EXISTENCE RESULTS FOR CLASSES OF p-LAPLACIAN SEMIPOSITONE EQUATIONS

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We study positive $C^1(\bar{\Omega})$ solutions to classes of boundary value problems of the form $-\Delta_p u = g(x,u,c)$ in Ω, $u = 0$ on $\partial\Omega$, where Δ_p denotes the p-Laplacian operator defined by $\Delta_p z := \mathrm{div}(|\nabla z|^{p-2}\nabla z)$; $p > 1$, $c > 0$ is a parameter, Ω is a bounded domain in R^N; $N \geq 2$ with $\partial\Omega$ of class C^2 and connected (if $N = 1$, we assume that Ω is a bounded open interval), and $g(x,0,c) < 0$ for some $x \in \Omega$ (semipositone problems). In particular, we first study the case when $g(x,u,c) = \lambda f(u) - c$ where $\lambda > 0$ is a parameter and f is a $C^1([0,\infty))$ function such that $f(0) = 0$, $f(u) > 0$ for $0 < u < r$ and $f(u) \leq 0$ for $u \geq r$. We establish positive constants $c_0(\Omega,r)$ and $\lambda^*(\Omega,r,c)$ such that the above equation has a positive solution when $c \leq c_0$ and $\lambda \geq \lambda^*$. Next we study the case when $g(x,u,c) = a(x)u^{p-1} - u^{\gamma-1} - ch(x)$ (logistic equation with constant yield harvesting) where $\gamma > p$ and a is a $C^1(\bar{\Omega})$ function that is allowed to be negative near the boundary of Ω. Here h is a $C^1(\bar{\Omega})$ function satisfying $h(x) \geq 0$ for $x \in \Omega$, $h(x) \not\equiv 0$, and $\max_{x\in\bar{\Omega}} h(x) = 1$. We establish a positive constant $c_1(\Omega,a)$ such that the above equation has a positive solution when $c < c_1$. Our proofs are based on subsuper solution techniques.

1. Introduction

We consider weak solutions to classes of boundary value problems of the form

$$\begin{aligned} -\Delta_p u &= g(x,u,c) \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

where Δ_p denotes the p-Laplacian operator defined by $\Delta_p z := \mathrm{div}(|\nabla z|^{p-2}\nabla z)$; $p > 1$, $c > 0$ is a parameter, Ω is a bounded domain in R^N; $N \geq 2$ with $\partial\Omega$ of class C^2 and connected (if $N = 1$, we assume that Ω is a bounded open interval) and $g(x,0,c) < 0$ for some $x \in \Omega$ (semipositone problems). By a weak solution to (1.1), we mean a function $u \in W_0^{1,p}(\Omega)$

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that satisfies

$$\int_{\Omega} |\nabla u|^{p-2}\nabla u \cdot \nabla w\, dx = \int_{\Omega} g(x,u,c) w\, dx, \quad \forall w \in C_0^{\infty}(\Omega). \tag{1.2}$$

However in this paper, we in fact study the existence of $C^1(\bar{\Omega})$ solutions that are strictly positive in Ω.

We first study the case when $g(x,u,c) = \lambda f(u) - c$ where $\lambda > 0$ is a parameter and f satisfies:

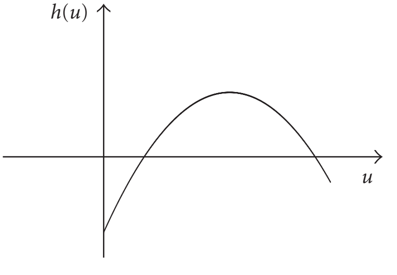
(A1) $f \in C^1([0,\infty))$, $f(0) = 0$, $f(u) > 0$ for $0 < u < r$ and $f(u) \leq 0$ for $u \geq r$ for some $r > 0$.

When $c = 0$ it is easy to establish the existence of a positive solution for large $\lambda > 0$. Here we consider the challenging semipositone case $c > 0$. Semipositone problems have been of great interest during the past two decades, and continue to pose mathematically difficult problems in the study of positive solutions (see [1–3, 10–12]). Also most of the results established to date are for the case when $p = 2$. Here we establish an existence result for $p > 1$ for a class of nonlinearities satisfying (A1). Namely, we prove the following theorem.

Theorem 1.1. *There exist positive constants $c_0 = c_0(\Omega, r)$ and $\lambda^* = \lambda^*(\Omega, r, c)$ such that (1.1) has a positive solution for $c \leq c_0$ and $\lambda \geq \lambda^*$.*

Remark 1.2. Refer to [2] where the authors study such a problem in the case when $p = 2$. In particular, when c is very small they establish an existence of a positive solution for $\tilde{\lambda}$ near the first eigenvalue λ_1 and then extend the existence for $\lambda \geq \tilde{\lambda}$. In this paper, we establish the existence of a positive solution directly for λ large. Our proof is new even in the case $p = 2$.

Remark 1.3. The case when $g(x,u,c) = \lambda[f(u) - c]$ with $h(u) = f(u) - c$ of the form



has been studied for the case when $p = 2$ in [6]. For $p \neq 2$ this remains a challenging semipositone problem for existence of positive solutions for large λ.

We next study the case when $g(x,u,c) = a(x)u^{p-1} - u^{\gamma-1} - ch(x)$ (Logistic equation with constant yield harvesting) where $\gamma > p$, a is a $C^1(\bar{\Omega})$ function that is allowed to be negative near the boundary of Ω, and h is a $C^1(\bar{\Omega})$ function satisfying $h(x) \geq 0$ for $x \in \Omega$, $h(x) \not\equiv 0$ and $\max_{x \in \bar{\Omega}} h(x) = 1$. Again for $c > 0$ this is a semipositone problem. In order to precisely state our result for this problem we introduce the region where we allow $a(x)$ to be negative. Let λ_1 be the first eigenvalue of the $-\Delta_p$ with Dirichlet boundary conditions and $\phi_1 \in C^1(\bar{\Omega})$ be a corresponding eigenfunction such that $\phi_1 > 0$ in Ω, $\partial\phi/\partial n < 0$ on $\partial\Omega$ and $\|\phi_1\|_\infty = 1$. Let $m > 0$, $\delta > 0$, and $\sigma > 0$ be such that

$$\begin{aligned} |\nabla\phi_1|^p - \lambda_1\phi_1^p \geq m \quad &\text{on } \bar{\Omega}_\delta, \\ \phi_1 \geq \sigma \quad &\text{on } \Omega \setminus \bar{\Omega}_\delta, \end{aligned} \tag{1.3}$$

where $\bar{\Omega}_\delta := \{x \in \Omega \mid d(x,\partial\Omega) \leq \delta\}$. Further assume that there exists a constant $a_0 > 0$ such that

$$a(x) \geq a_0 \quad \text{in } \Omega \setminus \bar{\Omega}_\delta \tag{1.4}$$

and let $\mu > 0$ be such that

$$a(x) \geq -\mu \quad \text{in } \bar{\Omega}_\delta. \tag{1.5}$$

Then we prove the following theorem.

Theorem 1.4. *Let $\mu < m(p/(p-1))^{p-1}$ and $a_0 > (p/(p-1))^{p-1}\lambda_1$. Then there exists a positive constant $c_1 = c_1(\Omega,\mu,a_0)$ such that (1.1) has a positive solution for $c \leq c_1$.*

Remark 1.5. Refer to [7] where they studied the case when $c = 0$ and $a(x)$ is a positive function throughout $\bar{\Omega}$.

We establish Theorems 1.1 and 1.4 by the method of sub- and super-solutions. By a super-solution ϕ of (1.1) we mean a function in $W^{1,p}(\Omega) \cap C(\bar{\Omega})$ such that $\phi = 0$ on $\partial\Omega$ and

$$\int_\Omega |\nabla\phi|^{p-2}\nabla\phi \cdot \nabla w\, dx \geq \int_\Omega g(x,\phi,c)w\, dx, \quad \forall w \in W, \tag{1.6}$$

where $W = \{v \in C_0^\infty(\Omega) \mid v \geq 0 \text{ in } \Omega\}$. And by a subsolution ψ of (1.1) we mean a function in $W^{1,p}(\Omega) \cap C(\bar{\Omega})$ such that $\psi = 0$ on $\partial\Omega$ and

$$\int_\Omega |\nabla\psi|^{p-2}\nabla\psi \cdot \nabla w\, dx \leq \int_\Omega g(x,\psi,c)w\, dx, \quad \forall w \in W, \tag{1.7}$$

where W is as defined before. Then if there exist sub- and super-solutions ψ and ϕ respectively such that $\psi \leq \phi$ in Ω then (1.1) has a $C^1(\bar{\Omega})$ solution u such that $\psi \leq u \leq \phi$ (see [7, 8]).

In semipositone problems it is well documented that finding a nonnegative subsolution is nontrivial. Recently in [4] an anti-maximum principle by [5, 8, 9] was used to create a crucial subsolution in the study of the problem when $g(x,u,c) = \lambda\tilde{f}(u) - c$ where $\tilde{f}$ satisfies $\tilde{f}(0) = 0$, $\tilde{f}(u) \geq 0$ and $\lim_{u\to\infty}(\tilde{f}(u)/u) = 0$. Namely, the authors exploited the $C^1(\bar{\Omega})$ solution of

$$\begin{aligned} -\Delta_p z_\alpha - \alpha z_\alpha^{p-1} = -1 \quad &\text{in } \Omega, \\ z_\alpha = 0 \quad &\text{on } \partial\Omega, \end{aligned} \tag{1.8}$$

which is positive in Ω by the anti-maximum principle for $\alpha \in (\lambda_1, \lambda_1 + \nu)$ for some $\nu > 0$ where λ_1 is the first eigenvalue of the $-\Delta_p$ with Dirichlet boundary conditions. However this requires a further restriction on $\tilde{f}$ namely: there exists $m > 0$ such that $\tilde{f}(v) > v^{p-1} - m^{p-1}\alpha^{p-2} + (c/\alpha)$, $\forall v \in [0, m\alpha\|z_\alpha\|_\infty]$. Moreover they obtain a positive a solution for λ near the first eigenvalue λ_1. In proving Theorem 1.1 we avoid the use of the anti-maximum principle in creating a crucial subsolution. Thus we avoid this above restriction on f for small u which seems unnatural when we look for positive solutions for large λ. In Theorem 1.1 we establish a subsolution by analyzing an appropriate power of the first eigenfunction of the $-\Delta_p$ with Dirichlet boundary conditions.

Also recently in [13] the Logistic equation with constant yield harvesting was studied via an anti-maximum principle in the case when $a(x)$ is a positive constant equal to A_0 $(> \lambda_1)$ throughout $\bar{\Omega}$. But in the case of Theorem 1.4, since we allow $a(x)$ to be negative near the boundary, the idea in [13] fails. Again we use an appropriate power of the eigenfunction to create the crucial subsolution needed to establish Theorem 1.4. We will prove Theorem 1.1 in Section 2 and Theorem 1.4 in Section 3.

2. Proof of Theorem 1.1

Here note that $g(x,u,c) = \lambda f(u) - c$ where f satisfies (A1). Let λ_1, ϕ_1, δ, m, σ, and Ω_δ be as described in Section 1.

We now construct our positive subsolution. Let $\psi := ((p-1)/p) r \phi_1^{p/(p-1)}$. (Note that $\|\psi\|_\infty < r$.) Then $\nabla\psi = r\phi_1^{1/(p-1)}\nabla\phi_1$ and ψ will be a subsolution if

$$\int_\Omega |\nabla\psi|^{p-2}\nabla\psi \cdot \nabla w\, dx \le \int_\Omega [\lambda f(\psi) - c] w\, dx, \quad \forall w \in W. \tag{2.1}$$

But

$$\begin{aligned} \int_\Omega |\nabla\psi|^{p-2}\nabla\psi \cdot \nabla w\, dx &= r^{p-1}\int_\Omega |\nabla\phi_1|^{p-2}\phi_1\nabla\phi_1 \cdot \nabla w\, dx \\ &= r^{p-1}\left[\int_\Omega |\nabla\phi_1|^{p-2}\nabla\phi_1 \cdot \nabla(\phi_1 w)\, dx - \int_\Omega |\nabla\phi_1|^p w\, dx\right] \\ &= r^{p-1}\int_\Omega \left[\lambda_1\phi_1^p - |\nabla\phi_1|^p\right] w\, dx. \end{aligned} \tag{2.2}$$

Now $r^{p-1}[\lambda_1\phi_1^p - |\nabla\phi_1|^p] \le -mr^{p-1}$ in $\bar{\Omega}_\delta$. Hence if $c \le c_0 = mr^{p-1}$ then $r^{p-1}[\lambda_1\phi_1^p - |\nabla\phi_1|^p] \le [\lambda f(\psi) - c]$ in $\bar{\Omega}_\delta$, since $f(\psi) \ge 0$.

Next in $\Omega - \bar{\Omega}_\delta$, $r^{p-1}[\lambda_1\phi_1^p - |\nabla\phi_1|^p] \le \lambda_1 r^{p-1}$ while

$$\lambda f(\psi) - c \ge \lambda\alpha - c, \tag{2.3}$$

where $\alpha = \inf\{f(s) \mid ((p-1)/p) r\sigma^{p/(p-1)} \le s \le ((p-1)/p) r\}$. Hence if $\lambda \ge \lambda^* = (\lambda_1 r^{p-1} + c)/\alpha$ then in $\Omega - \bar{\Omega}_\delta$,

$$r^{p-1}\left[\lambda_1\phi_1^p - |\nabla\phi_1|^p\right] \le \lambda f(\psi) - c. \tag{2.4}$$

Hence if $c \le c_0$ and $\lambda \ge \lambda^*$ then (2.1) is satisfied and ψ is a subsolution.

We next construct a super-solution ϕ such that $\phi \geq \psi$. Let $\phi := M\phi_0$ where $\phi_0 \in C^1(\Omega)$ is the solution of

$$-\Delta_p \phi_0 = 1 \quad \text{in } \Omega, \qquad \phi_0 = 0 \quad \text{on } \partial\Omega. \tag{2.5}$$

Now ϕ will be a super-solution if

$$\int_\Omega |\nabla \phi|^{p-2} \nabla \phi \cdot \nabla w \, dx \geq \int_\Omega [\lambda f(\phi) - c] w \, dx, \quad \forall w \in W. \tag{2.6}$$

But $\int_\Omega |\nabla \phi|^{p-2}\nabla\phi \cdot \nabla w \, dx = M^{p-1}\int_\Omega w\, dx \geq \int_\Omega [\lambda f(\phi) - c] w \, dx$, provided $M^{p-1} \geq \lambda \sup_{[0,r]} f(s) := M(\lambda)$ (say). That is, if $M \geq (M(\lambda))^{1/(p-1)}$ then (2.6) is satisfied and ϕ is a super-solution. Since $\phi_0 > 0$ in Ω and $\partial\phi_0/\partial n < 0$ on $\partial\Omega$, we can choose M large enough so that $\phi \geq \psi$ is also satisfied. Hence Theorem 1.1 is proven.

Remark 2.1. We have, in the proof of Theorem 1.1, an explicit expression for both $c_0(\Omega, r)$ and $\lambda^*(\Omega, r, c)$.

3. Proof of Theorem 1.4

Here note that $g(x,u,c) = a(x)u^{p-1} - u^{\gamma-1} - ch(x)$. Let λ_1, ϕ_1, m, σ, δ, a_0, μ, and Ω_δ be as described in Section 1.

Let $\psi = \varepsilon\phi_1^{p/(p-1)}$ where ε will be chosen small enough later. (Note that $\|\psi\|_\infty \leq \varepsilon$.) Then ψ will be a subsolution if

$$\int_\Omega |\nabla \psi|^{p-2}\nabla\psi \cdot \nabla w \, dx \leq \int_\Omega \left[a(x)\psi^{p-1} - \psi^{\gamma-1} - ch(x)\right] w \, dx, \quad \forall w \in W. \tag{3.1}$$

Using a calculation similar to the one in the proof of Theorem 1.1, we have

$$\int_\Omega |\nabla\psi|^{p-2}\nabla\psi \cdot \nabla w \, dx = \varepsilon^{p-1}\left(\frac{p}{p-1}\right)^{p-1} \int_\Omega \left[\lambda_1 \phi_1^p - |\nabla \phi_1|^p\right] w \, dx. \tag{3.2}$$

Hence inequality (3.1) will be satisfied if *both*

$$\varepsilon^{p-1}\left(\frac{p}{p-1}\right)^{p-1}(-m) \leq -\mu\varepsilon^{p-1} - \varepsilon^{\gamma-1} - c \quad (\text{considering } \bar{\Omega}_\delta), \tag{3.3}$$

$$\varepsilon^{p-1}\left(\frac{p}{p-1}\right)^{p-1}\lambda_1\phi_1^p \leq a_0\varepsilon^{p-1}\phi_1^p - \varepsilon^{\gamma-1} - c \quad (\text{considering } \Omega \setminus \bar{\Omega}_\delta) \tag{3.4}$$

are satisfied. Note that since $\mu < m(p/(p-1))^{p-1}$ inequality (3.3) will be satisfied if

$$\varepsilon < \alpha_1 = \left\{ m\left(\frac{p}{p-1}\right)^{p-1} - \mu \right\}^{1/(\gamma-p)}, \qquad c \leq \tilde{c}_1(\varepsilon) = \varepsilon^{p-1}\left\{ m\left(\frac{p}{p-1}\right)^{p-1} - \mu - \varepsilon^{\gamma-p}\right\}. \tag{3.5}$$

Note that $\widetilde{c}_1(\varepsilon) > 0$. Similarly, since $a_0 > (p/(p-1))^{p-1}\lambda_1$, inequality (3.4) will be satisfied if

$$\begin{gathered}\varepsilon \leq \alpha_2\left[\left\{a_0 - \left(\frac{p}{p-1}\right)^{p-1}\lambda_1\right\}\sigma^p\right]^{1/(\gamma-p)}, \\ c \leq \widetilde{c}_2(\varepsilon) = \varepsilon^{p-1}\left[\left\{a_0 - \left(\frac{p}{p-1}\right)^{p-1}\lambda_1\right\}\sigma^p - \varepsilon^{\gamma-p}\right].\end{gathered} \tag{3.6}$$

Note that $\widetilde{c}_2(\varepsilon) > 0$. Choose $\alpha = \min\{\alpha_1, \alpha_2\}$ and $\varepsilon = \alpha/2$. Then simplifying, both $\widetilde{c}_1(\varepsilon)$ and $\widetilde{c}_2(\varepsilon)$ are greater than $(\alpha/2)^{\gamma-1}[2^{\gamma-p}-1]$. Hence if $c \leq (\alpha/2)^{\gamma-1}[2^{\gamma-p}-1] = c_1(\Omega, a_0, \mu)$ then ψ is a subsolution.

We next construct a super-solution ϕ such that $\phi \geq \psi$. Let $\phi := M\phi_0$ where $\phi_0 \in C^1(\bar{\Omega})$ is the solution of (2.5). Now ϕ will be a super-solution if

$$\int_\Omega |\nabla\phi|^{p-2}\nabla\phi\cdot\nabla w\,dx \geq \int_\Omega \left[a(x)\phi^{p-1} - \phi^{\gamma-1} - ch(x)\right]w\,dx, \quad \forall w \in W. \tag{3.7}$$

But $\int_\Omega |\nabla\phi|^{p-2}\nabla\phi\cdot\nabla w\,dx = M^{p-1}\int_\Omega w\,dx \geq \int_\Omega[a(x)\phi^{p-1} - \phi^{\gamma-1} - ch(x)]w\,dx$, provided $M^{p-1} \geq \sup_{[0,k]}[\|a\|_\infty s^{p-1} - s^{\gamma-1}] := M_1$ (say) where $k = \|a\|_\infty^{1/(\gamma-p)}$. That is, if $M \geq M_1^{1/(p-1)}$ then (3.7) is satisfied and ϕ is a super-solution. Since $\phi_0 > 0$ in Ω and $\partial\phi_0/\partial n < 0$ on $\partial\Omega$, we can choose M large enough so that $\phi \geq \psi$ is also satisfied. Hence Theorem 1.4 is proven.

Remark 3.1. We have, in the proof of Theorem 1.4, an explicit expression for $c_1(\Omega, a_0, \mu)$.

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