THE FIRST EIGENVALUE OF p-LAPLACIAN SYSTEMS WITH NONLINEAR BOUNDARY CONDITIONS

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We study the properties of the positive principal eigenvalue and the corresponding eigenspaces of two quasilinear elliptic systems under nonlinear boundary conditions. We prove that this eigenvalue is simple, unique up to positive eigenfunctions for both systems, and isolated for one of them.

1. Introduction

Let Ω be an unbounded domain in $\mathbb{R}^N$, $N \geq 2$, with a noncompact and smooth boundary $\partial\Omega$. In this paper we prove certain properties of the principal eigenvalue of the following quasilinear elliptic systems

$$\begin{aligned} -\Delta_p u &= \lambda a(x)|u|^{p-2}u + \lambda b(x)|u|^{\alpha-1}|v|^{\beta+1}u, \quad \text{in } \Omega, \\ -\Delta_q v &= \lambda d(x)|v|^{q-2}v + \lambda b(x)|u|^{\alpha+1}|v|^{\beta-1}v, \quad \text{in } \Omega, \end{aligned} \tag{1.1}$$

$$\begin{aligned} -\Delta_p u &= \lambda a(x)|u|^{p-2}u + \lambda b(x)|u|^{\alpha}|v|^{\beta}v \quad \text{in } \Omega, \\ -\Delta_q v &= \lambda d(x)|v|^{q-2}v + \lambda b(x)|u|^{\alpha}|v|^{\beta}u \quad \text{in } \Omega \end{aligned} \tag{1.2}$$

satisfying the nonlinear boundary conditions

$$\begin{aligned} |\nabla u|^{p-2}\nabla u \cdot \eta + c_1(x)|u|^{p-2}u &= 0 \quad \text{on } \partial\Omega, \\ |\nabla v|^{q-2}\nabla v \cdot \eta + c_2(x)|v|^{q-2}v &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{1.3}$$

where η is the unit outward normal vector on $\partial\Omega$. As it will be clear later, under condition (H1), $1 < p,q < N$, $\alpha,\beta \geq 0$ and

$$\frac{\alpha+1}{p} + \frac{\beta+1}{q} = 1, \quad \alpha+1 < \frac{pq^*}{N}, \ \beta+1 < \frac{p^*q}{N}, \tag{1.4}$$

systems (1.1), (1.2) are in fact nonlinear eigenvalue problems. Our procedure here will be based on the proper space setting provided in [14], (see Section 2). In this section, we also state the assumptions on the coefficient functions.

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Problems of such a type arise in a variety of applications, for example, non-Newtonian fluids, reaction-diffusion problems, theory of superconductors, biology, and so forth, (see [2, 15] and the references therein). As a consequence, there are many works treating nonlinear systems from different points of view, for example, [4, 7, 9, 11, 13].

Properties of the principal eigenvalue are of prime interest since for example they are closely associated with the dynamics of the associated evolution equations (e.g., global bifurcation, stability) or with the description of the solution set of corresponding perturbed problems (e.g., [17]). These properties are: *existence, positivity, simplicity, uniqueness up to eigenfunctions which do not change sign and isolation*, which hold in the case of the Laplacian operator in a bounded domain. It is well known that these properties also hold for the p-Laplacian scalar eigenvalue problem (in both bounded and unbounded domains) and were recently obtained in [12] under nonlinear boundary conditions while the case of some (p,q)-Laplacian systems with Dirichlet boundary conditions was also successfully treated in [1, 10, 16, 18].

Note that we discuss the case of a *potential (or gradient) system*, which is a restriction. However, this is in some sense natural because the aforementioned properties of the principal eigenvalue are stronger than in the scalar equation case; for example the principal eigenvalue of the system is the only eigenvalue which admits a nonnegative eigenfunction in the sense that both components do not change sign. It is also remarkable that the associated "eigenspaces" are generally not linear subspaces.

Starting with the system (1.1)–(1.3), we proceed as follows: in Section 2, we give the space setting and the assumptions on the coefficient functions. In Section 3, using the compactness of the corresponding operators we prove the existence and positivity of λ_1 and we state a regularity result based on the iterative procedure of [5]. In Section 4, we prove the simplicity and the uniqueness up to positive (componentwise) eigenfunctions. This is done by using the Picone's identity (see [1]). Finally, in Section 5, we prove Theorem 2.3 by establishing the connection between the two systems with respect to existence and simplicity of the common principal eigenvalue λ_1 as well as the regularity of the eigenfunctions. In addition, we show that λ_1 is isolated for the system (1.2)-(1.3).

2. Preliminaries and statement of the results

Let Ω be an unbounded domain in R^N, $N \geq 2$, with a noncompact and smooth boundary $\partial\Omega$. For $m > 0$ and $r \in (1,+\infty)$ let $w_m(x) = 1/(1+|x|)^m$ and assume that the space $L^r(w_m,\Omega) := \{u : \int_\Omega (1/(1+|x|)^m)|u|^r < +\infty\}$ is supplied with the norm

$$\|u\|_{w_m,r} = \left(\int_\Omega \frac{1}{(1+|x|)^m}|u|^r\right)^{1/r}. \tag{2.1}$$

We require the following hypotheses:

(H1) $1 < p,q < N$, $\alpha,\beta \geq 0$ with $(\alpha+1)/p + (\beta+1)/q = 1$, $\alpha + 1 < pq^*/N$ and $\beta + 1 < p^*q/N$.

Here p^* and q^* are the critical Sobolev exponents defined by

$$p^* = \frac{pN}{N-p}, \qquad q^* = \frac{qN}{N-q}. \tag{2.2}$$

(H2)

(i) There exists positive constants α_1, A_1 with $\alpha_1 \in (p+((\beta+1)(N-p)/q^*),N)$ and

$$0 < a(x) \leq A_1 w_{\alpha_1}(x) \quad \text{a.e. in } \Omega, \tag{2.3}$$

(ii) there exists positive constants α_2, D_1 with $\alpha_2 \in (q+((\alpha+1)(N-q)/p^*),N)$ and

$$0 < d(x) \leq D_1 w_{\alpha_2}(x) \quad \text{a.e. in } \Omega, \tag{2.4}$$

(iii) $m\{x \in \Omega : b(x) > 0\} > 0$ and

$$0 \leq b(x) \leq B_1 w_s(x) \quad \text{a.e. in } \Omega, \tag{2.5}$$

where $B_1 > 0$ and $s \in (\max\{p,q\},N)$.

(H3) $c_1(\cdot)$ and $c_2(\cdot)$ are positive and continuous functions defined on R^N with

$$\begin{aligned} k_1 w_{p-1}(x) &\leq c_1(x) \leq K_1 w_{p-1}(x), \\ l_1 w_{q-1}(x) &\leq c_2(x) \leq L_1 w_{q-1}(x), \end{aligned} \tag{2.6}$$

for some positive constants k_1, K_1, l_1, L_1.

Let $C_\delta^\infty(\Omega)$ be the space of $C_0^\infty(R^N)$-functions restricted to Ω. For $m \in (1,+\infty)$, the weighted Sobolev space E_m is the completion of $C_\delta^\infty(\Omega)$ in the norm

$$|||u|||_m = \left(\int_\Omega |\nabla u|^m + \int_\Omega \frac{1}{(1+|x|)^m} |u|^m \right)^{1/m}. \tag{2.7}$$

By [14, Lemma 2] we see that if $c(\cdot)$ is a positive continuous function defined on R^n then the norm

$$\|u\|_{1,m} = \left(\int_\Omega |\nabla u|^m + \int_{\partial\Omega} c(x)|u|^m \right)^{1/m} \tag{2.8}$$

is equivalent to $|||\cdot|||_m$. The proof of the following lemma is also provided in [14].

Lemma 2.1. (i) *If*

$$p \leq r \leq \frac{pN}{N-p}, \quad N > \alpha \geq N - r\frac{N-p}{p}, \tag{2.9}$$

then the embedding $E \subseteq L^r(w_\alpha,\Omega)$ is continuous. If the upper bound for r in the first inequality and the lower bound in the second is strict, then the embedding is compact.

(ii) *If*

$$p \le m \le \frac{p(N-1)}{N-p}, \quad N > \beta \ge N-1-m\frac{N-p}{p}, \tag{2.10}$$

then the embedding $E \subseteq L^m(w_\beta,\partial\Omega)$ is continuous. If the upper bounds for m are strict, then the embedding is compact.

It is natural to consider our systems on the space $E = E_p \times E_q$ supplied with the norm

$$\left|\left|(u,v)\right|\right|_{pq} = \|u\|_{1,p} + \|v\|_{1,q}. \tag{2.11}$$

We now define the functionals Φ, I, $J : E \to R$ as follows:

$$\begin{aligned}
\Phi(u,v) &= \frac{\alpha+1}{p}\int_\Omega |\nabla u|^p + \frac{\alpha+1}{p}\int_{\partial\Omega} c_1(x)|u|^p + \frac{\beta+1}{q}\int_\Omega |\nabla v|^p + \frac{\beta+1}{q}\int_{\partial\Omega} c_2|v|^q \\
&\quad - \lambda\frac{\alpha+1}{p}\int_\Omega a(x)|u|^p - \lambda\frac{\beta+1}{q}\int_\Omega d(x)|v|^q - \lambda\int_\Omega b(x)|u|^{\alpha+1}|v|^{\beta+1}, \\
I(u,v) &= \frac{\alpha+1}{p}\int_\Omega |\nabla u|^p + \frac{\beta+1}{q}\int_\Omega |\nabla v|^p + \frac{\alpha+1}{p}\int_{\partial\Omega} c_1(x)|u|^p + \frac{\beta+1}{q}\int_{\partial\Omega} c_2|v|^q, \\
J(u,v) &= \frac{\alpha+1}{p}\int_\Omega a(x)|u|^p + \frac{\beta+1}{q}\int_\Omega d(x)|v|^q + \int_\Omega b(x)|u|^{\alpha+1}|v|^{\beta+1}.
\end{aligned} \tag{2.12}$$

In view of (H1)–(H3), the functionals Φ, I, J are well defined and continuously differentiable on E. By a *weak solution* of (1.1) we mean an element (u_0,v_0) of E which is a critical point of the functional Φ.

The main results of this work are the following theorems.

Theorem 2.2. *Let Ω be an unbounded domain in R^N, $N \ge 2$, with a noncompact and smooth boundary $\partial\Omega$. Assume that the hypotheses (H1), (H2), and (H3) hold. Then*

(i) *System (1.1)–(1.3) admits a positive principal eigenvalue λ_1 given by*

$$\lambda_1 = \inf\{I(u,v) : J(u,v) = 1\}. \tag{2.13}$$

Each component of the associated normalized eigenfunction (u_1,v_1) is positive in Ω and of class $C^{1,\delta}_{\text{loc}}(\Omega)$ for some $\delta \in (0,1)$.

(ii) *The set of eigenfunctions corresponding to λ_1 forms a one dimensional manifold $E_1 \subseteq E$ defined by*

$$E_1 = \{(cu_1, \pm|c|^{p/q}v_1) : c \in R\backslash\{0\}\}. \tag{2.14}$$

Furthermore, a componentwise positive eigenfunction always corresponds to λ_1.

THEOREM 2.3. *Assume that the hypotheses of Theorem 2.2 hold.*

(a) *System (1.2)-(1.3) shares the same positive principal eigenvalue* λ_1 *and the same properties of the associated eigenfunctions with (1.1)–(1.3).*

(b) *The set of eigenfunctions corresponding to* λ_1 *forms a one dimensional manifold* $E_2 \subseteq E$ *defined by*

$$E_2 = \{ \pm (cu_1, c^{p/q}v_1) : c > 0\}. \tag{2.15}$$

(c) λ_1 *is isolated for the system (1.2)-(1.3), in the sense that there exists* $\eta > 0$ *such that the interval* $(0, \lambda_1 + \eta)$ *does not contain any other eigenvalue than* λ_1.

3. Existence and regularity

In this section, we prove the existence of a positive principal eigenvalue and the regularity of the corresponding eigenfunctions for the system (1.1)–(1.3).

Existence. The operators I, J are continuously Fréchet differentiable, I is coercive on $E \cap \{J(u,v) \leq \text{const}\}$, J is compact and $J'(u,v) = 0$ only at $(u,v) = 0$. So the assumptions of Theorem 6.3.2 in [3] are fulfilled implying the existence of a principal eigenvalue λ_1, satisfying

$$\lambda_1 = \inf_{J(u,v)=1} I(u,v). \tag{3.1}$$

Moreover, if (u_1, v_1) is a minimizer of (2.13) then $(|u_1|, |v_1|)$ should be also a minimizer. Hence, we may assume that there exists an eigenfunction (u_1, v_1) corresponding to λ_1, such that $u_1 \geq 0$ and $v_1 \geq 0$, a.e. in Ω.

Regularity. We show first that $w_p u_1$ and $w_q v_1$ are essentially bounded in Ω. To that purpose define $u_M(x) := \min\{u_1(x), M\}$. It is clear that $u_M^{kp+1} \in E_p$, for $k \geq 0$. Multiplying the first equation of (1.1) by u_M^{kp+1} and integrating over Ω, we get

$$\begin{aligned} &\int_\Omega |\nabla u_1|^{p-2} \nabla u_1 \cdot \nabla \left(u_M^{kp+1}\right) dx + \int_{\partial\Omega} c_1(x) u_1^{p-1} u_M^{kp+1}\, dx \\ &\quad \leq \lambda_1 \int_\Omega a(x) u_1^{(k+1)p}\, dx + \lambda_1 \int_\Omega b(x) v_1^{\beta+1} u_1{}^{kp+\alpha+1}\, dx. \end{aligned} \tag{3.2}$$

Note that

$$\int_\Omega |\nabla u_1|^{p-2} \nabla u_1 \cdot \nabla \left(u_M^{kp+1}\right) dx = (kp+1) \int_\Omega |\nabla u_M|^p u_M^{kp}\, dx = \frac{kp+1}{(k+1)^p} \int_\Omega |\nabla u_M^{k+1}|^p dx, \tag{3.3}$$

so since $(kp+1)/(k+1)^p \leq 1$, then

$$\begin{aligned} &\int_\Omega |\nabla u_1|^{p-2} \nabla u_1 \cdot \nabla \left(u_M^{kp+1}\right) dx + \int_{\partial\Omega} c_1(x) u_1^{p-1} u_M^{kp+1}\, dx \\ &\quad \geq c_3 \frac{kp+1}{(k+1)^p} \left(\int_\Omega \frac{1}{(1+|x|)^p} u_M^{(k+1)p^*}\, dx \right)^{p/p^*} \end{aligned} \tag{3.4}$$

due to Lemma 2.1(i) and (2.8). Let $t = p(1-(\beta+1/q^*))^{-1}$, which is less than p^* because of H(1). Then H(2)(i) and Hölder inequality imply that

$$\begin{aligned}\int_\Omega a(x)u_1^{(k+1)p}dx &\le A_1\int_\Omega \frac{1}{(1+|x|)^{\alpha_1}}u_1^{(k+1)p}dx\\ &= A_1\int_\Omega \frac{1}{(1+|x|)^{\alpha_1-p^2/t}}\frac{u_1^{(k+1)p}}{(1+|x|)^{p^2/t}}dx\\ &\le A_1\left(\int_\Omega \frac{1}{(1+|x|)^{(t\alpha_1-p^2)/(t-p)}}dx\right)^{(t-p)/t}\left(\int_\Omega \frac{1}{(1+|x|)^{p}}u_1^{(k+1)t}dx\right)^{p/t}\end{aligned} \tag{3.5}$$

(observe that $(t\alpha_1-p^2)/(t-p) > N$ by H(2)(i)). Also, because of (H1), we may assume that

$$\int_\Omega b(x)v_1^{\beta+1}u_1^{kp+\alpha+1}dx \le \int_\Omega b(x)v_1^{\beta+1}u_1^{(k+1)p}dx, \tag{3.6}$$

otherwise we could consider

$$u_M(x) = \begin{cases}\min\{u_1(x), M\}, & u_1(x)\ge 1,\\ 0, & u_1(x) < 1\end{cases} \tag{3.7}$$

as a test function. So

$$\begin{aligned}\int_\Omega b(x)v_1^{\beta+1}u_1^{(k+1)p}dx &\le B_1\int_\Omega \frac{1}{(1+|x|)^{s}}v_1^{\beta+1}u_1^{(k+1)p}dx\\ &= B_1\int_\Omega \frac{v_1^{\beta+1}}{(1+|x|)^{s(1-(p/t))}}\frac{u_1^{(k+1)p}}{(1+|x|)^{s(p/t)}}dx\\ &\le B_1\left(\int_\Omega \frac{v_1^{(\beta+1)(t/t-p)}}{(1+|x|)^{s}}dx\right)^{(t-p)/t}\left(\int_\Omega \frac{u_1^{(k+1)t}}{(1+|x|)^{s}}dx\right)^{p/t}\\ &\le B_1\left(\int_\Omega \frac{1}{(1+|x|)^{q}}v_1^{q^*}dx\right)^{(t-p)/t}\left(\int_\Omega \frac{1}{(1+|x|)^{p}}u_1^{(k+1)t}dx\right)^{p/t},\end{aligned} \tag{3.8}$$

by H(2)(iii). On combining (3.2)–(3.8), we conclude that

$$\|u_M\|_{w_p,(k+1)p^*} \le C^{1/(k+1)}\left[\frac{k+1}{(kp+1)^{1/p}}\right]^{1/(k+1)}\|u_1\|_{w_p,(k+1)t}, \tag{3.9}$$

where C is independent of M and k. We now follow the same steps as in the proof of [8, Theorem 2] or [5, Lemma 3.2]. Let $k_1 = (p^*/t)-1$. Since $(k_1p+1)/(k_1+1)^p \le 1$, we can choose $k = k_1$ in (3.9) to get

$$\|u_M\|_{w_p,(k_1+1)p^*} \le C^{1/(k_1+1)} \left[\frac{k_1+1}{(k_1p+1)^{1/p}} \right]^{1/(k_1+1)} \|u_1\|_{w_p,p^*}, \tag{3.10}$$

while by letting $M \to \infty$ we obtain that

$$\|u_1\|_{w_p,(k_1+1)p^*} \le C^{1/(k_1+1)} \left[\frac{k_1+1}{(k_1p+1)^{1/p}} \right]^{1/(k_1+1)} \|u_1\|_{w_p,p^*}. \tag{3.11}$$

Hence, $u_1 \in L^{(k_1+1)p^*}(w_p,\Omega)$. Note that if $k \ge k_1$ then $(kp+1)/(k+1)^p \le 1$. Choosing in (1.1) $k = k_2$ with $(k_2+1)t = (k_1+1)p^*$, that is, $k_2 = (p^*/t)^2 - 1$, we have

$$\|u_1\|_{w_p,(k_2+1)p^*} \le C^{1/(k_1+1)} \left[\frac{k_2+1}{(k_2p+1)^{1/p}} \right]^{1/(k_2+1)} \|u_1\|_{w_p,(k_1+1)p^*}. \tag{3.12}$$

Hence, $u_1 \in L^{(k_2+1)p^*}(w_p,\Omega)$. Proceeding by induction we arrive at

$$\|u_1\|_{w_p,(k_n+1)p^*} \le C^{1/(k_n+1)} \left[\frac{k_n+1}{(k_np+1)^{1/p}} \right]^{1/(k_n+1)} \|u_1\|_{w_p,(k_{n-1}+1)p^*}. \tag{3.13}$$

From (3.10) and (3.13) we conclude that

$$\begin{aligned} \|u_1\|_{w_p,(k_n+1)p^*} &\le C^{\sum_{i=1}^n 1/(k_i+1)} \prod_{i=1}^n \left[\frac{k_i+1}{(k_ip+1)^{1/p}} \right]^{1/(k_i+1)} \|u_1\|_{w_p,p^*} \\ &= C^{\sum_{i=1}^n 1/(k_i+1)} \prod_{i=1}^n \left[\left(\frac{k_i+1}{(k_ip+1)^{1/p}} \right)^{1/\sqrt{k_i+1}} \right]^{1/\sqrt{k_i+1}} \|u_1\|_{w_p,p^*}. \end{aligned} \tag{3.14}$$

Since $(y + 1/(yp+1)^{1/p})^{1/\sqrt{y+1}} > 1$ for $y > 0$, and $\lim_{y\to\infty}(y + 1/(yp+1)^{1/p})^{1/\sqrt{y+1}} = 1$, there exists $K > 1$ independent of k_n such that

$$\|u_1\|_{w_p,(k_n+1)p^*} \le C^{\sum_{i=1}^n 1/(k_i+1)} K^{\sum_{i=1}^n 1/\sqrt{k_i+1}} \|u_1\|_{w_p,p^*}, \tag{3.15}$$

where $1/(k_i+1) = (t/p^*)^i$ and $1/\sqrt{k_i+1} = (\sqrt{t/p^*})^i$. Letting now $n \to \infty$ we conclude that

$$\|u_1\|_{w_p,\infty} \le c\|u_1\|_{w_p,p^*}, \tag{3.16}$$

for some positive constant c. By [8], $u_1 \in C^{1,\delta}_{\text{loc}}(\Omega)$. Similarly $v_1 \in C^{1,\delta}_{\text{loc}}(\Omega)$.

Finally, we notice that for the principal eigenvalue, each component of an eigenfunction is either positive or negative in Ω due to the Harnack inequality [8] and if we assume that $u_1(x_0) = 0$ for some $x_0 \in \partial\Omega$ then by [19, Theorem 5] we have $|\nabla u_1(x_0)|^{p-2}\nabla u_1(x_0) \cdot \eta(x_0) < 0$, contradicting (1.3). Thus $u_1 > 0$ (or $u_1 < 0$) on $\overline{\Omega}$. Similarly $v_1 > 0$ (or $v_1 < 0$) on $\overline{\Omega}$.

4. The eigenfunctions corresponding to λ_1

In this section, we complete the proof of Theorem 2.2 establishing the simplicity of λ_1. More precisely, we show that if (u_2, v_2) is another pair of eigenfunctions corresponding to λ_1, then there exists $c \in R\backslash\{0\}$ such that $(u_2, v_2) = (cu_1, \pm|c|^{p/q}v_1)$. To that end, we employ a technique similar to the one described in [1]. Namely, we will prove that if (w_1, w_2) is a positive on $\bar{\Omega}$ solution of the problem

$$\begin{gathered} -\Delta_p u \le \lambda a(x)|u|^{p-2}u + \lambda b(x)|u|^{\alpha-1}|v|^{\beta+1}u, \quad \text{in } \Omega, \\ -\Delta_q v \le \lambda d(x)|v|^{q-2}v + \lambda b(x)|u|^{\alpha+1}|v|^{\beta-1}v, \quad \text{in } \Omega, \\ |\nabla u|^{p-2}\nabla u \cdot \eta + c_1(x)|u|^{p-2}u = 0, \quad \text{on } \partial\Omega, \\ |\nabla v|^{q-2}\nabla v \cdot \eta + c_2(x)|v|^{q-2}v = 0, \quad \text{on } \partial\Omega, \end{gathered} \tag{4.1}$$

for some $\lambda > 0$, and (w_1', w_2') is a positive on $\bar{\Omega}$ solution of

$$\begin{gathered} -\Delta_p u \ge \lambda a(x)|u|^{p-2}u + \lambda b(x)|u|^{\alpha-1}|v|^{\beta+1}u \quad \text{in } \Omega, \\ -\Delta_q v \ge \lambda d(x)|v|^{q-2}v + \lambda b(x)|u|^{\alpha+1}|v|^{\beta-1}v \quad \text{in } \Omega, \\ |\nabla u|^{p-2}\nabla u \cdot \eta + c_1(x)|u|^{p-2}u = 0 \quad \text{on } \partial\Omega, \\ |\nabla v|^{q-2}\nabla v \cdot \eta + c_2(x)|v|^{q-2}v = 0 \quad \text{on } \partial\Omega \end{gathered} \tag{4.2}$$

then $(w_1', w_2') = (cw_1, c^{p/q}w_2)$ for a constant $c > 0$.

Let $\varphi \in C_\delta^\infty(\Omega)$, $\varphi > 0$, then $\varphi^p/(w_1')^{p-1} \in E_p$. By Picone's identity [1], we get

$$\begin{aligned} 0 \le \int_\Omega R(\varphi, w_1') &= \int_\Omega |\nabla\varphi|^p - \int_\Omega \nabla\left(\frac{\varphi^p}{(w_1')^{p-1}}\right) \cdot |\nabla w_1'|^p \nabla w_1' \\ &= \int_\Omega |\nabla\varphi|^p + \int_\Omega \frac{\varphi^p}{(w_1')^{p-1}}\Delta_p w_1' - \int_{\partial\Omega} \frac{\varphi^p}{(w_1')^{p-1}} |\nabla w_1'|^p \nabla w_1' \cdot \eta \\ &\le \int_\Omega |\nabla\varphi|^p - \lambda\int_\Omega \frac{\varphi^p}{(w_1')^{p-1}}\left(a(x)(w_1')^{p-1} + b(x)(w_1')^\alpha (w_2')^{\beta+1}\right) \\ &\quad - \int_{\partial\Omega} \frac{\varphi^p}{(w_1')^{p-1}} |\nabla w_1'|^p \nabla w_1' \cdot \eta \\ &= \int_\Omega |\nabla\varphi|^p - \lambda\int_\Omega a(x)\varphi^p \frac{(w_1')^{p-1}}{(w_1')^{p-1}} - \lambda\int_\Omega b(x)\varphi^p \frac{(w_1')^\alpha}{(w_1')^{p-1}}(w_2')^{\beta+1} \\ &\quad - \int_{\partial\Omega} \frac{\varphi^p}{(w_1')^{p-1}} |\nabla w_1'|^p \nabla w_1' \cdot \eta, \end{aligned} \tag{4.3}$$

while the boundary conditions imply that

$$\begin{aligned} 0 \le &\int_\Omega |\nabla\varphi|^p - \lambda\int_\Omega a(x)\varphi^p \frac{(w_1')^{p-1}}{(w_1')^{p-1}} - \lambda\int_\Omega b(x)\varphi^p \frac{(w_1')^\alpha}{(w_1')^{p-1}}(w_2')^{\beta+1} \\ &+ \int_{\partial\Omega} c_1(x)\frac{\varphi^p}{(w_1')^{p-1}}(w_1')^{p-1}. \end{aligned} \tag{4.4}$$

Letting $\varphi \to w_1$ in E_p we obtain

$$0 \le \int_\Omega |\nabla w_1|^p - \lambda \int_\Omega a(x) w_1^p - \lambda \int_\Omega b(x) w_1^p (w_1')^{\alpha - p + 1} (w_2')^{\beta+1} + \int_{\partial\Omega} c_1(x) w_1^p. \tag{4.5}$$

Note also that

$$\int_\Omega |\nabla w_1|^p + \int_{\partial\Omega} c_1(x) w_1^p \le \lambda \int_\Omega a(x) w_1^p + \lambda \int_\Omega b(x) w_1^{\alpha+1} w_2^{\beta+1}. \tag{4.6}$$

On combining (4.5) and (4.6) we get

$$0 \le \int_\Omega b(x) \Big(w_1^{\alpha+1} w_2^{\beta+1} - w_1^p (w_1')^{\alpha-p+1} (w_2')^{\beta+1} \Big). \tag{4.7}$$

Similarly,

$$0 \le \int_\Omega b(x) \Big(w_1^{\alpha+1} w_2^{\beta+1} - w_2^q (w_2')^{\beta+1-q} (w_1')^{\alpha+1} \Big). \tag{4.8}$$

We can now work as in Theorem 2.7 in [1] to get the desired result.

Returning to our problem, we obtain E_1 as the set of eigenfunctions corresponding to λ_1, simply by applying the previous result to the case of our system with $\lambda = \lambda_1$, and taking (u_1, v_1) instead of (w_1, w_2). One has now to combine the fact that the nonnegative solutions are given by $(cu_1, c^{p/q} v_1)$, $c > 0$, with the trivial observation that if (u, v) is an eigenfunction then $(-u, v)$, $(u, -v)$, $(-u, -v)$ are also eigenfunctions.

The same technique can be used for proving that nonnegative solutions in Ω correspond only to the first eigenvalue. Assume, for instance, that there exists an eigenpair (λ^*, u_2, v_2) for the problem (1.1) such that $\lambda^* > \lambda_1$, $u_2 \ge 0$ and $v_2 \ge 0$, a.e. in Ω. Then (u_1, v_1) is a solution of (1.2) with $\lambda = \lambda^*$ and (u_2, v_2) is a solution of (1.3). Then $(u_2, v_2) = (cu_1, c^{p/q} v_1)$, for some $c > 0$, which is a contradiction.

5. The second system

In this section, we present the proof of Theorem 2.3.

(a) Since for positive solutions systems (1.1) and (1.2) coincide, we deduce that (λ_1, u_1, v_1) is also an eigenpair for the system (1.2). Assume that there exists another nontrivial eigenpair (λ_*, u_*, v_*) of (1.2), such that $0 < \lambda_* < \lambda_1$. Then the following equality must be satisfied

$$\lambda_* = \frac{I(u_*, v_*)}{\tilde{J}(u_*, v_*)}, \tag{5.1}$$

with $\tilde{J}(u_*, v_*) > 0$, where $\tilde{J}(\cdot, \cdot)$ is defined by

$$\tilde{J}(u, v) = \frac{\alpha+1}{p} \int_\Omega a(x)|u|^p + \frac{\beta+1}{q} \int_\Omega d(x)|v|^q + \int_\Omega b(x)|u|^\alpha |v|^\beta uv. \tag{5.2}$$

Note that $\tilde{J}(\cdot,\cdot)$ is also compact. From (5.1) we also have that

$$\lambda_* = \frac{I(u_*,v_*)}{J(u_*,v_*)}\frac{J(u_*,v_*)}{\tilde{J}(u_*,v_*)} \geq \frac{I(u_*,v_*)}{J(u_*,v_*)}, \tag{5.3}$$

since

$$\frac{J(u_*,v_*)}{\tilde{J}(u_*,v_*)} \geq 1. \tag{5.4}$$

Normalizing (u_*,v_*) by setting

$$u^* =: \frac{|u_*|}{[J(u_*,v_*)]^{1/p}}, \qquad v^* =: \frac{|v_*|}{[J(u_*,v_*)]^{1/q}}, \tag{5.5}$$

we get that

$$I(u^*,v^*) = \frac{I(u_*,v_*)}{J(u_*,v_*)}, \tag{5.6}$$

$$J(u^*,v^*) = 1. \tag{5.7}$$

From relations (5.3)–(5.7) we conclude that

$$\lambda_* \geq \frac{I(u_*,v_*)}{J(u_*,v_*)} = I(u^*,v^*) \geq \lambda_1, \tag{5.8}$$

a contradiction.

(b) Let (u,v) be an eigenfunction of (1.2) corresponding to λ_1. If $uv \geq 0$ a.e., then the right-hand sides of (1.1) and (1.2) are equal, so (u,v) is an eigenfunction of (1.1), and we are done. On the other hand we cannot have $uv < 0$ on a set of positive measure, because then

$$\lambda_1 = \frac{I(u,v)}{\tilde{J}(u,v)} > \frac{I(u,v)}{J(u,v)} = \lambda_1, \tag{5.9}$$

a contradiction.

(c) Suppose that there exists a sequence of eigenpairs (λ_n,u_n,v_n) of (1.2) with $\lambda_n \to \lambda_1$. By the variational characterization of λ_1 we know that $\lambda_n \geq \lambda_1$. So we may assume that $\lambda_n \in (\lambda_1,\lambda_1+\eta)$ for each $n \in \mathbb{N}$. Furthermore, without loss of generality, we may assume that $\|(u_n,v_n)\| = 1$, for all $n \in \mathbb{N}$. Hence, there exists $(\tilde{u},\tilde{v}) \in E$ such that $(u_n,v_n) \rightharpoonup (\tilde{u},\tilde{v})$. The simplicity of λ_1 implies that $(\tilde{u},\tilde{v}) = (u_1,v_1)$ or $(\tilde{u},\tilde{v}) = (-u_1,-v_1)$. Let us suppose that $(u_n,v_n) \rightharpoonup (u_1,v_1)$ in E. For any two pairs of eigenfunctions (u_n,v_n), (u_m,v_m), multiplying the first equation by $u_n - u_m$ and integrating by parts we derive

$$\begin{aligned}
&\int_\Omega \Big(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u_m|^{p-2}\nabla u_m\big)(\nabla u_n - \nabla u_m\Big)dx \\
&\qquad + \int_{\partial\Omega} c_1(x)\Big(|u_n|^{p-2}u_n - |u_m|^{p-2}u_m\Big)(u_n - u_m)dx \\
&\quad = \lambda_n \int_\Omega a(x)\Big(|u_n|^{p-2}u_n - |u_m|^{p-2}u_m\Big)(u_n - u_m)dx \\
&\qquad + \lambda_n \int_\Omega b(x)\Big(|u_n|^{\alpha}|v_n|^{\beta}v_n - |u_m|^{\alpha}|v_m|^{\beta}v_m\Big)(u_n - u_m)dx \\
&\qquad + (\lambda_n - \lambda_m)\Big[\int_\Omega a(x)|u_m|^{p-2}u_m(u_n - u_m)dx + \int_\Omega b(x)|u_m|^{\alpha}|v_m|^{\beta}v_m dx\Big].
\end{aligned} \tag{5.10}$$

From the second equation we similarly derive

$$\begin{aligned}
&\int_\Omega \Big(|\nabla v_n|^{q-2}\nabla v_n - |\nabla v_m|^{q-2}\nabla v_m\Big)(\nabla v_n - \nabla v_m)dx \\
&\qquad + \int_{\partial\Omega} c_2(x)\Big(|v_n|^{q-2}v_n - |v_m|^{q-2}v_m\Big)(v_n - v_m)dx \\
&\quad = \lambda_n \int_\Omega d(x)\Big(|v_n|^{q-2}v_n - |v_m|^{q-2}v_m\Big)(v_n - v_m)dx \\
&\qquad + \lambda_n \int_\Omega b(x)\Big(|u_n|^{\alpha}|v_n|^{\beta}u_n - |u_m|^{\alpha}|v_m|^{\beta}u_m\Big)(v_n - v_m)dx \\
&\qquad + (\lambda_n - \lambda_m)\Big[\int_\Omega a(x)|v_m|^{q-2}v_m(v_n - v_m)dx + \int_\Omega b(x)|u_m|^{\alpha}|v_m|^{\beta}u_m dx\Big].
\end{aligned} \tag{5.11}$$

From (5.10) and (5.11), by using the compactness of the operator $\tilde{J}$ and the monotonicity of the p-Laplacian operator [6], we obtain

$$\begin{aligned}
\int_\Omega |\nabla u_n|^p dx &\longrightarrow \int_\Omega |\nabla u_1|^p dx, \\
\int_\Omega |\nabla v_n|^q dx &\longrightarrow \int_\Omega |\nabla v_1|^q dx.
\end{aligned} \tag{5.12}$$

Exploiting the strict convexity of E_p and E_q we get that $(u_n,v_n) \to (u_1,v_1)$ in E. For a fixed $n \in N$ and for every $(\phi,\psi) \in E$ we have

$$\begin{aligned}
&\int_\Omega |\nabla u_n|^{p-2}\nabla u_n \nabla\phi\, dx + \int_{\partial\Omega} c_1(x)|u_n|^{p-2}u_n\phi\, dx \\
&\qquad = \lambda_n \int_\Omega a(x)|u_n|^{p-2}u_n\phi\, dx + \lambda_n \int_\Omega b(x)|u_n|^{\alpha}|v_n|^{\beta}v_n\phi\, dx, \\
&\int_\Omega |\nabla v_n|^{q-2}\nabla v_n \nabla\psi\, dx + \int_{\partial\Omega} c_1(x)|v_n|^{p-2}v_n\psi\, dx \\
&\qquad = \lambda_n \int_\Omega d(x)|v_n|^{q-2}v_n\psi\, dx + \lambda_n \int_\Omega b(x)|u_n|^{\alpha}|v_n|^{\beta}u_n\psi\, dx,
\end{aligned} \tag{5.13}$$

Let $\mathcal{U}_n^- =: \{x \in \overline{\Omega} : u_n(x) < 0\}$ and $\mathcal{V}_n^- =: \{x \in \overline{\Omega} : v_n(x) < 0\}$. By (c) we must have $m(\Omega_n^-) > 0$, with $\Omega_n^- = \mathcal{U}_n^- \cup \mathcal{V}_n^-$. Denoting by $u_n^- = \min\{0, u_n\}$ and $v_n^- = \min\{0, v_n\}$ and choosing $\phi \equiv u_n^-$ and $\psi \equiv v_n^-$, it follows that

$$\begin{aligned}
&\int_{\mathcal{U}_n^-} |\nabla u_n^-|^p dx + \int_{\partial\Omega\cap\mathcal{U}_n^-} c_1(x)|u_n^-|^p dx \\
&\qquad = \lambda_n \int_{\mathcal{U}_n^-} a(x)|u_n^-|^p dx + \lambda_n \int_{\mathcal{U}_n^-} b(x)|u_n^-|^\alpha |v_n|^\beta u_n^- v_n dx, \\
&\int_{\mathcal{V}_n^-} |\nabla v_n^-|^q dx + \int_{\partial\Omega\cap\mathcal{V}_n^-} c_1(x)|v_n^-|^q dx \\
&\qquad = \lambda_n \int_{\mathcal{V}_n^-} d(x)|v_n^-|^q dx + \lambda_n \int_{\mathcal{V}_n^-} b(x)|u_n|^\alpha |v_n^-|^\beta u_n v_n^- dx.
\end{aligned} \tag{5.14}$$

Since the products $u_n^- v_n^+$ and $u_n^+ v_n^-$ are negative, from the above system of equations we obtain

$$\begin{aligned}
&\int_{\mathcal{U}_n^-} |\nabla u_n^-|^p dx + \int_{\partial\Omega\cap\mathcal{U}_n^-} c_1(x)|u_n^-|^p dx \\
&\qquad \le \lambda_n \int_{\mathcal{U}_n^-} a(x)|u_n^-|^p dx + \lambda_n \int_{\mathcal{U}_n^-} b(x)|u_n^-|^\alpha |v_n^-|^\beta u_n^- v_n^- dx, \\
&\int_{\mathcal{V}_n^-} |\nabla v_n^-|^q dx + \int_{\partial\Omega\cap\mathcal{V}_n^-} c_2(x)|v_n^-|^q dx \\
&\qquad \le \lambda_n \int_{\mathcal{V}_n^-} d(x)|v_n^-|^q dx + \lambda_n \int_{\mathcal{V}_n^-} b(x)|u_n^-|^\alpha |v_n^-|^\beta u_n^- v_n^- dx.
\end{aligned} \tag{5.15}$$

From Hölder and Young inequalities we derive that

$$\begin{aligned}
&\int_{\mathcal{U}_n^-} b(x)|u_n^-|^\alpha |v_n^-|^\beta u_n^- v_n^- dx \\
&\qquad \le B_1 \int_{\mathcal{U}_n^-} \frac{1}{(1+|x|)^s} |u_n^-|^\alpha |v_n^-|^\beta u_n^- v_n^- dx \\
&\qquad = B_1 \int_{\mathcal{U}_n^-} \frac{1}{(1+|x|)^s} |u_n^-|^{\alpha+1} |v_n^-|^{\beta+1} dx \\
&\qquad \le c_3 \left(\int_{\mathcal{U}_n^-} \frac{1}{(1+|x|)^s} |u_n^-|^p dx + \int_{\mathcal{U}_n^-} \frac{1}{(1+|x|)^s} |v_n^-|^q dx \right).
\end{aligned} \tag{5.16}$$

Thus

$$\|u_n^-\|_{1,p}^p \le c_4(\lambda_1 + \eta)\left[\|u_n^-\|_{L^p(w_s,\mathcal{U}_n^-)}^p + \|v_n^-\|_{L^q(w_s,\mathcal{U}_n^-)}^q\right]. \tag{5.17}$$

Similarly,

$$\|v_n^-\|_{1,p}^q \le c_5(\lambda_1 + \eta)\left[\|v_n^-\|_{L^q(w_s,\mathcal{V}_n^-)}^q + \|u_n^-\|_{L^p(w_s,\mathcal{V}_n^-)}^p\right]. \tag{5.18}$$

For $r>0$ let B_r denote the open ball with radius r centered at $0\in R^n$. For $\varepsilon>0$ let $r_\varepsilon>0$ be such that

$$\begin{aligned}||u_n^-||_{1,p}^p &\le c_4(\lambda_1+\eta)\Big(||u_n^-||^p_{L^p(w_s,\mathcal{U}_n^-\cap B_{r_\varepsilon})}+||v_n^-||^q_{L^q(w_s,\mathcal{U}_n^-\cap B_{r_\varepsilon})}+\varepsilon\Big),\\ ||v_n^-||_{1,q}^q &\le c_5(\lambda_1+\eta)\Big(||v_n^-||^q_{L^q(w_s,\mathcal{V}_n^-\cap B_{r_\varepsilon})}+||u_n^-||^p_{L^p(w_s,\mathcal{V}_n^-\cap B_{r_\varepsilon})}+\varepsilon\Big).\end{aligned}\tag{5.19}$$

Let $0<\delta<\min\{p^*-p,q^*-q\}$ and suppose that $\gamma_1\in(N(p^*-p-\delta)/p^*,s-(N-p)(\delta/p))$ and $\gamma_2\in(N(q^*-q-\delta)/q^*,s-(N-q)(\delta/q))$. Lemma 2.1 implies that $E_p\subseteq L^{pp^*/(p+\delta)}(w_{\zeta_1},\Omega)$ and $E_q\subseteq L^{qq^*/(q+\delta)}(w_{\zeta_2},\Omega)$, where $\zeta_1=(s-\gamma_1)p^*/(p+\delta)$ and $\zeta_2=(s-\gamma_2)q^*/(q+\delta)$. Applying once again the Hölder inequality we derive that

$$\begin{aligned}||u_n^-||^p_{L^p(w_s,\mathcal{U}_n^-\cap B_{r_\varepsilon})} &\le \left(\int_{\mathcal{U}_n^-\cap B_{r_\varepsilon}}\frac{1}{(1+|x|)^{\gamma_1p^*/(p^*-p-\delta)}}dx\right)^{(p^*-p-\delta)/p^*}\\ &\quad\times\left(\int_{\mathcal{U}_n^-\cap B_{r_\varepsilon}}\frac{1}{(1+|x|)^{(s-\gamma_1)p^*/(p+\delta)}}|u_n^-|^{pp^*/(p+\delta)}dx\right)^{(p+\delta)/p^*}\\ &\le c_6\left(\int_{\mathcal{U}_n^-\cap B_{r_\varepsilon}}\frac{1}{(1+|x|)^{\gamma_1p^*/(p^*-p-\delta)}}dx\right)^{(p^*-p-\delta)/p^*}||u_n^-||^p_{1,p},\end{aligned}\tag{5.20}$$

(note that $\gamma_1p^*/(p^*-p-\delta)>N$). A similar inequality also holds for v_n^- :

$$||v_n^-||^q_{L^q(w_s,\mathcal{U}_n^-\cap B_{r_\varepsilon})}\le c_7\left(\int_{\mathcal{U}_n^-\cap B_{r_\varepsilon}}\frac{1}{(1+|x|)^{\gamma_2q^*/(q^*-q-\delta)}}dx\right)^{(q^*-q-\delta)/q^*}||v_n^-||^q_{1,q}.\tag{5.21}$$

Combining (5.19), (5.20), and (5.21) we get

$$\begin{aligned}&||u_n^-||^p_{1,p}-c_8\varepsilon\\ &\quad\le c_9\left(\int_{\mathcal{U}_n^-\cap B_{r_\varepsilon}}\frac{1}{(1+|x|)^{\gamma_1p^*/(p^*-p-\delta)}}dx\right)^{(p^*-p-\delta)/p^*}||u_n^-||^p_{1,p}\\ &\qquad+c_{10}\left(\int_{\mathcal{U}_n^-\cap B_{r_\varepsilon}}\frac{1}{(1+|x|)^{\gamma_2q^*/(q^*-q-\delta)}}dx\right)^{(q^*-q-\delta)/q^*}||v_n^-||^q_{1,q}\\ &\quad\le c_{11}\Big[||u_n^-||^p_{1,p}+||v_n^-||^q_{1,q}\Big]\left\{\left(\int_{\mathcal{U}_n^-\cap B_{r_\varepsilon}}\frac{1}{(1+|x|)^{\gamma_1p^*/(p^*-p-\delta)}}dx\right)^{(p^*-p-\delta)/p^*}\right.\\ &\qquad\left.+\left(\int_{\mathcal{U}_n^-\cap B_{r_\varepsilon}}\frac{1}{(1+|x|)^{\gamma_2q^*/(q^*-q-\delta)}}dx\right)^{(q^*-q-\delta)/q^*}\right\}.\end{aligned}\tag{5.22}$$

Similarly,

$$\begin{aligned}
&||v_n^-||_{1,q}^q - c_{12}\varepsilon \\
&\quad \leq c_{13}\Big[||u_n^-||_{1,p}^p + ||v_n^-||_{1,q}^q\Big]\Bigg\{\bigg(\int_{\mathcal{V}_n^-\cap B_{r_\varepsilon}} \frac{1}{(1+|x|)^{\gamma_1 q^*/(p^*-p-\delta)}}dx\bigg)^{(p^*-p-\delta)/p^*} \\
&\qquad + \bigg(\int_{\mathcal{V}_n^-\cap B_{r_\varepsilon}} \frac{1}{(1+|x|)^{\gamma_2 q^*/(q^*-q-\delta)}}dx\bigg)^{(q^*-q-\delta)/q^*}\Bigg\}.
\end{aligned} \tag{5.23}$$

We can now add inequalities (5.22), (5.23) to get

$$\begin{aligned}
1-\varepsilon' \leq c_{14}&\bigg(\int_{\mathcal{U}_n^-\cap B_{r_\varepsilon}} \frac{1}{(1+|x|)^{\gamma_1 p^*/(p^*-p-\delta)}}dx\bigg)^{(p^*-p-\delta)/p^*} \\
&+ c_{15}\bigg(\int_{\mathcal{V}_n^-\cap B_{r_\varepsilon}} \frac{1}{(1+|x|)^{\gamma_2 q^*/(q^*-q-\delta)}}dx\bigg)^{(q^*-q-\delta)/q^*}.
\end{aligned} \tag{5.24}$$

By taking ε sufficiently small we see that

$$m(\Omega_n^- \cap B_{r_\varepsilon}) > c_{16} > 0, \tag{5.25}$$

where the constant c_{16} is independent of λ_n and u_n. Since $u_n \to u_1$ in E_p and $v_n \to v_1$ in E_q, we have that $u_n \to u_1$ in $L^{p^*}(w_1,\Omega)$ and $v_n \to v_1$ in $L^{q^*}(w_2,\Omega)$. Consequently, $u_n \to u_1$ in $L^{p^*}(w_1,B_K(0))$ and $v_n \to v_1$ in $L^{q^*}(w_2,B_K(0))$. By Egorov's theorem we conclude that $u_n(x)$ ($v_n(x)$) converges uniformly to $u_1(x)$ (resp., $v_1(x)$) on $B_{r_\varepsilon}(0)$ with the exception of a set with arbitrarily small measure. But this contradicts (5.25) and the conclusion follows. The proof is complete.

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