A MODIFIED QUASI-BOUNDARY VALUE METHOD FOR A CLASS OF ABSTRACT PARABOLIC ILL-POSED PROBLEMS

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Received 14 October 2004; Accepted 9 August 2005

We study a final value problem for first-order abstract differential equation with positive self-adjoint unbounded operator coefficient. This problem is ill-posed. Perturbing the final condition, we obtain an approximate nonlocal problem depending on a small parameter. We show that the approximate problems are well posed and that their solutions converge if and only if the original problem has a classical solution. We also obtain estimates of the solutions of the approximate problems and a convergence result of these solutions. Finally, we give explicit convergence rates.

1. Introduction

We consider the following final value problem (FVP)

$$u'(t) + Au(t) = 0, \quad 0 \le t < T \tag{1.1}$$

$$u(T) = f \tag{1.2}$$

for some prescribed final value f in a Hilbert space H; where A is a positive self-adjoint operator such that $0 \in \rho(A)$. Such problems are not well posed, that is, even if a unique solution exists on $[0,T]$ it need not depend continuously on the final value f. We note that this type of problems has been considered by many authors, using different approaches. Such authors as Lavrentiev [8], Lattès and Lions [7], Miller [10], Payne [11], and Showalter [12] have approximated (FVP) by perturbing the operator A.

In [1, 4, 13] a similar problem is treated in a different way. By perturbing the final value condition, they approximated the problem (1.1), (1.2), with

$$u'(t) + Au(t) = 0, \quad 0 < t < T, \tag{1.3}$$

$$u(T) + \alpha u(0) = f. \tag{1.4}$$

Hindawi Publishing Corporation
Boundary Value Problems
Volume 2006, Article ID 37524, Pages 1–8
DOI 10.1155/BVP/2006/37524

A similar approach known as the method of auxiliary boundary conditions was given in [6, 9]. Also, we have to mention that the non standard conditions of the form (1.4) for parabolic equations have been considered in some recent papers [2, 3].

In this paper, we perturbe the final condition (1.2) to form an approximate nonlocal problem depending on a small parameter, with boundary condition containing a derivative of the same order than the equation, as follows:

$$u'(t) + Au(t) = 0, \quad 0 < t < T, \tag{1.5}$$

$$u(T) - \alpha u'(0) = f. \tag{1.6}$$

Following [4], this method is called quasi-boundary value method, and the related approximate problem is called quasi-boundary value problem (QBVP). We show that the approximate problems are well posed and that their solutions u_α converge in $C^1([0,T],H)$ if and only if the original problem has a classical solution. We show that this method gives a better approximation than many other quasi reversibility type methods, for example, [1, 4, 7]. Finally, we obtain several other results, including some explicit convergence rates. The case where the operator A has discrete spectrum has been treated in [5].

2. The approximate problem

Definition 2.1. A function $u : [0,T] \to H$ is called a classical solution of the (FVP) problem (resp., (QBVP) problem) if $u \in C^1([0,T],H)$, $u(t) \in D(A)$ for every $t \in [0,T]$ and satisfies (1.1) and the final condition (1.2) (resp., the boundary condition (1.6)).

Now, let $\{E_\lambda\}_{\lambda>0}$ be a spectral measure associated to the operator A in the Hilbert space H, then for all $f \in H$, we can write

$$f = \int_0^\infty dE_\lambda f. \tag{2.1}$$

If the (FVP) problem (resp., (QBVP) problem) admits a solution u (resp., u_α), then this solution can be represented by

$$u(t) = \int_0^\infty e^{\lambda(T-t)} dE_\lambda f, \tag{2.2}$$

respectively,

$$u_\alpha(t) = \int_0^\infty \frac{e^{-\lambda t}}{\alpha\lambda + e^{-\lambda T}} dE_\lambda f. \tag{2.3}$$

Theorem 2.2. *For all $f \in H$, the functions u_α given by (2.3) are classical solutions to the (QBVP) problem and we have the following estimate*

$$||u_\alpha(t)|| \le \frac{T}{\alpha(1+\ln(T/\alpha))} \|f\|, \quad \forall t \in [0,T], \tag{2.4}$$

where $\alpha < eT$.

Proof. If we assume that the functions u_α given in (2.3) are defined for all $t \in [0,T]$, then, it is easy to show that $u_\alpha \in C^1([0,T],H)$ and

$$u'_\alpha(t) = \int_0^\infty \frac{-\lambda e^{-\lambda t}}{\alpha\lambda + e^{-\lambda T}} dE_\lambda f. \tag{2.5}$$

From

$$||Au_\alpha(t)||^2 = \int_0^\infty \frac{\lambda^2 e^{-2\lambda t}}{(\alpha\lambda + e^{-\lambda T})^2} d||E_\lambda f||^2 \le \frac{1}{\alpha^2}\int_0^\infty d||E_\lambda f||^2 = \frac{1}{\alpha^2}\|f\|^2, \tag{2.6}$$

we get $u_\alpha(t) \in D(A)$ and so $u_\alpha \in C([0,T],D(A))$. This shows that the function u_α is a classical solution to the (QBVP) problem.

Now, using (2.3), we have

$$||u_\alpha(t)||^2 \le \int_0^\infty \frac{1}{(\alpha\lambda + e^{-\lambda T})^2} d||E_\lambda f||^2, \tag{2.7}$$

if we put

$$h(\lambda) = (\alpha\lambda + e^{-\lambda T})^{-1}, \quad \text{for } \lambda > 0, \tag{2.8}$$

then,

$$\sup_{\lambda>0} h(\lambda) = h\left(\frac{\ln(T/\alpha)}{T}\right), \tag{2.9}$$

and this yields

$$||u_\alpha(t)||^2 \le \left[\frac{T}{\alpha(1+\ln(T/\alpha))}\right]^2 \int_0^\infty d||E_\lambda f||^2 = \left[\frac{T}{\alpha(1+\ln(T/\alpha))}\right]^2 \|f\|^2. \tag{2.10}$$

This shows that the integral defining $u_\alpha(t)$ exists for all $t \in [0,T]$ and we have the desired estimate. □

Remark 2.3. One advantage of this method of regularization is that the order of the error, introduced by small changes in the final value f, is less than the order given in [4].

Now, we give the following convergence result.

THEOREM 2.4. *For every $f \in H$, $u_\alpha(T)$ converges to f in H, as α tends to zero.*

Proof. Let $\varepsilon > 0$, choose $\eta > 0$ for which

$$\int_\eta^\infty d||E_\lambda f||^2 < \frac{\varepsilon}{2}. \tag{2.11}$$

From (2.3), we have

$$||u_\alpha(T) - f||^2 \le \alpha^2 \int_0^\eta \frac{\lambda^2}{(\alpha\lambda + e^{-\lambda T})^2} d||E_\lambda f||^2 + \frac{\varepsilon}{2}, \tag{2.12}$$

so by choosing α such that

$$\alpha^2 < \varepsilon\left(2\int_0^\eta \lambda^2 e^{2\lambda T}||E_\lambda f||^2\right)^{-1}, \tag{2.13}$$

we obtain the desired result. □

Theorem 2.5. *For every $f \in H$, the (FVP) problem has a classical solution u given by (2.2), if and only if the sequence $(u'_\alpha(0))_{\alpha>0}$ converge in H. Furthermore, we then have that $u_\alpha(t)$ converges to $u(t)$ in $C^1([0,T],H)$ as α tends to zero.*

Proof. If we assume that the (FVP) problem has a classical solution u, then we have

$$\begin{aligned} ||u'_\alpha(0)-u'(0)||^2 &= \int_0^\infty \frac{\alpha^2\lambda^4 e^{2\lambda T}}{(\alpha\lambda+e^{-\lambda T})^2}||dE_\lambda f||^2 \\ &\le \alpha^2\int_0^\eta \lambda^4 e^{4\lambda T}d||E_\lambda f||^2 + \int_\eta^\infty \frac{\alpha^2\lambda^4 e^{2\lambda T}}{\alpha^2\lambda^2}d||E_\lambda f||^2 \\ &< \alpha^2\int_0^\eta \lambda^4 e^{4\lambda T}d||E_\lambda f||^2 + \frac{\varepsilon}{2}, \end{aligned} \tag{2.14}$$

so by choosing α such that $\alpha^2 < \varepsilon(2\int_0^\eta \lambda^4 e^{4\lambda T}d\|E_\lambda f\|^2)^{-1}$, we obtain

$$||u'_\alpha(0)-u'(0)||^2 < \varepsilon, \tag{2.15}$$

this shows that $\|u'_\alpha(0)-u'(0)\|$ tends to zero as α tends to zero. Since

$$\begin{aligned} ||u'_\alpha(t)-u'(t)||^2 &\le \int_0^\infty \lambda^2\left(\frac{1}{\alpha\lambda+e^{-\lambda T}}-e^{\lambda T}\right)^2 d||E_\lambda f||^2 \\ &= ||u'_\alpha(0)-u'(0)||^2, \end{aligned} \tag{2.16}$$

then $u'_\alpha(t)$ converges to $u'(t)$ uniformly in $[0,T]$ as α tends to zero.

Since

$$||u_\alpha(0)-u(0)||^2 \le \alpha^2\int_0^\eta \lambda^2 e^{4\lambda T}d||E_\lambda f||^2 + \frac{\varepsilon}{2}, \tag{2.17}$$

for η quite large. Then by choosing α such that $\alpha^2 < (2\int_0^\eta \lambda^2 e^{4\lambda T}d\|E_\lambda f\|^2)^{-1}$, we get

$$||u_\alpha(0)-u(0)||^2 < \varepsilon. \tag{2.18}$$

Thus $u_\alpha(0)$ converges to $u(0)$, which in turn gives that $u_\alpha(t)$ converges to $u(t)$ uniformly in $[0,T]$ as α tends to zero. Combining all these convergence results, we conclude that $u_\alpha(t)$ converges to $u(t)$ in $C^1([0,T],H)$.

Now, assume that $(u'_\alpha(0))_{\alpha>0}$ converges in H. Since u_α is a classical solution to the (QBVP) problem, then we have

$$||u'_\alpha(0)||^2 = \int_0^\infty \frac{\lambda^2}{(\alpha\lambda+e^{-\lambda T})^2}d||E_\lambda f||^2, \tag{2.19}$$

and it is easy to show that

$$\left|\left|\lim_{\alpha\downarrow 0} u'_\alpha(0)\right|\right|^2 = \int_0^\infty \lambda^2 e^{2\lambda T} d||E_\lambda f||^2, \tag{2.20}$$

and so the function $u(t)$ defined by

$$u(t) = \int_0^\infty e^{\lambda(T-t)} dE_\lambda f, \tag{2.21}$$

is a classical solution to the (FVP) problem. This ends the proof of the theorem. □

Theorem 2.6. *If the function u given by (2.2) is a classical solution of the (FVP) problem, and u_α^δ is a solution of the (QBVP) problem for $f = f_\delta$, such that $\| f - f_\delta\| < \delta$, then we have*

$$||u(0) - u_\alpha^\delta(0)|| \le c\left(1 + \ln\frac{T}{\delta}\right)^{-1}, \tag{2.22}$$

where $c = T(1 + \|Au(0)\|)$.

Proof. Suppose that the function u given by (2.2) is a classical solution to the (FVP) problem, and let's denote by u_α^δ a solution of the (QBVP) problem for $f = f_\delta$, such that

$$||f - f_\delta|| < \delta. \tag{2.23}$$

Then, $u_\alpha^\delta(t)$ is given by

$$u_\alpha^\delta(t) = \int_0^\infty \frac{e^{-\lambda t}}{\alpha\lambda + e^{-\lambda T}} dE_\lambda f_\delta, \quad \forall t \in [0, T]. \tag{2.24}$$

From (2.2) and (2.24), we have

$$||u(0) - u_\alpha^\delta(0)|| \le \Delta_1 + \Delta_2, \tag{2.25}$$

where $\Delta_1 = \|u(0) - u_\alpha(0)\|$, and $\Delta_2 = \|u_\alpha(0) - u_\alpha^\delta(0)\|$. Using (2.9), we get

$$\begin{aligned} \Delta_1 &\le \frac{T}{(1 + \ln(T/\alpha))}\left(\int_0^\infty \lambda^2 e^{2\lambda T} d||E_\lambda f||^2\right)^{1/2}, \\ \Delta_2 &\le \frac{T}{\alpha(1 + \ln(T/\alpha))}||f - f_\delta||, \end{aligned} \tag{2.26}$$

then,

$$\begin{aligned} \Delta_1 &\le \frac{T||Au(0)||}{1 + \ln(T/\alpha)}, \\ \Delta_2 &\le \frac{T\delta}{\alpha(1 + \ln(T/\alpha))}. \end{aligned} \tag{2.27}$$

From (2.27), we obtain

$$||u_\alpha(0) - u_\alpha^\delta(0)||^2 \le \frac{T||Au(0)||}{(1 + \ln(T/\alpha))} + \frac{T\delta}{\alpha(1 + \ln(T/\alpha))}, \tag{2.28}$$

then, for the choice $\alpha = \delta$, we get

$$||u_\alpha(0) - u_\alpha^\delta(0)||^2 \le \frac{T(1+||Au(0)||)}{(1+\ln(T/\alpha))}. \tag{2.29}$$

□

Remark 2.7. From (2.22), for $T > e^{-1}$ we get

$$||u(0) - u_\alpha^\delta(0)|| \le c\left(\ln\frac{1}{\delta}\right)^{-1}, \tag{2.30}$$

Remark 2.8. Under the hypothesis of the above theorem, if we denote by U_α^δ the solution of the approximate (FVP) problem for $f = f_\delta$, using the quasireversibility method [7], we obtain the following estimate

$$||u(0) - U_\alpha^\delta(0)|| \le c_1\left(\ln\frac{1}{\delta}\right)^{-2/3}. \tag{2.31}$$

Proof. A proof can be given in a similar way as in [9]. □

THEOREM 2.9. *If there exists an $\varepsilon \in]0,2[$ so that*

$$\int_0^\infty \lambda^\varepsilon e^{\varepsilon\lambda T}||dE_\lambda f||^2, \tag{2.32}$$

converges, then $u_\alpha(T)$ converges to f with order $\alpha^\varepsilon \varepsilon^{-2}$ as α tends to zero.

Proof. Let $\varepsilon \in]0,2[$ such that $\int_0^\infty \lambda^\varepsilon e^{\varepsilon\lambda T}\|dE_\lambda f\|^2$ converges, and let $\beta \in]0,2[$. For a fix $\lambda > 0$, and if we define a function $g_\lambda(\alpha) = \alpha^\beta/(\alpha\lambda + e^{-\lambda T})^2$. Then we can show that

$$g_\lambda(\alpha) \le g_\lambda(\alpha_0), \quad \forall \alpha > 0, \tag{2.33}$$

where $\alpha_0 = \beta e^{-\lambda T}/(2-\beta)\lambda$. Furthermore, from (2.3), we have

$$||u_\alpha(T) - f||^2 = \alpha^{2-\beta}\int_0^\infty \lambda^2 g_\lambda(\alpha)dE_\lambda f. \tag{2.34}$$

Hence from (2.33) and (2.34) we obtain

$$||u_\alpha(T) - f||^2 \le \alpha^{2-\beta}\left(\frac{\beta}{2-\beta}\right)^\beta \int_0^\infty \lambda^{2-\beta}e^{(2-\beta)\lambda T}d||E_\lambda f||^2. \tag{2.35}$$

If we choose $\beta = (2-\varepsilon)$, we have

$$||u_\alpha(T) - f||^2 \le \alpha^\varepsilon \varepsilon^{-2}\left(4\int_0^\infty \lambda^\varepsilon e^{\varepsilon\lambda T}d||E_\lambda f||^2\right), \tag{2.36}$$

hence

$$||u_\alpha(T) - f||^2 \le c_\varepsilon \alpha^\varepsilon \varepsilon^{-2} \tag{2.37}$$

with $c_\varepsilon = 4\int_0^\infty \lambda^\varepsilon e^{\varepsilon\lambda T}d\|E_\lambda f\|^2$. □

Now, we give the following corollary.

Corollary 2.10. *If there exists an* $\varepsilon \in]0,2[$ *so that*

$$\int_0^\infty \lambda^{(\varepsilon+2\gamma)} e^{(\varepsilon+2)\lambda T} d\|E_\lambda f\|^2, \tag{2.38}$$

where $\gamma = \overline{0,1}$, *converges, then* u_α *converges to* u *in* $C^1([0,T],H)$ *with order of convergence* $\alpha^\varepsilon \varepsilon^{-2}$.

Proof. If we assume that (2.38) is satisfied, then

$$\int_0^\infty \lambda^2 e^{2\lambda T} d\|E_\lambda f\|^2, \tag{2.39}$$

converges, and so the function $u(t)$ given by (2.2) is a classical solution of the (FVP) problem. Let $u_\alpha^{(\gamma)}$, $u^{(\gamma)}$ denote the derivatives of order γ ($\gamma = \overline{0,1}$) of the functions u_α and u, respectively. Using the following inequalities

$$\begin{aligned} \left\| u_\alpha^{(\gamma)}(0) - u^{(\gamma)}(0) \right\|^2 &= \int_0^\infty \frac{\alpha^2 \lambda^{(2+2\gamma)} e^{2\lambda T}}{(\alpha\lambda + e^{-\lambda T})^2} d\|E_\lambda f\|^2 \\ &\le \alpha^{2-\beta} \left(\frac{\beta}{2-\beta} \right)^\beta \int_0^\infty \lambda^{(2+2\gamma-\beta)} e^{(4-\beta)\lambda T} d\|E_\lambda f\|^2, \end{aligned} \tag{2.40}$$

and setting $\beta = 2 - \varepsilon$, in (2.40), we obtain

$$\left\| u_\alpha^{(\gamma)}(0) - u^{(\gamma)}(0) \right\|^2 \le c_{\varepsilon,\gamma} \alpha^\varepsilon \varepsilon^{-2}, \tag{2.41}$$

where $c_{\varepsilon,\gamma} = 4\int_0^\infty \lambda^{(\varepsilon+2\gamma)} e^{(\varepsilon+2)\lambda T} d\|E_\lambda f\|^2$.

And since

$$\left\| u_\alpha^{(\gamma)}(t) - u^{(\gamma)}(t) \right\|^2 \le \left\| u_\alpha^{(\gamma)}(0) - u^{(\gamma)}(0) \right\|^2, \tag{2.42}$$

then $u_\alpha^{(\gamma)}(t)$ converges to $u^{(\gamma)}(t)$ uniformly in $[0,T]$, with order of convergence $\alpha^\varepsilon \varepsilon^{-2}$, and so u_α converges to u in $C^1([0,T],H)$, with order $\alpha^\varepsilon \varepsilon^{-2}$. □

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