EXISTENCE OF A POSITIVE SOLUTION FOR A p-LAPLACIAN SEMIPOSITONE PROBLEM

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We consider the boundary value problem $-\Delta_p u = \lambda f(u)$ in Ω satisfying $u = 0$ on $\partial\Omega$, where $u = 0$ on $\partial\Omega$, $\lambda > 0$ is a parameter, Ω is a bounded domain in $\mathbb{R}^n$ with C^2 boundary $\partial\Omega$, and $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ for $p > 1$. Here, $f : [0,r] \to \mathbb{R}$ is a C^1 nondecreasing function for some $r > 0$ satisfying $f(0) < 0$ (semipositone). We establish a range of λ for which the above problem has a positive solution when f satisfies certain additional conditions. We employ the method of subsuper solutions to obtain the result.

1. Introduction

Consider the boundary value problem

$$\begin{aligned} -\Delta_p u &= \lambda f(u) \quad \text{in } \Omega, \\ u &> 0 \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

where $\lambda > 0$ is a parameter, Ω is a bounded domain in $\mathbb{R}^n$ with C^2 boundary $\partial\Omega$ and $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ for $p > 1$. We assume that $f \in C^1[0,r]$ is a nondecreasing function for some $r > 0$ such that $f(0) < 0$ and there exist $\beta \in (0,r)$ such that $f(s)(s-\beta) \geq 0$ for $s \in [0,r]$. To precisely state our theorem we first consider the eigenvalue problem

$$\begin{aligned} -\Delta_p v &= \lambda |v|^{p-2} v \quad \text{in } \Omega, \\ v &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{1.2}$$

Let $\phi_1 \in C^1(\overline{\Omega})$ be the eigenfunction corresponding to the first eigenvalue λ_1 of (1.2) such that $\phi_1 > 0$ in Ω and $\|\phi_1\|_\infty = 1$. It can be shown that $\partial\phi_1/\partial\eta < 0$ on $\partial\Omega$ and hence, depending on Ω, there exist positive constants m, δ, σ such that

$$\begin{aligned} |\nabla\phi_1|^p - \lambda_1 \phi_1^p &\geq m \quad \text{on } \overline{\Omega}_\delta, \\ \phi_1 &\geq \sigma \quad \text{on } \Omega \setminus \overline{\Omega}_\delta, \end{aligned} \tag{1.3}$$

where $\overline{\Omega}_\delta := \{x \in \Omega \mid d(x,\partial\Omega) \leq \delta\}$.

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We will also consider the unique solution, $e \in C^1(\overline{\Omega})$, of the boundary value problem

$$\begin{aligned} -\Delta_p e = 1 \quad & \text{in } \Omega, \\ e = 0 \quad & \text{on } \partial\Omega \end{aligned} \tag{1.4}$$

to discuss our result. It is known that $e > 0$ in Ω and $\partial e/\partial\eta < 0$ on $\partial\Omega$. Now we state our theorem.

Theorem 1.1. *Assume that there exist positive constants $l_1, l_2 \in (\beta, r]$ satisfying*
(a) $l_2 \geq k l_1$,
(b) $|f(0)|\lambda_1/mf(l_1) < 1$, *and*
(c) $l_2^{p-1}/f(l_2) > \mu(l_1^{p-1}/f(l_1))$,
where $k = k(\Omega) = \lambda_1^{1/(p-1)}(p/(p-1))\sigma^{(p-1)/p}\|e\|_\infty$ *and* $\mu = \mu(\Omega) = (p\|e\|_\infty/(p-1))^{p-1}(\lambda_1/\sigma^p)$. *Then there exist $\hat{\lambda} < \lambda^*$ such that (1.1) has a positive solution for $\hat{\lambda} \leq \lambda \leq \lambda^*$.*

Remark 1.2. A simple prototype example of a function f satisfying the above conditions is

$$f(s) = r\left[(s+1)^{1/2} - 2\right]; \quad 0 \leq s \leq r^4 - 1 \tag{1.5}$$

when r is large.

Indeed, by taking $l_1 = r^2 - 1$ and $l_2 = r^4 - 1$ we see that the conditions $\beta(=3) < l_1 < l_2$ and (a) are easily satisfied for r large. Since $f(0) = -r$, we have

$$\frac{|f(0)|\lambda_1}{mf(l_1)} = \frac{\lambda_1}{m(r-2)}. \tag{1.6}$$

Therefore (b) will be satisfied for r large. Finally,

$$\frac{l_2^{p-1}/f(1_2)}{l_1^{p-1}/f(l_1)} = \frac{(r^4-1)^{p-1}(r-2)}{(r^2-1)^{p-1}(r^2-1)} \sim \frac{r^{4p-3}}{r^{2p}} \sim r^{2p-3} \tag{1.7}$$

for large r and hence (c) is satisfied when $p > 3/2$.

Remark 1.3. Theorem 1.1 holds no matter what the growth condition of f is, for large u. Namely, f could satisfy p-superlinear, p-sublinear or p-linear growth condition at infinity.

It is well documented in the literature that the study of positive solution is very challenging in the semipostone case. See [5] where positive solution is obtained for large λ when f is p-sublinear at infinity. In this paper, we are interested in the existence of a positive solution in a range of λ without assuming any condition on f at infinity.

We prove our result by using the method of subsuper solutions. A function ψ is said to be a subsolution of (1.1) if it is in $W^{1,p}(\Omega) \cap C^0(\overline{\Omega})$ such that $\psi \leq 0$ on $\partial\Omega$ and

$$\int_\Omega |\nabla\psi|^{p-2}\nabla\psi \cdot \nabla w \leq \int_\Omega \lambda f(\psi) w \quad \forall w \in W, \tag{1.8}$$

where $W = \{w \in C_0^\infty(\Omega) \mid w \geq 0 \text{ in } \Omega\}$ (see [4]). A function $\phi \in W^{1,p}(\Omega) \cap C^0(\overline{\Omega})$ is said to be a supersolution if $\phi \geq 0$ on $\partial\Omega$ and satisfies

$$\int_\Omega |\nabla \phi|^{p-2} \nabla \phi \cdot \nabla w \geq \int_\Omega \lambda f(\phi) w \quad \forall w \in W. \tag{1.9}$$

It is known (see [2, 3, 4]) that if there is a subsolution ψ and a supersolution ϕ of (1.1) such that $\psi \leq \phi$ in Ω then (1.1) has a $C^1(\overline{\Omega})$ solution u such that $\psi \leq u \leq \phi$ in Ω.

For the semipositone case, it has always been a challenge to find a nonnegative subsolution. Here we employ a method similar to that developed in [5, 6] to construct a positive subsolution. Namely, we decompose the domain Ω by using the properties of eigenfunction corresponding to the first eigenvalue of $-\Delta_p$ with Dirichlet boundary conditions to construct a subsolution. We will prove Theorem 1.1 in Section 2.

2. Proof of Theorem 1.1

First we construct a positive subsolution of (1.1). For this, we let $\psi = l_1 \sigma^{p/(1-p)} \phi_1^{p/(p-1)}$. Since $\nabla \psi = p/(p-1) l_1 \sigma^{p/(1-p)} \phi_1^{1/(p-1)} \nabla \phi_1$,

$$\begin{aligned}
&\int_\Omega |\nabla \psi|^{p-2} \nabla \psi . \nabla w \\
&\quad = \left(\frac{p}{p-1} l_1 \sigma^{p/(1-p)}\right)^{p-1} \int_\Omega \phi_1 |\nabla \phi_1|^{p-2} \nabla \phi_1 \cdot \nabla w \\
&\quad = \left(\frac{p}{p-1} l_1 \sigma^{p/(1-p)}\right)^{p-1} \int_\Omega |\nabla \phi_1|^{p-2} \nabla \phi_1 [\nabla(\phi_1 w) - w \nabla \phi_1] \\
&\quad = \left(\frac{p}{p-1} l_1 \sigma^{p/(1-p)}\right)^{p-1} \int_\Omega |\nabla \phi_1|^{p-2} \nabla \phi_1 . \nabla(\phi_1 w) - \left(\frac{p}{p-1} l_1 \sigma^{p/(1-p)}\right)^{p-1} \\
&\qquad \times \int_\Omega |\nabla \phi_1|^p w \\
&\quad = \left(\frac{p}{p-1} l_1 \sigma^{p/(1-p)}\right)^{p-1} \int_\Omega \lambda_1 |\phi_1|^{p-2} \phi_1 (\phi_1 w) - \left(\frac{p}{p-1} l_1 \sigma^{p/(1-p)}\right)^{p-1} \\
&\qquad \times \int_\Omega |\nabla \phi_1|^p w \quad \text{(by (1.2))} \\
&\quad = \left(\frac{p}{p-1} l_1 \sigma^{p/(1-p)}\right)^{p-1} \int_\Omega \left[\lambda_1 |\phi_1|^p - |\nabla \phi_1|^p\right] w \quad \forall w \in W.
\end{aligned} \tag{2.1}$$

Thus ψ is a subsolution if

$$\left(\frac{p}{p-1} l_1 \sigma^{p/(1-p)}\right)^{p-1} \int_\Omega \left[\lambda_1 \phi_1^p - |\nabla \phi_1|^p\right] w \leq \lambda \int_\Omega f(\psi) w. \tag{2.2}$$

On $\overline{\Omega}_\delta$

$$|\nabla\phi_1|^p - \lambda\phi_1^p \geq m \tag{2.3}$$

and therefore

$$\left(\frac{p}{p-1}l_1\sigma^{p/(1-p)}\right)^{p-1}\left[\lambda_1\phi_1^p - |\nabla\phi_1|^p\right] \leq -m\left(\frac{p}{p-1}l_1\sigma^{p/(1-p)}\right)^{p-1} \leq \lambda f(\psi) \tag{2.4}$$

if

$$\lambda \leq \tilde{\lambda} := \frac{m((p/(p-1))l_1\sigma^{p/(1-p)})^{p-1}}{|f(0)|}. \tag{2.5}$$

On $\Omega \setminus \overline{\Omega}_\delta$ we have $\phi_1 \geq \sigma$ and therefore

$$\psi = l_1\sigma^{p/(1-p)}\phi_1^{p/(p-1)} \geq l_1\sigma^{p/(1-p)}\sigma^{p/(p-1)} = l_1. \tag{2.6}$$

Thus

$$\left(\frac{p}{p-1}l_1\sigma^{p/(1-p)}\right)^{p-1}\left[\lambda_1\phi_1^p - |\nabla\phi_1|^p\right] \leq \lambda f(\psi) \tag{2.7}$$

if

$$\lambda \geq \hat{\lambda} := \frac{\lambda_1(p/(1-p)l_1\sigma^{p/(1-p)})^{p-1}}{f(l_1)}. \tag{2.8}$$

We get $\hat{\lambda} < \tilde{\lambda}$ by using (b). Therefore ψ is a subsolution for $\hat{\lambda} \leq \lambda \leq \tilde{\lambda}$.

Next we construct a supersolution. Let $\phi = l_2/(\|e\|_\infty)e$. Then ϕ is a supersolution if

$$\int_\Omega |\nabla\phi|^{p-2}\nabla\phi.\nabla w = \int_\Omega\left(\frac{l_2}{\|e\|_\infty}\right)^{p-1} w \geq \lambda\int_\Omega f(\phi)w \quad \forall w \in W. \tag{2.9}$$

But $f(\phi) \leq f(l_2)$ and hence ϕ is a super solution if

$$\lambda \leq \overline{\lambda} := \frac{l_2^{p-1}}{\|e\|_\infty^{p-1}f(l_2)}. \tag{2.10}$$

Recalling (c), we easily see that $\hat{\lambda} < \overline{\lambda}$. Finally, using (2.1), (2.9) and the weak comparison principle [3], we see that $\psi \leq \phi$ in Ω when (a) is satisfied. Therefore (1.1) has a positive solution for $\hat{\lambda} \leq \lambda \leq \lambda^*$ where $\lambda^* = \min\{\tilde{\lambda}, \overline{\lambda}\}$.

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