c2), and (c5) ensure the existence of a (sufficiently large) constant $c_0>0$ such that

$$|f(x,t)|\le c_0|t|^\tau,\qquad |F(x,t)|\le c_0|t|^{\tau+2}\quad \forall (x,t)\in\mathbb{R}^N\times\mathbb{R}. \tag{4.20}$$

By Remark 2.3 it is known that the space E is continuously embedded in $\mathcal{D}^{1,2}_\alpha(\mathbb{R}^N)$. Applying [11, Proposition 3.4], it follows that E is compactly embedded in $L^{\tau+2}(\mathbb{R}^N)$ because $2<\tau+2<2^*_\alpha$. Consequently, up to a subsequence, we may suppose that $\{u_n\}$ converges to u_0 strongly in $L^{\tau+2}(\mathbb{R}^N)$ and a.e. in $\mathbb{R}^N$, and there is a function $h\in L^{\tau+2}(\mathbb{R}^N)$ such that $|u_n(x)|\le h(x)$ for almost all $x\in\mathbb{R}^N$. Therefore

$$\begin{aligned}&\frac{1}{2}f(x,u_n)u_n^2-F(x,u_n)\longrightarrow\frac{1}{2}f(x,u_0)u_0^2-F(x,u_0)\quad\text{for a.e. } x\in\mathbb{R}^N,\\ &\left|\frac{1}{2}f(x,u_n(x))u_n(x)^2-F(x,u_n(x))\right|\le\frac{3}{2}c_0h(x)^{\tau+2}\quad\text{for a.e. } x\in\mathbb{R}^N.\end{aligned} \tag{4.21}$$

Thus one may apply Fatou's lemma to the sequence $\{(1/2)f(x,u_n)u_n^2 - F(x,u_n)\}$. Using also that u_0 solves problem (2.8), we then obtain

$$\begin{aligned} c &\leq \int_{\mathbb{R}^N} \limsup_{n\to\infty} \left[\frac{1}{2}f(x,u_n)u_n^2 - F(x,u_n)\right]dx \\ &= \int_{\mathbb{R}^N} \left[\frac{1}{2}f(x,u_0)u_0^2 - F(x,u_0)\right]dx \\ &= I(u_0) - \frac{1}{2}\langle I'(u_0),u_0\rangle_E = I(u_0). \end{aligned} \tag{4.22}$$

Since $c \leq I(u_0)$ and $c > 0$ (as remarked at the beginning of this section), we conclude that $u_0 \not\equiv 0$.

5. Proof of Theorem 2.9

We first state a minimax-type lemma which will be used in the sequel. A version of this result under the Palais-Smale condition has been given in [20]. Applications to different classes of variational bifurcation problems can be found in [8, 19] and in [21, Chapter 9].

Lemma 5.1. *Let E be a real Banach space, let a function $G : E\times\mathbb{R} \to \mathbb{R}$ be of class C^1, and let two positive numbers $\rho < r$ be such that the following condition is fulfilled:*

$$\inf_{v\in E} G(v,\rho) > \max\{G(0,0),G(0,r)\}. \tag{5.1}$$

Denoting

$$c := \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} G(\gamma(t)), \tag{5.2}$$

with

$$\Gamma = \{\gamma \in C([0,1],E\times\mathbb{R}) : \gamma(0) = (0,0),\ \gamma(1) = (0,r)\}, \tag{5.3}$$

then there exists a sequence $\{u_n\} \subset E\times\mathbb{R}$ such that $G(u_n) \to c$ and $G'(u_n) \to 0$.

Proof. We apply [6, Theorem 1] for $X = E\times\mathbb{R}$, $K^* = \{(0,0),(0,r)\}$, p^* being the identity map on K^*, and K equal to the closed segment joining in $E\times\mathbb{R}$ the points $(0,0)$ and $(0,r)$. It is possible to apply this result because the imposed assumption ensures that for every $p \in C(K,X)$ with $p = p^*$ on K^* one has $\max_{\xi\in K} G(p(\xi)) > \max_{\xi\in K^*} G(p^*(\xi))$. Then the desired conclusion follows. □

We proceed with the proof of Theorem 2.9. Choose positive numbers $\rho < r$ and a function $\beta \in C^1(\mathbb{R})$ with the properties

$$\beta(0) = \beta(r) = 0, \tag{5.4}$$

$$\rho \geq \sqrt{pa_2}, \qquad \beta(\rho) > \frac{pa_1}{2}, \tag{5.5}$$

$$\beta(t) \longrightarrow +\infty \quad \text{as } |t| \longrightarrow +\infty, \tag{5.6}$$

$$\beta'(t) < 0 \Longleftrightarrow t < 0 \quad \text{or} \quad \rho < t < r. \tag{5.7}$$

Considering the Hilbert space E introduced in Section 2 and the function J in problem (2.31), we define $G: E\times\mathbb{R}\to\mathbb{R}$ by

$$G(u,t)=\frac{1}{p}t^2\|u\|^p+\frac{2}{p}\beta(t)-J(u)+\frac{\nu}{2}\|u\|^2 \quad \forall(u,t)\in E\times\mathbb{R}, \tag{5.8}$$

where $\|\cdot\|$ is the norm on E given in Section 2 and $\nu>0$ is the number prescribed in the statement of Theorem 2.9. Since E is a Hilbert space and $J\in C^1(E,\mathbb{R})$, it is clear that $G\in C^1(E\times\mathbb{R})$ and its partial gradients have the expressions

$$\begin{aligned} G_u(u,t)&=t^2\|u\|^{p-2}u-\nabla J(u)+\nu u \quad \forall(u,t)\in E\times\mathbb{R},\\ G_t(u,t)&=\frac{2}{p}\left(t\|u\|^p+\beta'(t)\right) \quad \forall(u,t)\in E\times\mathbb{R}. \end{aligned} \tag{5.9}$$

It follows readily from (5.8), (5.4), and the first relation in (2.27) that

$$\begin{aligned} G(0,0)&=\frac{2}{p}\beta(0)-J(0)\le 0,\\ G(0,r)&=\frac{2}{p}\beta(r)-J(0)\le 0. \end{aligned} \tag{5.10}$$

Moreover, from (5.8), (2.28), and (5.5), we get the estimate

$$G(u,\rho)\ge\left(\frac{1}{p}\rho^2-a_2\right)\|u\|^p+\frac{2}{p}\beta(\rho)-a_1\ge\frac{2}{p}\beta(\rho)-a_1>0 \quad \forall u\in E. \tag{5.11}$$

Therefore the requirement in Lemma 5.1 is fulfilled. Applying Lemma 5.1 provides a sequence $\{(v_n,t_n)\}\subset E\times\mathbb{R}$ such that $G(v_n,t_n)\to c$, $G_u(v_n,t_n)\to 0$ in E and $G_t(v_n,t_n)\to 0$ in $\mathbb{R}$. Taking into account relations (5.8), (5.9), these convergences read as

$$\frac{1}{p}t_n^2\|v_n\|^p+\frac{2}{p}\beta(t_n)-J(v_n)+\frac{\nu}{2}\|v_n\|^2\longrightarrow c, \tag{5.12}$$

$$t_n^2\|v_n\|^{p-2}v_n-\nabla J(v_n)+\nu v_n\longrightarrow 0, \tag{5.13}$$

$$t_n\|v_n\|^p+\beta'(t_n)\longrightarrow 0. \tag{5.14}$$

By (5.12) and (2.28) we see that

$$\begin{aligned} c+o(1)&=\frac{1}{p}t_n^2\|v_n\|^p+\frac{2}{p}\beta(t_n)-J(v_n)+\frac{\nu}{2}\|v_n\|^2\\ &\ge\left(\frac{1}{p}t_n^2-a_2\right)\|v_n\|^p+\frac{2}{p}\beta(t_n)-a_1. \end{aligned} \tag{5.15}$$

Then (5.6) enables us to deduce that the sequence $\{t_n\}$ is bounded in $\mathbb{R}$. Thus there is $\bar{t}\in\mathbb{R}$ such that along a relabelled subsequence we may suppose

$$t_n\longrightarrow\bar{t} \quad \text{in } \mathbb{R} \text{ as } n\longrightarrow\infty. \tag{5.16}$$

We claim that

$$\{v_n\} \text{ is bounded in } E. \tag{5.17}$$

In order to prove (5.17), we first consider the case $\bar{t} \neq 0$. In this situation, for n sufficiently large, writing (5.14) in the form

$$\|v_n\|^p = \frac{1}{t_n}\left(o(1) - \beta'(t_n)\right), \tag{5.18}$$

it results that assertion (5.17) is verified because $\{t_n\}$ is bounded away from zero.

Assume now that $\bar{t} = 0$. Then (5.14) and (5.4) ensure that $t_n\|v_n\|^p \to 0$. In view of (5.16), we then have $t_n^2\|v_n\|^p \to 0$. It turns out from (5.12) that

$$-J(v_n) + \frac{\nu}{2}\|v_n\|^2 \longrightarrow c. \tag{5.19}$$

On the other hand, from $t_n\|v_n\|^p \to 0$ we deduce

$$|t_n|^{(p-1)/p}\|v_n\|^{p-1} = \left(|t_n|\,\|v_n\|^p\right)^{(p-1)/p} \longrightarrow 0, \tag{5.20}$$

which in turn, using (5.16), yields $t_n^2\|v_n\|^{p-1} \to 0$. Combining with (5.13) implies

$$-\nabla J(v_n) + \nu v_n \longrightarrow 0. \tag{5.21}$$

By relations (5.19), (5.21) and assumption (2.29), we may write

$$\begin{aligned} c + o(1) + \frac{1}{\gamma}\|v_n\| &\geq \nu\left(\frac{1}{2} - \frac{1}{\gamma}\right)\|v_n\|^2 + \frac{1}{\gamma}\langle \nabla J(v_n), v_n\rangle_E - J(v_n) \\ &\geq \left[\nu\left(\frac{1}{2} - \frac{1}{\gamma}\right) - b_2\right]\|v_n\|^2 - b_1, \end{aligned} \tag{5.22}$$

if n is sufficiently large. Since

$$b_2 < \nu\left(\frac{1}{2} - \frac{1}{\gamma}\right), \tag{5.23}$$

we arrive at the conclusion (5.17) in the situation $\bar{t} = 0$, too. The verification of the claim in (5.17) is complete.

On the basis of (5.17) we know that there is $u \in E$ such that along a relabelled subsequence one has $v_n \rightharpoonup u$ in E. According to (5.13), we note

$$\langle \eta_n v_n - \nabla J(v_n), \varphi\rangle_E \longrightarrow 0 \tag{5.24}$$

for all $\varphi \in C_c^\infty(\mathbb{R}^N \setminus \{0\})$, where

$$\eta_n = \nu + t_n^2 ||v_n||^{p-2}. \tag{5.25}$$

Passing eventually to a subsequence, from (5.17) we may assume that there exists $\theta := \lim_{n\to\infty} \|v_n\|$ and

$$\|u\| \leq \liminf_{n\to\infty} ||v_n|| \leq \theta. \tag{5.26}$$

Letting $n \to \infty$ in (5.24) and (5.25), and using (2.30) and (5.16), we obtain

$$\langle u - \mu \nabla J(u), \varphi \rangle_E = 0, \quad \forall \varphi \in C_c^\infty(\mathbb{R}^N \setminus \{0\}), \tag{5.27}$$

with

$$\mu = \frac{1}{\nu + \bar{t}\theta^{p-2}}. \tag{5.28}$$

Taking into account the definition of the inner product on the space E, it is clear that equality (5.27) is just (2.31) with the eigenvalue μ in (5.28).

In addition, from (5.14), (5.16) and the definition of θ, we obtain the equality

$$\bar{t}\theta^p + \beta'(\bar{t}) = 0. \tag{5.29}$$

This implies that

$$\bar{t}\beta'(\bar{t}) \leq 0. \tag{5.30}$$

Notice that $\bar{t} \neq 0$. Indeed, if $\bar{t} = 0$, then (5.28) yields that $1/\nu$ is an eigenvalue of (2.31), which contradicts (2.32). Further, we observe from (5.26), in conjunction with the assumption $\nabla J(0) \neq 0$ in (2.27) and relation (5.27), that $\theta \neq 0$. Since $\bar{t} \neq 0$, we deduce from (5.30) and (5.7) that $\rho \leq \bar{t} \leq r$. Knowing by (5.7) that $\beta'(\rho) = \beta'(r) = 0$, it follows from (5.29) that the inequality $\rho \leq \bar{t} \leq r$ can be sharpened to

$$\rho < \bar{t} < r. \tag{5.31}$$

Thus (5.28) and (5.31) allow us to write

$$\frac{1}{\nu + r^2\theta^{p-2}} < \mu < \frac{1}{\nu + \rho^2\theta^{p-2}}. \tag{5.32}$$

For $p = 2$, (5.32) represents just (2.34). For $p > 2$, the relations (5.26) and (5.32) entail (2.33). This completes the proof.

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