RADIAL SOLUTIONS FOR A NONLOCAL BOUNDARY VALUE PROBLEM

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We consider the boundary value problem for the nonlinear Poisson equation with a non-local term $-\Delta u = f(u, \int_U g(u))$, $u|_{\partial U} = 0$. We prove the existence of a positive radial solution when f grows linearly in u, using Krasnoselskii's fixed point theorem together with eigenvalue theory. In presence of upper and lower solutions, we consider monotone approximation to solutions.

1. Introduction

Let us consider the following nonlocal BVP in a ball $U = B(0, R)$ of $\mathbb{R}^n$:

$$-\Delta u = f\left(u, \int_U g(u)\right),$$

$$u|_{\partial U} = 0, \tag{1.1}$$

where f and g are continuous functions. For simplicity we shall take $R = 1$. We want to study the existence of positive radial solutions

$$u(x) = v(\|x\|), \tag{1.2}$$

of (1.1). This may be seen as the stationary problem corresponding to a class of nonlocal evolution (parabolic) boundary value problems related to relevant phenomena in engineering and physics. The literature dealing with such problems has been growing in the last decade. The reader may find some hints on the motivation for the study of this mathematical model, for example, in the paper by Bebernes and Lacey [1]. For more recent developments, see [2] and the references therein.

Here we are considering a nonlocal term inserted in the right-hand side of the equation. Note, however, that it is also of interest to study boundary value problems where the nonlocal expression appears in a boundary condition. We refer the reader to the recent paper by Yang [13] and its references.

When dealing with a nonlinear term with rather general dependence on the nonlocal functional as in (1.1) new difficulties arise with respect to the treatment of standard boundary value problems. Differences of behaviour which are met in general elliptic and parabolic problems are already present in simple models as those we shall analyse in this paper. For instance, the use of the powerful lower and upper solution method (good accounts of which can be consulted in the monographs of Pao [10] and De Coster and Habets [3]) is limited by the absence of general maximum principles. Even for linear problems with nonlocal terms the issue of positivity is far from trivial and may require a detailed study via the analysis of the Green's operator, as in Freitas and Sweers [6].

The purpose of this paper is twofold. First, we want to improve a quite recent result of Fijałkowski and Przeradzki [5]: these authors have obtained existence of positive radial solutions of (1.1) by using Krasnoselskii's fixed point theorem in cones; the main assumption is that f may grow at most like $Au + B$, the bound on A being computed by means of a Green's function. By using a similar theoretical background, together with the consideration of the eigenvalues of the underlying linear problem, we show that an improvement of that bound is possible. This is done in Theorem 3.2. Second, while remaining in the same simple general setting, we will handle (1.1) from the point of view of the upper and lower solution method. We establish a nonlocal maximum principle (Lemma 4.6) and we use it as a device to obtain a monotone approximation scheme for the radial solutions of (1.1) in presence of lower and upper solutions (Theorem 4.10). We follow an idea used by Jiang et al. [9] in studying a fourth-order periodic problem.

Note that we could use similar methods to consider the case where $U = B(0,1) \backslash \overline{B}(0,\rho)$, with $0 < \rho < 1$. Similar results could then be reached. We remark also that for special classes of functions f and g different approaches are needed. For instance, in [8] variational methods have been used to study existence and multiplicity when $f(u,v) = g(u)/v^p$ $(p > 0)$ and g behaves as an exponencial function.

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2. Some auxiliary results

It is well known that the existence of a solution for some boundary value problems is equivalent to the existence of a fixed point of a certain operator. For our purpose we need to consider a second-order ordinary differential equation of the form

$$-\left(p(t)u'(t)\right)' = p(t)f(t, u(t)), \tag{2.1}$$

with boundary conditions

$$u'(0) = u(1) = 0, \tag{2.2}$$

where f is a continuous function in $[0,1] \times \mathbb{R}$ and $p \in C[0,1]$ is positive and increasing in $]0,1]$. If $p > 0$ in $[0,1]$, it is well known that the problem is fully regular, having a standard reduction to a fixed point problem:

$$u = Tf(\cdot, u(\cdot)) \quad \text{in } C[0,1], \tag{2.3}$$

where T is the linear operator that takes $v \in C[0,1]$ into the unique solution u of

$$-(p(t)u'(t))' = p(t)v(t), \qquad u'(0) = u(1) = 0. \tag{2.4}$$

In addition we can write explicitly

$$Tv(t) = \int_0^1 G(t,s)v(s)ds, \tag{2.5}$$

where $G(t,s)$ is the Green's function associated to the problem. The Green's function is continuous in $[0,1] \times [0,1]$, so T is a completely continuous linear operator in $C[0,1]$.

We are interested in the case where $p(t) > 0$ in $]0,1]$ only. Under certain assumptions we still have a continuous Green's function for the linear problem (2.4). The reader can find a more general approach in [7], but for completeness we include here a simple version which is sufficient for our purpose.

Lemma 2.1. *Let p be continuous, increasing in $[0,1]$, $p(0) = 0$ and $p > 0$ in $]0,1]$. If the function*

$$p(s) \int_s^1 \frac{1}{p(\tau)} d\tau \tag{2.6}$$

is continuously extendible to $[0,1]$, then the operator $T : C[0,1] \to C[0,1]$ previously considered is well defined, linear, and completely continuous.

Proof. Consider the equation

$$-(p(t)u'(t))' = p(t)v(t), \tag{2.7}$$

with boundary conditions (2.2). Integrating both sides we get

$$-p(t)u'(t) = \int_0^t p(s)v(s)ds. \tag{2.8}$$

Integrating again, we obtain

$$\begin{aligned}
u(t) &= \int_t^1 \frac{d\tau}{p(\tau)} \int_0^\tau p(s)v(s)ds \\
&= \int_0^t p(s)v(s)ds \int_t^1 \frac{1}{p(\tau)} d\tau + \int_t^1 p(s)v(s)\,ds \int_s^1 \frac{1}{p(\tau)} d\tau \\
&= \int_0^1 G(t,s)v(s)ds,
\end{aligned} \tag{2.9}$$

where

$$G(t,s) = \begin{cases} p(s) \int_t^1 \dfrac{1}{p(\tau)} d\tau, & t \geq s \\[12pt] p(s) \int_s^1 \dfrac{1}{p(\tau)} d\tau, & t \leq s \end{cases} \qquad (2.10)$$

is clearly continuous in $[0,1] \times [0,1]$, so that the operator

$$Tv(t) = \int_0^1 G(t,s)v(s)ds = \int_0^t p(s) \int_t^1 \frac{1}{p(\tau)} d\tau v(s)ds + \int_t^1 p(s) \int_s^1 \frac{1}{p(\tau)} d\tau v(s)\, ds \qquad (2.11)$$

is completely continuous in $C[0,1]$.

It is trivial to see that $Tv(1) = 0$ and if we differentiate the expression for $Tv(t)$ we obtain

$$(Tv)'(t) = p(t) \int_t^1 \frac{1}{p(\tau)} d\tau v(t) + \int_0^t -\frac{p(s)v(s)}{p(t)} ds - p(t) \int_t^1 \frac{1}{p(\tau)} d\tau v(t) \qquad (2.12)$$

$$= -\int_0^t \frac{p(s)v(s)}{p(t)} ds,$$

and thus

$$(Tv)'(0) = \lim_{t \to 0} -\int_0^t \frac{p(s)v(s)}{p(t)} ds = -\lim_{t \to 0} v(0) \frac{\int_0^t p(s)}{p(t)} = 0. \qquad (2.13)$$

$\square$

Remark 2.2. The continuous functions $p(t) = t^n$, with $n > 0$, satisfy the assumptions of the lemma.

The following fixed point theorem of Krasnoselskii will be used in the next section (see [4]).

THEOREM 2.3. *Let P be a cone in a Banach space and $S : P \to P$ a completely continuous operator. If there exist positive constants $r < R$ such that (compression case)*

$$\|Sx\| \geq \|x\|, \quad \forall x \in P \text{ such that } \|x\| = r, \qquad \|Sx\| \leq \|x\|, \quad \forall x \in P \text{ such that } \|x\| = R, \qquad (2.14)$$

then S has a fixed point x in P such that $r < \|x\| < R$.

3. Nonlinearities with linear growth in u: a positive solution

Let $f : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}^+$ and $g : \mathbb{R}^+ \to \mathbb{R}$ be continuous functions. The radial solutions v of the problem (1.1) solve the ordinary differential equation

$$-v''(r) - \frac{n-1}{r} v'(r) = f\left(v(r), \omega_n \int_0^1 s^{n-1} g(v(s)) ds\right) \qquad (3.1)$$

which is equivalent to

$$-\left(r^{n-1}v'(r)\right)' = r^{n-1}f\left(v(r),\omega_n\int_0^1 s^{n-1}g(v(s))ds\right), \tag{3.2}$$

with boundary conditions

$$\lim_{r\to 0^+}v'(r) = v(1) = 0, \tag{3.3}$$

where ω_n is the measure of the unit sphere in $\mathbb{R}^n$.

The homogeneous equation $-v'' - (n-1)v'/r = 0$, with the boundary conditions (3.3), has only the trivial solution, and therefore there exists a Green's function associated to the linear problem. In fact, the Green's function may be written according to Lemma 2.1 (see also [5]):

(i) for $n > 2$,

$$G(r,t) = \frac{t^{n-1}}{n-2}\left(\frac{1}{\max(r,t)^{n-2}} - 1\right); \tag{3.4}$$

(ii) and for $n = 2$,

$$G(r,t) = -t\ln\left(\max(r,t)\right). \tag{3.5}$$

Hence the boundary value problem (3.1)–(3.3) is equivalent to the integral equation

$$v(r) = \int_0^1 G(r,t)f\left(v(t),\omega_n\int_0^1 s^{n-1}g(v(s))ds\right)dt. \tag{3.6}$$

In $C[0,1]$, the Banach space of continuous functions in $[0,1]$ with the usual norm, let P be the cone of the nonnegative functions. The radial solutions of (1.1) are exactly the fixed points of the completely continuous operator $S : P \to P$, defined by

$$S(v)(r) = \int_0^1 G(r,t)f\left(v(t),\omega_n\int_0^1 s^{n-1}g(v(s))ds\right)dt. \tag{3.7}$$

In [5], the following theorem is proved.

THEOREM 3.1. *Let $f : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}^+$ and $g : \mathbb{R}^+ \to \mathbb{R}$ be continuous functions, and*

$$\gamma = \sup_{r\in[0,1]}\int_0^1 G(r,s)ds. \tag{3.8}$$

Suppose there exist constants $A, B \in \mathbb{R}$ such that $0 \le A < \gamma^{-1}$ and

$$f(v,y) \le Av + B \tag{3.9}$$

for all $v \ge 0$ and $y \in \mathbb{R}$.

Then the problem (1.1) has a positive radial solution.

We will show that the estimate on the constant A in the previous result can be improved.

Consider the problem (3.1)–(3.3) and the associated eigenvalue problem:

$$-v''(r) - \frac{n-1}{r}v'(r) = \lambda v(r), \quad \text{with } \lim_{r \to 0^+} v'(r) = 0, \qquad v(1) = 0. \tag{3.10}$$

We have

$$-v''(r) - \frac{n-1}{r}v'(r) = \lambda v(r) \Longleftrightarrow \left(r^{n-1}v'(r)\right)' + \lambda r^{n-1}v(r) = 0. \tag{3.11}$$

To find the eigenvalues, it is useful to consider the auxiliar initial value problem:

$$\left(r^{n-1}v'(r)\right)' + r^{n-1}v(r) = 0, \qquad v(0) = 1, \qquad v'(0) = 0. \tag{3.12}$$

The solution v to this problem is well defined in $[0,+\infty[$, oscillates, and has zeros $\{\xi_n \mid n \in \mathbb{N}\}$ such that $0 < \xi_1 < \xi_2 < \cdots \to +\infty$, with $\xi_{n+1} - \xi_n \to \pi$ (see [12]).

Define $u(r) = v(\beta r)$. Then

$$u'(r) = \beta v'(\beta r), \qquad u''(r) = \beta^2 v''(\beta r). \tag{3.13}$$

Using (3.12) we have

$$(n-1)(\beta r)^{n-2}v'(\beta r) + (\beta r)^{n-1}v''(\beta r) + (\beta r)^{n-1}v(\beta r) = 0 \Longleftrightarrow \left(r^{n-1}u'(r)\right)' + \beta^2 r^{n-1}u(r) = 0. \tag{3.14}$$

It is obvious that $u'(0) = 0$, so it remains to find β such that $u(1) = 0$. As $u(1) = v(\beta)$, we get $\beta = \xi_n$ for some $n \in \mathbb{N}$, hence $\beta = \xi_n$ and, therefore, the eigenvalues of (3.10) are

$$\lambda_n = \beta^2 = \xi_n^{\,2}. \tag{3.15}$$

Let us identify the zeros of the unique solution of (3.12). We have

$$\left(r^{n-1}v'(r)\right)' + r^{n-1}v(r) = 0 \Longleftrightarrow r^{n-3}\left(r^2 v'' + (n-1)rv' + r^2 v\right) = 0, \tag{3.16}$$

and the last equation has the form

$$t^2 u'' + atu' + (b + ct^m)u = 0, \tag{3.17}$$

which is easily reduced to a Bessel equation (cf. [11]). Using the new independent variable

$$y = r^{(n-2)/2}v \tag{3.18}$$

we obtain the transformed equation

$$r^2 y'' + ry' + \left(r^2 - \left(\frac{n-2}{2}\right)^2\right)y = 0, \tag{3.19}$$

whose solutions are well known, and thus we get

(i) $v(r) = c_1 r^{(-n-2)/2} J_{(n-2)/2}(r) + c_2 r^{(-n-2)/2} K_{(n-2)/2}(r)$ if n is even, or

(ii) $v(r) = c_1 r^{(-n-2)/2} J_{(n-2)/2}(r) + c_2 r^{(-n-2)/2} J_{(2-n)/2}(r)$ if n is odd,

where c_1, c_2 are constants and J_i, K_i are Bessel functions of order i, of the first and second kind, respectively.

Taking into consideration the boundary conditions, the constant c_2 must be zero in both cases (otherwise we would have $\lim_{r\to 0^+} v(r) = \infty$), so that

$$v(r) = c_1 r^{(-n-2)/2} J_{(n-2)/2}(r). \tag{3.20}$$

For our boundary value problem we know that $\gamma^{-1} = 2n$ (see [5]). If we compare $\sqrt{2n}$ with ξ_1—the zeros of these Bessel functions are well known—we can see that

$$\sqrt{2n} < \xi_1 \tag{3.21}$$

and hence,

$$\gamma^{-1} < \lambda_1 \quad \text{(first eigenvalue of (3.10)).} \tag{3.22}$$

For instance, for $n = 2$ or $n = 4$ we have

$$\sqrt{4} = 2{,}000 < \xi_1(J_0) \approx 2{,}404,$$
$$\sqrt{8} \approx 2{,}828 < \xi_1(J_1) \approx 3{,}832. \tag{3.23}$$

By adapting the approach of [5] we will prove the following improved version of Theorem 3.1.

THEOREM 3.2. *Let $f : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}^+$ and $g : \mathbb{R}^+ \to \mathbb{R}$ be continuous functions, and λ_1 defined as above.*

Suppose there exist constants $A, B \in \mathbb{R}$ such that $0 \le A < \lambda_1$, and

$$f(v,y) \le Av + B, \quad \forall v \ge 0, \ y \in \mathbb{R}. \tag{3.24}$$

Then the problem (1.1) has a positive radial solution.

Let ϕ be an eigenfunction associated with the first eigenvalue λ_1. We have

$$-\phi'' - \frac{n-1}{r}\phi' = \lambda_1\phi, \qquad \phi'(0) = 0 = \phi(1). \tag{3.25}$$

Since our computation above shows that we may assume that $\phi(t) = v(\xi_1 r)$ where $v(r) = r^{-n-2/2} J_{n-2/2}(r)$, it is clear that $\phi > 0$ in $[0,1[$, (and, by the way, $\phi'(1) < 0$). We may therefore consider the norm

$$\|v(r)\|_X = \sup_{[0,1[} \frac{|v(r)|}{\phi(r)}, \tag{3.26}$$

in the Banach space

$$X = \left\{ v \in C([0,1]) : \frac{|v(r)|}{\phi(r)} \text{ bounded} \right\}. \tag{3.27}$$

Then, as stated before, we can write problem (3.1)–(3.3) as $v = Sv$, where

$$S(v)(r) = \int_0^1 G(r,t) f\left(v(t), \omega_n \int_0^1 s^{n-1} g(v(s)) \, ds \right) dt, \quad \text{for } v \in X. \tag{3.28}$$

Let T denote the operator introduced in Section 2, with $p(s) = s^{n-1}$. This operator acts in $C[0,1]$. Let K be the restriction of T to X and $v \in X$. Since

$$|v(t)| \leq \|v\|_X \phi(t),$$

$$\int_0^1 G(r,t)\phi(t) \, dt = \frac{\phi(r)}{\lambda_1}, \tag{3.29}$$

we have

$$|K(v)(r)| \leq \int_0^1 G(r,t) |v(t)| \, dt \leq \|v\|_X \int_0^1 G(r,t)\phi(t) \, dt \tag{3.30}$$

so that

$$\frac{|K(v)(r)|}{\phi(r)} \leq \frac{\|v\|_X}{\lambda_1}. \tag{3.31}$$

Taking the least upper bound in the left-hand side of the last inequality, we obtain

$$\|K(v)\|_X \leq \frac{\|v\|_X}{\lambda_1}. \tag{3.32}$$

This estimate, which is the main reason to work in the functional space X, will be used in the proof of Theorem 3.2 in a crucial way.

LEMMA 3.3. *The operator $S : X \to X$ is completely continuous.*

Proof. Since the embedding $i_1 : X \to C[0,1]$ is continuous, the Nemytskii operator $N : X \to C[0,1]$ given, for each $v \in X$, by

$$N(v) = f\left(v, \omega_n \int_0^1 s^{n-1} g(v(s)) \, ds \right) \tag{3.33}$$

is continuous. Moreover it takes bounded sets into bounded sets.

Now let us consider the following decomposition of T:

$$C[0,1] \xrightarrow{T_*} C_*^2[0,1] \xrightarrow{i_2} C_*^1[0,1] \xrightarrow{i_3} X, \tag{3.34}$$

where

$$C_*^2[0,1] = \{u \in C^2[0,1] : u'(0) = u(1) = 0\},$$
$$C_*^1[0,1] = \{u \in C^1[0,1] : u(1) = 0\}, \tag{3.35}$$

i_2, i_3 are embeddings, and T_* is the operator T acting between those two spaces.

The operator $(T_*)^{-1}$ takes u into $-u'' - ((n-1)/r)u'$; it is obviously linear continuous and bijective and, therefore, using the open map theorem, we get that T_* is continuous. The embedding i_2 is a well-known completely continuous operator and using L'Hospital's rule we can prove that i_3 is also continuous. Since $S = i_3 i_2 T_* i_1$, the conclusion of the lemma is now straightforward. $\qquad\square$

Proof of Theorem 3.2. The proof is similar to that of **Theorem 3.1** and so we only outline it. If $f(0, \omega_n g(0)/n) = 0$, then $v \equiv 0$ is obviously a fixed point of the operator S, so let us suppose that $f(0, \omega_n g(0)/n) > 0$. Then there exist positive constants M and δ such that

$$f\left(v(t), \omega_n \int_0^1 s^{n-1} g(v(s)) ds\right) \geq M, \quad \forall \|v\|_X \leq \delta. \tag{3.36}$$

A simple computation yields

$$\|Sv\|_X \geq M \sup_{r \in]0,1[} \int_0^1 \frac{G(r,t)}{\phi(r)} dt = M\epsilon, \tag{3.37}$$

if $\|v\|_X \leq \delta$, where we have set $\epsilon := \sup_{r \in]0,1[} \int_0^1 (G(r,t)/\phi(r)) dt$.

If we define $\Omega_1 = \{v \in X \mid \|v\|_X < \min(M\epsilon/2, \delta)\}$, in $\partial \Omega_1$ we have

$$\|Sv\|_X \geq M\epsilon > \|v\|_X. \tag{3.38}$$

Defining $\Omega_2 = \{v \in X \mid \|v\|_X < \|TB\|_X/(1 - A'/\lambda_1)\}$ with $A < A' < \lambda_1$, then for $v \in P \cap \partial \Omega_2$ we have (using the positivity of T and the estimate (3.32))

$$\|Sv\|_X \leq \|T(Av + B)\|_X \leq \|AKv\|_X + \|TB\|_X$$
$$< \frac{A'/\lambda_1 \|TB\|_X}{1 - A'/\lambda_1} + \frac{\|TB\|_X - A'/\lambda_1 \|TB\|_X}{1 - A'/\lambda_1} = \|v\|_X. \tag{3.39}$$

Applying Krasnoselskii's fixed point **Theorem 2.3** (compression version) we find a fixed point of S, and therefore a positive radial solution of (1.1). $\qquad\square$

In both theorems above, as mentioned in [5], the condition on f does not depend on the second variable, and, therefore, nothing is restraining the behaviour of g. The arguments used there are also valid for the same problem with $f(v(r), \alpha(v))$, for any continuous functional α in X.

A similar procedure allows us to prove a result in the spirit of the one considered in [5] where g is restrained, but the condition on f is weakened.

THEOREM 3.4. *Let $f : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}^+$ and $g : \mathbb{R}^+ \to \mathbb{R}$ be continuous functions.*
  Suppose there exist positive constants $A < \lambda_1$, B, C, D, p, and q with $pq \leq 1$ such that

$$f(v, y) \leq Av + B + C|y|^p \quad \forall v \geq 0, \ y \in \mathbb{R},$$
$$|g(v)| \leq D|v|^q \quad \forall v \in \mathbb{R},$$

(3.40)

where ϕ is the eigenfunction associated with λ_1.
  Then the problem (1.1) has a positive radial solution.

Remark 3.5. We could have considered in (3.1) a right-hand side of the form $f(r, v(r), \omega_n \int_0^1 s^{n-1} g(v(s)) ds)$, continuous in $[0, 1] \times \mathbb{R} \times \mathbb{R}$. Indeed we might even work with a nonlinear nonnegative function $f(r, v, w)$ continuous in (v, w) for a.e. $r \in [0, 1]$, and measurable in r for all $(v, w) \in \mathbb{R} \times \mathbb{R}$. However in this case, defining

$$L_k^p(0, 1) = \left\{ u : u \text{ is measurable in }]0, 1[, \int_0^1 s^k |u(s)| ds < +\infty \right\}$$

(3.41)

we should confine ourselves to $L_{n-1}^p(0, 1)$ Carathéodory functions f, that is,

$$\forall M > 0 \quad \sup_{|v| + |w| \leq M} |f(\cdot, v, w)| \in L_{n-1}^p(0, 1),$$

(3.42)

where $p > n$ is fixed.
  Under this restriction, it can still be shown that the analogue of Lemma 3.3 holds, because we can obtain an analogue of T acting compactly from $L_{n-1}^p(0, 1)$ to $C_*^1[0, 1]$.

4. Lower and upper solutions and monotone approximation

We will now apply the lower and upper solution method to find solutions of the boundary value problem (3.1)–(3.3).
  We should point that in [10, page 695] a monotone method approach using lower and upper solutions is applied to an epidemic problem with diffusion. The problem considered in there is a second-order system of two PDE with a nonlocal term, under assumptions related to those we use below (in particular a Lipschitz condition) and where uniqueness is obtained as well.
  We will use two different types of conditions concerning the given functions f and g, and construct monotone convergent sequences to solutions of the problem.
  Let us define the linear operator

$$Lu(r) = -u''(r) - \frac{n-1}{r} u'(r) + \lambda u(r).$$

(4.1)

LEMMA 4.1. *Let $\lambda \geq 0$, and $u \in C^1[0, 1] \cap C^2]0, 1[$ be such that $Lu(r) \geq 0$ in $]0, 1]$, $u'(0) \leq 0$ and $u(1) \geq 0$. Then $u(r) \geq 0$ for all $r \in [0, 1]$.*

Proof. Towards a contradiction, assume that $u(r_0) < 0$ for some $r_0 \in]0,1[$. There are two cases to consider:

(i) $u(r) < 0$ in some interval $]c,d[\subset [0,1]$, with $u(c) = u(d) = 0$.

Let us consider first the case where $\lambda > 0$. Then there must exist $p \in]c,d[$ such that $u'(p) = 0$ and $u''(p) \geq 0$, and since $u(p) < 0$, we get $Lu(p) < 0$, which is a contradiction.

If $\lambda = 0$, integrating in $[c,d]$, we get the contradiction

$$0 < d^{n-1}u'(d) - c^{n-1}u'(c) \leq 0; \tag{4.2}$$

(ii) $u(r) < 0$ in some interval $[0,c[\subset [0,1]$, with $u(c) = 0$.

If $u'(0) < 0$, the same argument applies. If $u'(0) = 0$, integrating in $[0,c]$, we get

$$0 > -c^{n-1}u'(c) + \lambda \int_0^c r^{n-1}u(r)\,dr \geq 0. \tag{4.3}$$

$\square$

From now on we assume that $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are continuous functions. Consider the boundary value problem

$$-u''(r) - \frac{n-1}{r}u'(r) = f\left(u(r), \omega_n \int_0^1 s^{n-1}g(u(s))\,ds\right) \quad \text{for } 0 < r \leq 1, \tag{4.4}$$

$$u'(0) = 0 = u(1). \tag{4.5}$$

We say that $\alpha(r)$ is a *lower solution* of (4.4)-(4.5) if

$$-\alpha''(r) - \frac{n-1}{r}\alpha'(r) \leq f\left(\alpha(r), \omega_n \int_0^1 s^{n-1}g(\alpha(s))\,ds\right) \quad \text{for } 0 < r \leq 1,$$

$$\alpha'(0) \geq 0, \qquad \alpha(1) \leq 0. \tag{4.6}$$

A function β satisfying the reversed inequalities is called an *upper solution*.

Let α_0 be a lower solution and β_0 an upper solution of (4.4)-(4.5). Consider the restriction L_0 of the operator L to the subspace $\{u \in C^1[0,1] \cap C^2]0,1[: u'(0) = u(1) = 0\}$. With the notations above, to get a solution of the problem (4.4)-(4.5) is equivalent to find a fixed point of the completely continuous operator in $C[0,1]$,

$$\Phi u \equiv L_0^{-1}\left(f\left(u, \omega_n \int_0^1 s^{n-1}g(u(s))\,ds\right) + \lambda u\right). \tag{4.7}$$

Let us define

$$R_f(u,v_1,v_2) = \frac{f(u,v_2) - f(u,v_1)}{v_2 - v_1}, \qquad R_g(u_1,u_2) = \frac{g(u_2) - g(u_1)}{u_2 - u_1}. \tag{4.8}$$

LEMMA 4.2. *Let α_0 be a lower solution and β_0 an upper solution of (4.4)-(4.5) such that $\alpha_0 \leq \beta_0$ in $[0,1]$. Suppose $f(u,v)$ is such that*

$$f(u_2,v) - f(u_1,v) \geq -\lambda(u_2 - u_1), \tag{4.9}$$

for some $\lambda \geq 0$, $v \in \mathbb{R}$, u_1, u_2 such that for some $r \in [0,1]$, $\alpha_0(r) \leq u_1 \leq u_2 \leq \beta_0(r)$, and R_f, R_g have the same sign for all u_1, u_2 such that $\alpha_0(r) \leq u_1, u_2 \leq \beta_0(r)$ for some $r \in [0,1]$.

Then, for any two functions $u_1(r), u_2(r) \in C[0,1]$ such that

$$\alpha_0(r) \leq u_1(r) \leq u_2(r) \leq \beta_0(r), \tag{4.10}$$

the following inequality holds:

$$\Phi u_1(r) \leq \Phi u_2(r). \tag{4.11}$$

Proof. The Green's function G_λ associated with L_0 is nonnegative according to Lemma 4.1. We have

$\Phi u_2(r) - \Phi u_1(r)$

$$= \int_0^1 G_\lambda(r,t) \left[f\left(u_2, \omega_n \int_0^1 s^{n-1} g(u_2(s)) ds\right) - f\left(u_1, \omega_n \int_0^1 s^{n-1} g(u_2(s)) ds\right) \right] dt$$

$$+ \int_0^1 G_\lambda(r,t) \left[f\left(u_1, \omega_n \int_0^1 s^{n-1} g(u_2(s)) ds\right) - f\left(u_1, \omega_n \int_0^1 s^{n-1} g(u_1(s)) ds\right) \right] dt$$

$$+ \int_0^1 G_\lambda(r,t) \lambda (u_2 - u_1) dt$$

$$\geq \int_0^1 G_\lambda(r,t) \left[-\lambda(u_2 - u_1) + \lambda(u_2 - u_1) \right] dt \geq 0.$$

$$\tag{4.12}$$

$\square$

Remark 4.3. Clearly if f and g are C^1 functions, the hypotheses of the last theorem are satisfied provided that

$$\frac{\partial f}{\partial u} \geq -\lambda, \qquad \frac{\partial f}{\partial v}, \frac{\partial g}{\partial u} \text{ have the same sign.} \tag{4.13}$$

THEOREM 4.4. *Suppose that f and g satisfy the assumptions of Lemma 4.2. Let α_0, β_0 be lower and upper solutions, respectively, of (4.4)-(4.5). Setting*

$$\alpha_{n+1} = \Phi \alpha_n, \quad \beta_{n+1} = \Phi \beta_n, \quad \forall n \in \mathbb{N}_0, \tag{4.14}$$

then

$$\alpha_0 \leq \alpha_1 \leq \cdots \leq \alpha_n \leq \cdots \leq \beta_n \leq \cdots \leq \beta_1 \leq \beta_0. \tag{4.15}$$

The monotone bounded sequences $(\alpha_n)_{n \in \mathbb{N}_0}$ and $(\beta_n)_{n \in \mathbb{N}_0}$ defined above are convergent in $C[0,1]$, respectively, to the minimal and maximal solutions of (4.4)-(4.5) in the interval $[\alpha_0, \beta_0]$.

Proof. Since

$$
L\alpha_1 = f\left(\alpha_0, \omega_n \int_0^1 s^{n-1} g(\alpha_0(s))\, ds\right) + \lambda\alpha_0,
$$

$$
L\alpha_0 \le f\left(\alpha_0, \omega_n \int_0^1 s^{n-1} g(\alpha_0(s))\, ds\right) + \lambda\alpha_0,
$$

(4.16)

we have

$$
L(\alpha_1 - \alpha_0) \ge 0, \qquad (\alpha_1 - \alpha_0)'(0) \le 0, \qquad (\alpha_1 - \alpha_0)(1) \ge 0, \tag{4.17}
$$

and therefore, by Lemma 4.1, we have $\alpha_0 \le \alpha_1$.

Using similar argumets, we can prove that $\alpha_1 \le \beta_0$.

We are now able to apply Lemma 4.2 to α_0 and α_1 which gives $\alpha_1 \le \alpha_2$. By iteration of this argument, we prove that $(\alpha_n)_{n \in \mathbb{N}_0}$ is an increasing sequence and stays below β_0. Analogously, we prove that $(\beta_n)_{n \in \mathbb{N}_0}$ is a decreasing sequence so that

$$
\alpha_0 \le \alpha_1 \le \cdots \le \alpha_n \le \cdots \le \beta_n \le \cdots \le \beta_1 \le \beta_0. \tag{4.18}
$$

Concerning the convergence of the sequences, as the cone of positive functions in $C[0,1]$ is normal (since $0 \le u \le v$ implies $\|u\| \le \|v\|$), we can use the standard argument ([14, page 283]), which gives the convergence of this iteration method to fixed points of Φ, and these are exactly the smallest and largest fixed points in $[\alpha_0, \beta_0] \subset C[0,1]$.   □

Example 4.5. Let us consider the nonlocal differential equation

$$
-u''(r) - \frac{2}{r} u'(r) = \frac{4}{3}\pi e^u \int_0^1 s^2 (u(s) + 1)\, ds \tag{4.19}
$$

with boundary conditions $u'(0) = u(1) = 0$.

In this case we have $n = 3$, $f(u,v) = e^u v$ and $g(u) = (u+1)/3$.

Consider $\alpha_0 \equiv 0$ and $\beta_0 = 1 - r$. Then

$$
-\alpha_0''(r) - \frac{2}{r}\alpha_0'(r) = 0 \le \frac{4}{9}\pi = \frac{4}{3}\pi e^0 \int_0^1 s^2\, ds \tag{4.20}
$$

and $\alpha_0'(0) = \alpha_0(1) = 0$, so α_0 is a lower solution of (4.5).

For $r \in [0,1]$ we have

$$
-(r^2 \beta_0')' = 2r \ge \frac{5}{9}\pi r^2 e^{1-r} = \frac{4}{3}\pi r^2 e^{1-r} \int_0^1 s^2 (1 - s + 1)\, ds, \tag{4.21}
$$

$\beta_0'(0) = -1$ and $\beta_0(1) = 0$. Therefore β_0 is an upper solution of (4.5)–(4.19).

The conditions in the Theorem 4.4 are satisfied for α_0 and β_0, so there exists a solution u of (4.5)–(4.19), such that

$$
0 \le u(r) \le 1 - r, \quad \forall r \in [0,1]. \tag{4.22}
$$

This solution is the limit of a monotone sequence constructed as in the statement of the theorem.

Let us now try another approach using the lower and upper solution method, where we drop a part of the monotonicity assumptions.

LEMMA 4.6. *Suppose that $u \in C^1[0,1] \cap C^2]0,1[$ satisfies*

$$-u''(r) - \frac{n-1}{r}u'(r) + \lambda u(r) + M \int_0^1 s^{n-1}|u(s)|ds \geq 0 \tag{4.23}$$

for some λ, $M > 0$ such that $\lambda + M < 1$ and $u'(0) \leq 0$, $u(1) \geq 0$. Then $u(r) \geq 0$ for all $r \in [0,1]$.

Proof. Suppose by contradiction that there exists a function u_0 that satisfies the assumptions above and is negative at some point.

Normalising u_0, we can assume that $\int_0^1 s^{n-1}|u_0(s)|ds = 1$ without loss of generality, which implies that $\|r^{n-1}u_0(r)\|_\infty \geq 1$.

Let us consider the auxiliary problem

$$-w''(r) - \frac{n-1}{r}w'(r) + M = 0, \quad w'(0) = w(1) = 0. \tag{4.24}$$

which is equivalent to

$$(r^{n-1}w'(r))' = r^{n-1}M, \qquad w'(0) = w(1) = 0. \tag{4.25}$$

Integrating (4.25), we get

$$w(r) = \frac{M}{2n}(r^2 - 1) \leq 0. \tag{4.26}$$

As u_0 satisfies

$$-u_0''(r) - \frac{n-1}{r}u_0'(r) + \lambda u_0(r) + M \geq 0, \tag{4.27}$$

with $u_0'(0) \leq 0$, $u_0(1) \geq 0$, we have

$$-(u_0 - w)'' - \frac{n-1}{r}(u_0 - w)' + \lambda(u_0 - w) \geq -\lambda w,$$
$$(u_0 - w)'(0) \leq 0, (u_0 - w)(1) \geq 0, \tag{4.28}$$

and, therefore, applying Lemma 4.1, we get $u_0 \geq w$.

We can easily see that $r^{n-1}u_0(r) \geq r^{n-1}w(r) \geq -M/2n > -1$, so the fact that $\|r^{n-1}u_0(r)\|_\infty \geq 1$ insures that there exists $a > 0$ such that $u_0(a) \geq 1/a^{n-1}$.

If u_0 is negative at $b > a$, there exists $c \in]a,b[$ such that $u_0(c) = 0$ (we can assume that $u_0'(b) = 0$). Using Lagrange's theorem, there exists $d \in [a,c]$ such that $u_0'(d) \leq -1/a^{n-1}$. As $d \geq a$, we have $d^{n-1}u_0'(d) \leq -1$ and therefore there exists $e \in [d,b]$ such that $(r^{n-1}u_0'(r))'|_{r=e} \geq 1$, (we can take e such that $e^{n-1}u_0(e) < 1$).

If u is negative at $b < a$, there exists $c < a$ such that $u_0(c) = 0$. As $u_0(a) > 1$, there exists $d \in]c,a[$ such that $u'_0(d) \geq 1$. Considering the boundary condition $u'_0(0) \leq 0$, there exists $e \in [0,d[$ such that $u'_0(e) = 0$ and $u'_0(r) > 0$ for all $r \in]e,d]$. Therefore there exists $f \in [e,d]$ such that $u''_0(f) \geq 1$ and $u'_0(f) > 0$ (we can take f such that $f^{n-1}u_0(f) < 1$).

In both cases, we know that for some r_0 we have $(r^{n-1}u'_0(r))'|_{r=r_0} \geq 1$, and $r_0^{n-1}u_0(r_0) < 1$. Therefore we would get

$$-(r^{n-1}u'_0(r))'|_{r=r_0} + \lambda r_0^{n-1}u_0(r_0) + M \leq -1 + \lambda + M < 0 \qquad (4.29)$$

which is a contradiction. $\qquad\square$

For a given function $u(r) \in C[0,1]$, consider the boundary value problem

$$-v''(r) - \frac{n-1}{r}v'(r) + \lambda v(r) = f\left(u(r), \omega_n \int_0^1 s^{n-1}g(v(s))ds\right) + \lambda u(r), \qquad (4.30)$$

with $v'(0) = 0 = v(1)$. Using the operator L defined in the beginning of this section, this equation is equivalent to the fixed point equation in $C[0,1]$,

$$v = L_0^{-1}\left(f\left(u, \omega_n \int_0^1 s^{n-1}g(v(s))ds\right) + \lambda u\right) \equiv \Phi_u v. \qquad (4.31)$$

Remark 4.7. Using a comparison method as the one in the proof of Lemma 4.6, we get $\|L_0^{-1}\| \leq 1/2n$ in $C[0,1]$.

LEMMA 4.8. *If $f(u,v)$ is k_1-Lipschitz in v, g is k_2-Lipschitz, and $k_1 k_2 \omega_n < 2n^2$, then Φ_u has a unique fixed point.*

Proof. We have

$$|\Phi_u v_2(r) - \Phi_u v_1(r)| \leq \frac{1}{2n}k_1\left|\omega_n \int_0^1 s^{n-1}g(v_2(s))ds - \omega_n \int_0^1 s^{n-1}g(v_1(s))ds\right|$$

$$\leq \frac{1}{2n}k_1 k_2 \omega_n \int_0^1 s^{n-1}|v_2(s) - v_1(s)|ds \leq \frac{k_1 k_2 \omega_n}{2n^2}\|v_2 - v_1\|_\infty, \qquad (4.32)$$

so that

$$\|\Phi_u v_2(r) - \Phi_u v_1(r)\|_\infty \leq \frac{k_1 k_2 \omega_n}{2n^2}\|v_2 - v_1\|_\infty \qquad (4.33)$$

and therefore Φ_u is a contraction mapping. $\qquad\square$

LEMMA 4.9. *Let f and g be functions defined as in Lemma 4.8, $\lambda > 0$ such that $k_1 k_2 \omega_n + \lambda < 1$, and suppose that*

$$f(u_2, v) - f(u_1, v) \geq -\lambda(u_2 - u_1), \qquad (4.34)$$

for all $r \in [0,1]$, $v \in \mathbb{R}$, and $u_1 \leq u_2$.

Let $u_1(r) \leq u_2(r)$ be two given functions defined in $[0,1]$ and $v_1(r)$, $v_2(r)$ the two respective solutions of (4.31). Then $v_1(r) \leq v_2(r)$.

Proof. We have

$$- (v_2 - v_1)'' - \frac{n-1}{r} (v_2 - v_1)' + \lambda (v_2 - v_1)$$

$$= \lambda (u_2 - u_1) + f \left(u_2, \omega_n \int_0^1 s^{n-1} g(v_2) ds \right) - f \left(u_1, \omega_n \int_0^1 s^{n-1} g(v_2) ds \right)$$

$$+ f \left(u_1, \omega_n \int_0^1 s^{n-1} g(v_2) ds \right) - f \left(u_1, \omega_n \int_0^1 s^{n-1} g(v_1) ds \right) \tag{4.35}$$

$$\geq -k_1 k_2 \omega_n \int_0^1 s^{n-1} |v_2 - v_1| \, ds.$$

The conclusion follows from Lemma 4.6. $\qquad\square$

THEOREM 4.10. *Suppose that $f(u,v)$ is k_1-Lipschitz in v, g is k_2-Lipschitz. Suppose that for some $\lambda > 0$ such that $k_1 k_2 \omega_n + \lambda < 1$,*

$$f(u_2, v) - f(u_1, v) \geq -\lambda (u_2 - u_1), \tag{4.36}$$

for all $v \in \mathbb{R}$, and $u_1 \leq u_2$. Let α_0 and β_0 be a lower solution and an upper solution of (4.4)-(4.5), respectively, with $\alpha_0 \leq \beta_0$ in $[0,1]$. Taking $(\alpha_n)_{n \in \mathbb{N}_0}$ and $(\beta_n)_{n \in \mathbb{N}_0}$ such that, according to Lemma 4.8,

$$\alpha_{n+1} = \Phi_{\alpha_n} \alpha_{n+1}, \quad \beta_{n+1} = \Phi_{\beta_n} \beta_{n+1}, \quad \forall n \in \mathbb{N}_0, \tag{4.37}$$

then

$$\alpha_0 \leq \alpha_1 \leq \cdots \leq \alpha_n \leq \cdots \leq \beta_n \leq \cdots \leq \beta_1 \leq \beta_0. \tag{4.38}$$

The monotone bounded sequences $(\alpha_n)_{n \in \mathbb{N}_0}$, $(\beta_n)_{n \in \mathbb{N}_0}$ defined above are convergent in $C[0,1]$ to solutions of (4.4)-(4.5).

Proof. The computation used here is similar to another one used in [9]. We have

$$L(\alpha_1 - \alpha_0) \geq f \left(\alpha_0, \omega_n \int_0^1 s^{n-1} g(\alpha_1(s)) ds \right) - f \left(\alpha_0, \omega_n \int_0^1 s^{n-1} g(\alpha_0(s)) ds \right)$$

$$\geq -k_1 k_2 \omega_n \int_0^1 |\alpha_1(s) - \alpha_0(s)| \, ds \tag{4.39}$$

with

$$(\alpha_1 - \alpha_0)'(0) \leq 0, \qquad (\alpha_1 - \alpha_0)(1) \geq 0, \tag{4.40}$$

and, therefore, using Lemma 4.6, we get $\alpha_0 \leq \alpha_1$. Let us prove that $\alpha_1 \leq \beta_0$. This comes

from

$$L(\beta_0 - \alpha_1) \geq f\left(\beta_0, \omega_n \int_0^1 s^{n-1} g(\beta_0(s)) \, ds\right) + \lambda\beta_0 - f\left(\alpha_0, \omega_n \int_0^1 s^{n-1} g(\beta_0(s)) \, ds\right)$$

$$+ f\left(\alpha_0, \omega_n \int_0^1 s^{n-1} g(\beta_0(s)) \, ds\right) - f\left(\alpha_0, \omega_n \int_0^1 s^{n-1} g(\alpha_1(s)) \, ds\right) - \lambda\alpha_0$$

$$\geq -\lambda(\beta_0 - \alpha_0) + \lambda(\beta_0 - \alpha_0) - k_1 k_2 \omega_n \int_0^1 |\beta_0(s) - \alpha_1(s)| \, ds$$

$$= -k_1 k_2 \omega_n \int_0^1 |\beta_0(s) - \alpha_1(s)| \, ds.$$

$$(4.41)$$

Applying this lemma in the following iterations, we prove that

$$\alpha_0 \leq \alpha_1 \leq \cdots \leq \alpha_n \leq \cdots \leq \beta_n \leq \cdots \leq \beta_1 \leq \beta_0 \tag{4.42}$$

as in the proof of Theorem 4.4.

Concerning the convergence of the sequences, there is a slight difference from the usual method, because in each iteration we use a different operator. But, as

$$\alpha_{n+1}(r) = L_0^{-1}\left(f\left(\alpha_n, \omega_n \int_0^1 s^{n-1} g(\alpha_{n+1}(s)) \, ds\right) + \lambda\alpha_n\right) \tag{4.43}$$

and $\|\alpha_n\|_\infty \leq \max(\|\alpha_0\|_\infty, \|\beta_0\|_\infty)$, we have that $\|\alpha_{n+1}\|_{C^1}$ is bounded, and, therefore, using Àrzela-Ascoli Theorem, there exists a convergent subsequence of α_n. Considering the monotonicity of α_n, we get the conclusion by the standard argument. $\square$

Remark 4.11. It is not difficult to prove that the monotone sequences defined in **Theorem 4.10** converge in fact to *extremal solutions* of the boundary value problem (4.4)-(4.5).

Example 4.12. Suppose that

$$\liminf_{(a,b) \to (0^+, 0^+)} \frac{f(a, (\omega_n/n) g(b))}{a} > \lambda_1. \tag{4.44}$$

and there exists $k > 0$ such that $f(k, \omega_n g(k)/n) < 0$. Suppose in addition that f and g satisfy the assumptions of Theorem 4.10.

Then there exists a *positive* solution of (4.4)-(4.5). This solution may be approximated by monotone sequences. In fact, a simple calculation shows that for $\epsilon > 0$ small enough, $\epsilon\phi$ is a positive lower solution of (4.4)-(4.5). The constant k is clearly an upper solution. The statement follows.

Acknowledgments

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