MULTIPLE SOLUTIONS FOR QUASILINEAR ELLIPTIC NEUMANN PROBLEMS IN ORLICZ-SOBOLEV SPACES

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We investigate the existence of multiple solutions to quasilinear elliptic problems containing Laplace like operators (ϕ-Laplacians). We are interested in Neumann boundary value problems and our main tool is Brézis-Nirenberg's local linking theorem.

1. Introduction

In this paper, we consider the following elliptic problem with Neumann boundary condition,

$$\begin{gathered} -\operatorname{div}\big(\alpha\big(\,|\nabla u(x)|\,\big)\nabla u(x)\big) = g(x,u) \quad \text{a.e. on } \Omega \\ \frac{\partial u}{\partial \nu} = 0 \quad \text{a.e. on } \partial\Omega. \end{gathered} \tag{1.1}$$

Here, Ω is a bounded domain with sufficiently smooth (e.g. Lipschitz) boundary $\partial\Omega$ and $\partial/\partial\nu$ denotes the (outward) normal derivative on $\partial\Omega$. We assume that the function $\phi:\mathbb{R}\to\mathbb{R}$, defined by $\phi(s)=\alpha(|s|)s$ if $s\neq 0$ and 0 otherwise, is an increasing homeomorphism from $\mathbb{R}$ to $\mathbb{R}$. Let $\Phi(s)=\int_0^s\phi(t)dt$, $s\in\mathbb{R}$. Then Φ is a Young function. We denote by L_Φ the Orlicz space associated with Φ and by $\|\cdot\|_\Phi$ the usual Luxemburg norm on L_Φ:

$$\|u\|_\Phi = \inf\left\{k>0:\int_\Omega \Phi\left(\frac{u(x)}{k}\right)dx \le 1\right\}. \tag{1.2}$$

Also, W^1L_Φ is the corresponding Orlicz-Sobolev space with the norm $\|u\|_{1,\Phi}=\|u\|_\Phi+\||\nabla u|\|_\Phi$. The boundary value problem (1.1) has the following weak formulation in W^1L_Φ:

$$u\in W^1L_\Phi : \int_\Omega \alpha(|\nabla u|)\nabla u\cdot\nabla v\,dx = \int_\Omega g(\cdot,u)v\,dx, \quad \forall v\in W^1L_\Phi. \tag{1.3}$$

Our goal in this short note is to prove the existence of two nontrivial solutions to our problem under some suitable conditions on g. The main tool that we are going to use is an abstract existence result of Brézis and Nirenberg [1], which is stated here for the sake of completeness.

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First, let us recall the well known Palais-Smale (PS) condition. Let X be a Banach space and $I : X \to \mathbb{R}$. We say that I satisfies the (PS) condition if any sequence $\{u_n\} \subseteq X$ satisfying

$$|I(u_n)| \leq M \qquad |\langle I'(u_n), \phi\rangle| \leq \varepsilon_n \|\phi\|_X, \tag{1.4}$$

with $\varepsilon_n \to 0$, has a convergent subsequence.

THEOREM 1.1 [1]. *Let X be a Banach space with a direct sum decomposition*

$$X = X_1 \oplus X_2 \tag{1.5}$$

with $\dim X_2 < \infty$. *Let J be a C^1 function on X with $J(0) = 0$, satisfying (PS) and, for some $R > 0$,*

$$\begin{aligned} J(u) &\geq 0, \quad \text{for } u \in X_1, \quad \|u\| \leq R, \\ J(u) &\leq 0, \quad \text{for } u \in X_2, \quad \|u\| \leq R. \end{aligned} \tag{1.6}$$

Assume also that J is bounded below and $\inf_X J < 0$. *Then J has at least two nonzero critical points.*

Note that our abstract main tool is the local linking theorem stated above. This method was first introduced by Liu and Li in [4] (see also [3]). It was generalized later by Silva in [6] and by Brézis and Nirenberg in [1]. The theorem stated above is a version of local linking theorems established in the last cited reference.

2. Existence result

First, let us state our assumptions on ϕ and g. Put

$$p^1 = \inf_{t>0} \frac{t\phi(t)}{\Phi(t)}, \qquad p_\Phi = \liminf_{t\to\infty} \frac{t\phi(t)}{\Phi(t)}, \qquad p^0 = \sup_{t>0} \frac{t\phi(t)}{\Phi(t)}. \tag{2.1}$$

(H(ϕ)) We assume that

$$1 < \liminf_{s\to\infty} \frac{s\phi(s)}{\Phi(s)} \leq \limsup_{s\to\infty} \frac{s\phi(s)}{\Phi(s)} < +\infty. \tag{2.2}$$

It is easy to check that under hypothesis (H(ϕ)), both Φ and its Hölder conjugate satisfy the Δ_2 condition.

Let $g : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function and let G be its anti-derivative:

$$G(x,u) = \int_0^u g(x,r)\,dr, \quad x \in \Omega,\ u \in \mathbb{R}. \tag{2.3}$$

(H(g)) We suppose that g and G satisfy the following hypotheses.

(i) There exist nonnegative constants a_1, a_2 such that $|g(x,s)| \leq a_1 + a_2|s|^{a-1}$, for all $s \in \mathbb{R}$, almost all $x \in \Omega$, with $p^0 < a < Np^1/(N-p^1)$.

(ii) We suppose that there exists $\delta > 0$ such that $G(x,u) \geq 0$, for a.e. $x \in \Omega$, all $u \in [-\delta,\delta]$.

(iii) Assume that

$$\lim_{u\to 0} \frac{G(x,u)}{|u|^{p^0}} = 0, \qquad \limsup_{u\to\infty} \frac{G(x,u)}{|u|^{p^1}} \leq 0, \tag{2.4}$$

uniformly for $x \in \Omega$.

(iv) Suppose that

$$\liminf_{|u|\to\infty} \frac{p^1 G(x,u) - g(x,u)u}{|u|} \geq k(x), \tag{2.5}$$

with $k \in L^1(\Omega)$, and such that $\int_\Omega k(x)dx > 0$.

(v) There exists some $t^* \in \mathbb{R}$ such that $\int_\Omega G(x,t^*)dx > 0$ and $G(x,u) \leq j(x)$ for $|u| > M$ with $M > 0$ and $j \in L^1(\Omega)$.

Our energy functional is $I : W^1L_\Phi \to \mathbb{R}$ with

$$I(u) = \int_\Omega \Phi(|\nabla u(x)|)dx - \int_\Omega G(x,u(x))dx. \tag{2.6}$$

It is easy to check that I is of class C^1 and the critical points of I are solutions of (1.3).

Let

$$V' = \left\{u \in W^{1,p^1}(\Omega) : \int_\Omega u(x)dx = 0\right\}, \tag{2.7}$$

and $V = V' \cap X$. It is clear that V' (resp., V) is the topological complement of $\mathbb{R}$ with respect to $W^{1,p^1}(\Omega)$ (resp., with respect to X). From the Poincaré-Wirtinger inequality, we have the following estimates in V':

$$\|u\|_{L^{p^1}(\Omega)} \leq C\||\nabla u|\|_{L^{p^1}(\Omega)}, \quad \forall u \in V', \tag{2.8}$$

(for some constant $C > 0$).

Lemma 2.1. *If hypotheses (H(ϕ)) and (H(g)) hold, then the energy functional I satisfies the (PS) condition.*

Proof. Let $X = W^1L_\Phi(\Omega)$. Suppose that there exists a sequence $\{u_n\} \subseteq X$ such that

$$|I(u_n)| \leq M, \tag{2.9}$$

$$|\langle I'(u_n),\phi\rangle| \leq \varepsilon_n\|\phi\|_{1,\Phi}, \tag{2.10}$$

for all $n \in \mathbb{N}$, all $\phi \in X$. We first show that $\{u_n\}$ is a bounded sequence in X. Suppose otherwise that the sequence is unbounded. By passing to a subsequence if necessary, we can assume that $\|u_n\|_{1,\Phi} \to \infty$. Let $y_n(x) = u_n(x)/\|u_n\|_{1,\Phi}$. Since $\{y_n\}$ is bounded in X, by passing once more to a subsequence, we can assume that $y_n \rightharpoonup y$ (weakly) in X and therefore

$$y_n \longrightarrow y \quad \text{(strongly) in } L_\Phi(\Omega). \tag{2.11}$$

From (2.9), we have

$$\int_\Omega \Phi(|\nabla u_n(x)|)dx - \int_\Omega G(x,u_n(x))dx \le M. \tag{2.12}$$

On the other hand, note that

$$\Phi(t) \ge \rho^{p^1}\Phi\left(\frac{t}{\rho}\right), \quad \forall t > 0,\ \rho > 1. \tag{2.13}$$

Indeed, from the definition of p^1, we have that $\Phi(t)p^1 \le t\phi(t)$ for $t > 0$. Thus,

$$\int_{t/\rho}^{t} \frac{p^1}{s}ds \le \int_{t/\rho}^{t} \frac{\phi(s)}{\Phi(s)}ds, \tag{2.14}$$

for all $t > 0$ and for $\rho > 1$. Simple calculations on these integrals give the above inequality.

It follows from (2.13) that

$$\int_\Omega \Phi(|\nabla y_n(x)|)dx \le \frac{1}{||u_n||_{1,\Phi}^{p^1}} \int_\Omega \Phi(|\nabla u_n(x)|)dx. \tag{2.15}$$

Dividing both sides of (2.12) by $\|u_n\|_{1,\Phi}^{p^1} > 1$ and making use of (2.15), we obtain

$$\int_\Omega \Phi(|\nabla y_n(x)|)dx \le \int_\Omega \frac{G(x,u_n(x))}{||u_n||_{1,\Phi}^{p^1}}dx + \frac{M}{||u_n||_{1,\Phi}^{p^1}}, \quad \forall n. \tag{2.16}$$

Next, let us prove that

$$\int_\Omega \frac{G(x,u_n(x))}{||u_n||_{1,\Phi}^{p^1}}dx \longrightarrow 0. \tag{2.17}$$

In fact, from (H(g))(iii) we have that for every $\varepsilon > 0$ there exists $M_1 > 0$ such that for $|u| > M_1$ we have $G(x,u)/|u|^{p^1} \le \varepsilon$ for almost all $x \in \Omega$. Thus,

$$\int_\Omega \frac{G(x,u_n(x))}{||u_n||_{1,\Phi}^{p^1}}dx \le \int_{\{x\in\Omega:|u_n(x)|\le M\}} \frac{G(x,u_n(x))}{||u_n||_{1,\Phi}^{p^1}}dx + \int_{\{x\in\Omega:|u_n(x)|\ge M\}} \varepsilon|y_n(x)|^{p^1}dx. \tag{2.18}$$

Because $p^1 \leq p^0 \leq a$, we have $W^1L_\Phi \hookrightarrow L^{p^1}(\Omega)$. From this embedding, one obtains

$$\int_\Omega \frac{G(x,u_n(x))}{\|u_n\|_{1,\Phi}^{p^1}}dx \leq \int_{\{x\in\Omega:|u_n(x)|\leq M\}} \frac{G(x,u_n(x))}{\|u_n\|_{1,\Phi}^{p^1}}dx + \varepsilon c\|y_n\|_{1,\Phi}^{p^1}. \tag{2.19}$$

Finally, noting that $\|y_n\|_{1,\Phi} = 1$, we obtain (2.17).

From (2.16) and (2.17), we have

$$\int_\Omega \Phi(|\nabla y_n(x)|)dx \longrightarrow 0, \tag{2.20}$$

and thus $\|\nabla y_n\|_\Phi \to 0$. The lower semicontinuity of the norm $\|\cdot\|_\Phi$ yields

$$(0 \leq)\|\nabla y\|_\Phi \leq \liminf_{n\to\infty} \|\nabla y_n\|_\Phi (= 0). \tag{2.21}$$

Hence, $\nabla y = 0$ a.e. on Ω, that is, $y \in \mathbb{R}$. This also implies that

$$\lim_{n\to\infty} \|\nabla(y_n - y)\|_\Phi = \lim_{n\to\infty} \|\nabla y_n\|_\Phi = 0. \tag{2.22}$$

From (2.11) and (2.22), we get

$$\|y_n - y\|_{1,\Phi} = \|y_n - y\|_\Phi + \|\nabla(y_n - y)\|_\Phi \longrightarrow 0 \quad \text{as } n \longrightarrow \infty, \tag{2.23}$$

that is, $y_n \to y$ (strongly) in X. Since $\|y_n\|_{1,\Phi} = 1$, we have $y \neq 0$. Furthermore, from the above arguments, $y = c \in \mathbb{R}$ with $c \neq 0$. From this we obtain that $|u_n(x)| \to \infty$.

Choosing $\phi = u_n$ in (2.10) and noting (2.9), we arrive at

$$\begin{aligned}&\int_\Omega p^1 G(x,u_n(x)) - g(x,u_n(x))u_n(x)dx \\ &\qquad + \int_\Omega \phi(|\nabla u_n|)|\nabla u_n| - p^1\Phi(|\nabla u_n|)dx \leq M + \varepsilon_n\|u_n\|_{1,\Phi}.\end{aligned} \tag{2.24}$$

From the definition of p^1 we have $p^1\Phi(t) \leq t\phi(t)$. Using this fact and dividing the last inequality by $\|u_n\|_{1,\Phi}$, one gets

$$\int_\Omega \frac{p^1G(x,u_n(x)) - g(x,u_n(x))u_n(x)}{|u_n(x)|}|y_n(x)|dx \leq \frac{M+\varepsilon_n\|u_n\|_{1,\Phi}}{\|u_n\|_{1,\Phi}}. \tag{2.25}$$

From this we can see that

$$\liminf_{n\to\infty} \int_\Omega \frac{p^1G(x,u_n(x)) - g(x,u_n(x))u_n(x)}{|u_n(x)|}|y_n(x)|dx \leq 0. \tag{2.26}$$

Using Fatou's lemma and (H(g))(iv) we obtain a contradiction, which shows that the sequence $\{u_n\}$ is bounded. Passing to a subsequence, we can assume that $u_n \rightharpoonup u$ weakly in X and thus $u_n \to u$ strongly in $L^a(\Omega)$.

In order to show the strong convergence of $\{u_n\}$ in X, we get back to (2.10) and choose $\phi = u_n - u$. We obtain

$$\left| \int_\Omega (\alpha(|\nabla u_n|)\nabla u_n - \alpha(|\nabla u|)\nabla u)(\nabla u_n - \nabla u)\,dx \right| \\ \le \int_\Omega f(x,u_n)(u_n - u)\,dx + \varepsilon_n \|u_n - u\|_{1,\Phi} - \int_\Omega \alpha(|\nabla u|)\nabla u(\nabla u_n - \nabla u)\,dx. \tag{2.27}$$

Using again the compact imbedding $X \hookrightarrow L^a(\Omega)$ and the fact that $u_n \to u$ weakly in X we arrive at

$$\int_\Omega (a(|\nabla u_n|)\nabla u_n - a(|\nabla u|)\nabla u)(\nabla u_n - \nabla u)\,dx \longrightarrow 0. \tag{2.28}$$

Using [2, Theorem 4] we obtain the strong convergence of $\{u_n\}$ in X. □

In the next result, we verify that under the above assumptions, the functional I satisfies the saddle conditions in Brézis-Nirenberg's theorem.

Lemma 2.2. *If hypotheses (H(ϕ)) and (H(g)) hold, then there exists $\rho > 0$ such that for all $u \in V$ with $\|u\|_{1,\Phi} \le \rho$ we have that $I(u) \ge 0$ and $I(e) \le 0$ for all $e \in \mathbb{R}$ with $|e| \le \rho$.*

Proof. Choose $u \in V$ with $\|u\|_{1,\Phi} = \rho$, with ρ sufficiently small, to be specified later. From (H(g))(iii) we have that for every $\varepsilon > 0$ there exists some $\delta > 0$ for which

$$G(x,u) \le \varepsilon |u|^{p^0} \quad \forall |u| \le \delta \text{ and almost all } x \in \Omega. \tag{2.29}$$

On the other hand, it follows from (H(g))(i) that there is $\tilde{a}_2 > 0$ such that

$$G(x,u) \le a_1 u + \tilde{a}_2 |u|^a \tag{2.30}$$

for all $u \in \mathbb{R}$ and almost all $x \in \Omega$. Together with (H(g))(iii), this shows that there is $\gamma > 0$ such that

$$G(x,u) \le \varepsilon |u|^{p^0} + \gamma |u|^a \tag{2.31}$$

for all $u \in \mathbb{R}$, almost all $x \in \Omega$. From the definition of p^0 we have $p^0/t \ge \phi(t)/\Phi(t)$. Integrating this inequality in $[t, t/\rho]$ with $\rho < 1$, $t > 0$ yields

$$\Phi(t) \ge \rho^{p^0} \Phi\left(\frac{t}{\rho}\right). \tag{2.32}$$

Recall also that from the definition of p^1 we can take for $t \geq 1$

$$\Phi(t) \geq \Phi(1)t^{p^1}, \tag{2.33}$$

thus, $L_\Phi \hookrightarrow L^{p^1}(\Omega)$ and there exists $k_0 > 0$ such that

$$\|u\|_{p^1} \leq k_0 \|u\|_\Phi, \tag{2.34}$$

for all $u \in L_\Phi$ ($\|\cdot\|_{p^1}$ is the usual Lebesgue norm on $L^{p^1}(\Omega)$).

Because $\|u\|_{1,\Phi} \leq 1$ we have also $\|\nabla u\|_\Phi \leq 1$. Then, we have the estimate

$$\int_\Omega \Phi(|\nabla u|)dx \geq \||\nabla u|\|_\Phi^{p^0} \geq C\||\nabla u|\|_{p^1}^{p^0}, \tag{2.35}$$

noting that $\int_\Omega \Phi(|\nabla u|/\|\nabla u\|_\Phi) = 1$ (see [5, Proposition 6, page 77]).

Using now the Poincaré-Wirtinger inequality, we arrive at

$$\int_\Omega \Phi(|\nabla u|)dx \geq C\|u\|_{1,p^1}^{p^0}. \tag{2.36}$$

Also,

$$\int_\Omega G(x,u)dx \leq \varepsilon\|u\|_{p^0}^{p^0} + \gamma_1\|u\|_{1,p^1}^a \leq \varepsilon c_1\|u\|_{1,p^1}^{p^0} + \gamma_1\|u\|_{1,p^1}^a. \tag{2.37}$$

Choosing small enough ε we arrive at $I(u) \geq C\|u\|_{1,p^1}^{p^0} - \gamma_1\|u\|_{1,p^1}^a$.

Therefore, we choose small enough ρ to obtain $I(u) \geq 0$ for $\|u\|_{1,\Phi} \leq \rho$.

For $t \in \mathbb{R}$ we have $I(t) = -\int_\Omega G(x,t)dx$. But from (H($g$))(ii) we have that $G(x,t) \geq 0$ for small enough $t \in \mathbb{R}$. Thus, for such a $t \in \mathbb{R}$ we obtain $I(t) \leq 0$. □

Finally from (H(v)) we have that I is bounded from below and that $\inf_X I < 0$, thus we are allowed to use the multiplicity theorem of Brézis-Nirenberg and have the following result.

Theorem 2.3. *Under hypotheses (H(ϕ)) and (H(g)) hold, the boundary value problem (1.3) has at least two nontrivial solutions.*

We conclude with a simple example to illustrate the above conditions and arguments.

Example 2.4. Let α and g be defined by

$$\alpha(s) = \ln(e+s^2), \quad \forall s \in \mathbb{R}, \tag{2.38}$$

$$g(u) = \begin{cases} 4u^3 & \text{if } |u| \leq \dfrac{1}{\sqrt{5}}, \\ u - u^3 & \text{if } |u| > \dfrac{1}{\sqrt{5}}. \end{cases} \tag{2.39}$$

It can be easily checked that $\Phi(s) = 1/2(e+s^2)[\ln(e+s^2) - 1]$ $(s \in \mathbb{R})$ and thus $p_\Phi = p^1 = 2$ and $p^0 \approx 2.6$. Because $G(u) = u^4$ for $|u|$ small and $G(u) \approx u^2/2 - u^4/4$ for $|u|$ large, we see that the conditions in (H(ϕ)) and (H(g)) are satisfied.

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