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Research Article

Solvability for a Class of Abstract Two-Point Boundary Value Problems Derived from Optimal Control

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The solvability for a class of abstract two-point boundary value problems derived from optimal control is discussed. By homotopy technique existence and uniqueness results are established under some monotonic conditions. Several examples are given to illustrate the application of the obtained results.

1. Introduction

This paper deals with the solvability of the following abstract two-point boundary value problem (BVP):

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + F\big(x(t), p(t), t\big), \quad x(a) = x_0, \\ \dot{p}(t) &= -A^*(t)p(t) + G\big(x(t), p(t), t\big), \quad p(b) = \xi\big(x(b)\big). \end{aligned} \tag{1.1}$$

Here, both $x(t)$ and $p(t)$ take values in a Hilbert space X for $t \in [a,b]$, F, $G : X \times X \times [a,b] \to X$, and $\xi : X \to X$ are nonlinear operators. $\{A(t) : a \le t \le b\}$ is a family of linear closed operators with adjoint operators $A^*(t)$ and generates a unique linear evolution system $\{U(t,s) : a \le s \le t \le b\}$ satisfying the following properties.

(a) For any $a \le s \le t \le b$, $U(t,s) \in \mathscr{L}(X)$, the Banach space of all bounded linear operators in X with uniform operator norm, also the mapping $(t,s) \to U(t,s)x$ is continuous for any $x \in X$;
(b) $U(t,s)U(s,\tau) = U(t,\tau)$ for $a \le \tau \le s \le t \le b$;
(c) $U(t,t) = I$ for $a \le t \le b$.

Equation (1.1) is motivated from optimal control theory; it is well known that a Hamiltonian system in the form

$$\dot{x}(t) = \frac{\partial H(x,p,t)}{\partial p}, \quad x(a) = x_0,$$
$$\dot{p}(t) = \frac{-\partial H(x,p,t)}{\partial x}, \quad p(b) = \xi(x(b)) \tag{1.2}$$

is obtained when the Pontryagin maximum principle is used to get optimal state feedback control. Here, $H(x,p,t)$ is a Hamiltonian function. Clearly, the solvability of system (1.2) is crucial for the discussion of optimal control. System (1.2) is also important in many applications such as mathematical finance, differential games, economics, and so on. The solvability of system (1.1), a nontrivial generalization of system (1.2), as far as I know, only a few results have been obtained in the literature; Lions [1, page 133] provided an existence and uniqueness result for a linear BVP:

$$\dot{x}(t) = A(t)x(t) + B(t)p(t) + \varphi(t), \quad x(a) = x_0,$$
$$\dot{p}(t) = -A^*(t)p(t) + C(t)x(t) + \psi(t), \quad p(b) = 0, \tag{1.3}$$

where $\varphi(\cdot),\psi(\cdot) \in L^2(a,b;X)$, $B(t),C(t) \in \mathcal{L}[X]$ are self-adjoint for each $t \in [a,b]$. Using homotopy approach, Hu and Peng [2] and Peng [3] discussed the existence and uniqueness of solutions for a class of forward-backward stochastic differential equations in finite dimensional spaces; that is, in the case $\dim X < \infty$. The deterministic version of stochastic systems discussed in [2, 3] has the form

$$\dot{x}(t) = F(x(t), p(t), t), \quad x(a) = x_0,$$
$$\dot{p}(t) = G(x(t), p(t), t), \quad p(b) = \xi(x(b)). \tag{1.4}$$

Note that systems (1.1) and (1.4) are equivalent in finite dimensional spaces since we may let $A(t) \equiv 0$ without loss of generality. However, in infinite dimensional spaces, (1.1) is more general than (1.4) because operators $A(t)$ and $A^*(t)$ are usually unbounded and hence $A(t)x$ and $A^*(t)p$ are not Lipschitz continuous with respect to x and p in X which is a typical assumption for F and G; see Section 2. Based on the idea of [2, 3], Wu [4] considered the solvability of (1.4) in finite spaces. Peng and Wu [5] dealt with the solvability for a class of forward-backward stochastic differential equations in finite dimensional spaces under G-monotonic conditions. In particular, $x(t)$ and $p(t)$ could take values in different spaces. In this paper, solvability of solutions of (1.1) are studied, some existence and uniqueness results are established. The obtained results extends some results of [2, 4] to infinite dimensional spaces. The technique used in this paper follows that of developed in [2, 3, 5].

The paper is organized as follows. In Section 2, main assumptions are imposed. In Section 3, an existence and uniqueness result of (1.1) with constant functions ξ is established. An existence and uniqueness result of (1.1) with general functions ξ is obtained in Section 4. Finally, some examples are given in Section 5 to illustrate the application of our results.

2. Assumptions

The inner product and the norm in the Hilbert space X are denoted by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$, respectively. Solutions of system (1.1) are always referred to mild solutions; that is, solution pairs $(x(\cdot),p(\cdot))\in C([a,b];X)\times C([a,b];X)$.

The following assumptions are imposed throughout the paper.

(A1) F and G are Lipschitz continuous with respect to x and p and uniformly in $t\in[a,b]$; that is, there exists a number $L>0$ such that for all $x_1,p_1,x_2,p_2\in X$ and $t\in[a,b]$, one has

$$\begin{aligned}||F(x_1,p_1,t)-F(x_2,p_2,t)||&\le L(||x_1-x_2||+||p_1-p_2||),\\ ||G(x_1,p_1,t)-G(x_2,p_2,t)||&\le L(||x_1-x_2||+||p_1-p_2||).\end{aligned}\tag{2.1}$$

Furthermore, $F(0,0,\cdot),G(0,0,\cdot)\in L^2(a,b;X)$.

(A2) There exist two nonnegative numbers α_1 and α_2 with $\alpha_1+\alpha_2>0$ such that

$$\begin{aligned}&\langle F(x_1,p_1,t)-F(x_2,p_2,t),p_1-p_2\rangle+\langle G(x_1,p_1,t)-G(x_2,p_2,t),x_1-x_2\rangle\\ &\quad\le-\alpha_1||x_1-x_2||^2-\alpha_2||p_1-p_2||^2\end{aligned}\tag{2.2}$$

for all $x_1,p_1,x_2,p_2\in X$ and $t\in[a,b]$.

(A3) There exists a number $c>0$ such that

$$\begin{aligned}||\xi(x_1)-\xi(x_2)||&\le c||x_1-x_2||,\\ \langle\xi(x_1)-\xi(x_2),x_1-x_2\rangle&\ge 0\end{aligned}\tag{2.3}$$

for all $x_1,x_2\in X$.

3. Existence and uniqueness: constant function ξ

In this section, we consider system (1.1) with a constant function $\xi(x)=\xi$; that is,

$$\begin{aligned}\dot{x}(t)&=A(t)x(t)+F(x(t),p(t),t),\quad x(a)=x_0,\\ \dot{p}(t)&=-A^*(t)p(t)+G(x(t),p(t),t),\quad p(b)=\xi.\end{aligned}\tag{3.1}$$

Two lemmas are proved first in this section and the solvability result follows.

Lemma 3.1. *Consider the following BVP:*

$$\begin{aligned}\dot{x}(t)&=A(t)x(t)+F_\beta(x(t),p(t),t)+\varphi(t),\quad x(a)=x_0,\\ \dot{p}(t)&=-A^*(t)p(t)+G_\beta(x(t),p(t),t)+\psi(t),\quad p(b)=\xi,\end{aligned}\tag{3.2}$$

where $\varphi(\cdot)$, $\psi(\cdot)\in L^2(a,b;X)$, $\xi,x_0\in X$, and

$$\begin{aligned}F_\beta(x,p,t)&=-(1-\beta)\alpha_2p+\beta F(x,p,t),\\ G_\beta(x,p,t)&=-(1-\beta)\alpha_1x+\beta G(x,p,t).\end{aligned}\tag{3.3}$$

Assume that for some number $\beta = \beta_0 \in [0,1)$, (3.2) has a solution in the space $L^2(a,b;X) \times L^2(a,b;X)$ for any φ and ψ. In addition, (A1) and (A2) hold. Then there exists $\delta > 0$ independent of β_0 such that problem (3.2) has a solution for any $\varphi,\psi,\beta \in [\beta_0,\beta_0+\delta]$, and ξ, x_0.

Proof. Given $\varphi(\cdot),\psi(\cdot),x(\cdot),p(\cdot) \in L^2(a,b;X)$, and $\delta > 0$. Consider the following BVP:

$$\begin{aligned}
\dot{X}(t) &= A(t)X(t) + F_{\beta_0}(X(t),P(t),t) + \alpha_2\delta p(t) + \delta F(x(t),p(t),t) + \varphi(t),\\
X(a) &= x_0,\\
\dot{P}(t) &= -A^*(t)P(t) + G_{\beta_0}(X(t),P(t),t) + \alpha_1\delta x(t) + \delta G(x(t),p(t),t) + \psi(t),\\
P(b) &= \xi.
\end{aligned} \tag{3.4}$$

It follows from (A1) that $\alpha_2\delta p(\cdot) + \delta F(x(\cdot),p(\cdot),\cdot) + \varphi(\cdot) \in L^2(a,b;X)$ and $\alpha_1\delta x(\cdot) + \delta G(x(\cdot),p(\cdot),\cdot) + \psi(\cdot) \in L^2(a,b;X)$. By the assumptions of Lemma 3.1, system (3.4) has a solution $(X(\cdot),P(\cdot))$ in $L^2(a,b;X) \times L^2(a,b;X)$. Therefore, the mapping $J : L^2(a,b;X) \times L^2(a,b;X) \to L^2(a,b;X) \times L^2(a,b;X)$ defined by $J(x(\cdot),p(\cdot)) := (X(\cdot),P(\cdot))$ is well defined.

We will show that J is a contraction mapping for sufficiently small $\delta > 0$. Indeed, let $J(x_1(t),p_1(t)) = (X_1(t),P_1(t))$ and $J(x_2(t),p_2(t)) = (X_2(t),P_2(t))$. Note that

$$\begin{aligned}
&\langle F_{\beta_0}(X_1(t),P_1(t),t) - F_{\beta_0}(X_2(t),P_2(t),t) + \alpha_2\delta(p_1(t) - p_2(t))\\
&\quad + \delta(F(x_1(t),p_1(t),t) - F(x_2(t),p_2(t),t)),\ P_1(t) - P_2(t)\rangle\\
&= -\alpha_2(1-\beta_0)\|P_1(t) - P_2(t)\|^2\\
&\quad + \beta_0\langle F(X_1(t),P_1(t),t) - F(X_2(t),P_2(t),t),\ P_1(t) - P_2(t)\rangle\\
&\quad + \alpha_2\delta\langle p_1(t) - p_2(t), P_1(t) - P_2(t)\rangle\\
&\quad + \delta\langle F(x_1(t),p_1(t),t) - F(x_2(t),p_2(t),t),\ P_1(t) - P_2(t)\rangle
\end{aligned} \tag{3.5}$$

and that

$$\begin{aligned}
&\langle G_{\beta_0}(X_1(t),P_1(t),t) - G_{\beta_0}(X_2(t),P_2(t),t) + \alpha_1\delta(x_1(t) - x_2(t))\\
&\quad + \delta(G(x_1(t),p_1(t),t) - G(x_2(t),p_2(t),t)),\ X_1(t) - X_2(t)\rangle\\
&= -\alpha_1(1-\beta_0)\|X_1(t) - X_2(t)\|^2\\
&\quad + \beta_0\langle G(X_1(t),P_1(t),t) - G(X_2(t),P_2(t),t), X_1(t) - X_2(t)\rangle\\
&\quad + \alpha_1\delta\langle x_1(t) - x_2(t), X_1(t) - X_2(t)\rangle\\
&\quad + \delta\langle G(x_1(t),p_1(t),t) - G(x_2(t),p_2(t),t), X_1(t) - X_2(t)\rangle.
\end{aligned} \tag{3.6}$$

We have from assumption (A2) that

$$\begin{aligned}
&\frac{d}{dt}\langle X_1(t)-X_2(t),P_1(t)-P_2(t)\rangle\\
&\quad=\langle F_{\beta_0}(X_1(t),P_1(t),t)-F_{\beta_0}(X_2(t),P_2(t),t)+\alpha_2\delta(p_1(t)-p_2(t))\\
&\qquad+\delta(F(x_1(t),p_1(t),t)-F(x_2(t),p_2(t),t)),\,P_1(t)-P_2(t)\rangle\\
&\qquad+\langle G_{\beta_0}(X_1(t),P_1(t),t)-G_{\beta_0}(X_2(t),P_2(t),t)+\alpha_1\delta(x_1(t)-x_2(t))\\
&\qquad+\delta(G(x_1(t),p_1(t),t)-G(x_2(t),p_2(t),t)),\,X_1(t)-X_2(t)\rangle\\
&\quad\le-\alpha_1\|X_1(t)-X_2(t)\|^2-\alpha_2\|P_1(t)-P_2(t)\|^2\\
&\qquad+\delta C_1(\|x_1(t)-x_2(t)\|^2+\|X_1(t)-X_2(t)\|^2)\\
&\qquad+\delta C_1(\|p_1(t)-p_2(t)\|^2+\|P_1(t)-P_2(t)\|^2),
\end{aligned}\tag{3.7}$$

where $C_1>0$ is a constant dependent of L, α_1, and α_2.

Integrating between a and b yields

$$\begin{aligned}
&\langle X_1(b)-X_2(b),\,P_1(b)-P_2(b)\rangle-\langle X_1(a)-X_2(a),\,P_1(a)-P_2(a)\rangle\\
&\quad\le(-\alpha_1+\delta C_1)\int_a^b\|X_1(t)-X_2(t)\|^2dt+(-\alpha_2+\delta C_1)\int_a^b\|P_1(t)-P_2(t)\|^2dt\\
&\qquad+\delta C_1\int_a^b(\|x_1(t)-x_2(t)\|^2+\|p_1(t)-p_2(t)\|^2)dt.
\end{aligned}\tag{3.8}$$

Since $\langle X_1(b)-X_2(b),\,P_1(b)-P_2(b)\rangle=0$ and $\langle X_1(a)-X_2(a),\,P_1(a)-P_2(a)\rangle=0$, (3.8) implies

$$\begin{aligned}
(\alpha_1-\delta C_1)\int_a^b\|X_1(t)-X_2(t)\|^2dt+(\alpha_2-\delta C_1)\int_a^b\|P_1(t)-P_2(t)\|^2dt\\
\le\delta C_1\int_a^b(\|x_1(t)-x_2(t)\|^2+\|p_1(t)-p_2(t)\|^2)dt.
\end{aligned}\tag{3.9}$$

Now, we consider three cases of the combinations of α_1 and α_2.

Case 1 ($\alpha_1>0$ and $\alpha_2>0$). Let $\underline{\alpha}=\min\{\alpha_1,\alpha_2\}$. From (3.9) we have

$$\begin{aligned}
(\underline{\alpha}-\delta C_1)\int_a^b(\|X_1(t)-X_2(t)\|^2+\|P_1(t)-P_2(t)\|^2)dt\\
\le\delta C_1\int_a^b(\|x_1(t)-x_2(t)\|^2+\|p_1(t)-p_2(t)\|^2)dt.
\end{aligned}\tag{3.10}$$

Choose δ such that $\underline{\alpha}-\delta C_1>0$ and $\delta C_1/(\underline{\alpha}-\delta C_1)<1/2$. Note that such a $\delta>0$ can be chosen independently of β_0. Then J is a contraction in this case.

Case 2 ($\alpha_1 = 0$ and $\alpha_2 > 0$). Apply the variation of constants formula to the equation

$$\begin{aligned}
\frac{d}{dt}(X_1(t) - X_2(t)) &= A(t)(X_1(t) - X_2(t)) - (1-\beta_0)\alpha_2(P_1(t) - P_2(t)) \\
&\quad + \beta_0(F(X_1(t), P_1(t), t) - F(X_2(t), P_2(t), t)) \\
&\quad + \alpha_2\delta(p_1(t) - p_2(t)) + \delta(F(x_1(t), p_1(t), t) - F(x_2(t), p_2(t), t)), \\
X_1(a) - X_2(a) &= 0,
\end{aligned} \tag{3.11}$$

and recall that $\beta_0 \in [0,1)$ and $M = \max\{\|U(t,s)\| : a \leq s \leq t \leq b\} < \infty$; then we have

$$\begin{aligned}
\|X_1(t) - X_2(t)\| &\leq M(\alpha_2 + L)\delta \int_a^b (\|x_1(s) - x_2(s)\| + \|p_1(s) - p_2(s)\|)ds \\
&\quad + M(\alpha_2 + L)\int_a^b \|P_1(s) - P_2(s)\|ds + ML\int_a^t \|X_1(s) - X_2(s)\|ds.
\end{aligned} \tag{3.12}$$

From Gronwall's inequality, we have

$$\begin{aligned}
&\|X_1(t) - X_2(t)\| \\
&\quad \leq e^{ML(b-a)}\Big(M(\alpha_2 + L)\delta\int_a^b (\|x_1(t) - x_2(t)\| + \|p_1(t) - p_2(t)\|)dt \\
&\qquad + M(\alpha_2 + L)\int_a^b \|P_1(t) - P_2(t)\|dt\Big).
\end{aligned} \tag{3.13}$$

Consequently, there exists a constant $C_2 \geq 1$ dependent of M, L, and α_2 such that

$$\begin{aligned}
&\int_a^b \|X_1(t) - X_2(t)\|^2 dt \\
&\quad \leq C_2\int_a^b \|P_1(t) - P_2(t)\|^2 dt + \delta C_2\int_a^b (\|x_1(t) - x_2(t)\|^2 + \|p_1(t) - p_2(t)\|^2)dt.
\end{aligned} \tag{3.14}$$

Choose a sufficiently small number $\delta > 0$ such that $(\alpha_2 - \delta C_1)/2 > \alpha_2/4C_2$ and $(\alpha_2 - \delta C_1)/2C_2 - \delta C_1 > \alpha_2/4C_2$. Taking into account (3.14), we have

$$\begin{aligned}
&-\delta C_1\int_a^b \|X_1(t) - X_2(t)\|^2 dt + (\alpha_2 - \delta C_1)\int_a^b \|P_1(t) - P_2(t)\|^2 dt \\
&\quad \geq \frac{\alpha_2}{4C_2}\int_a^b (\|X_1(t) - X_2(t)\|^2 + \|P_1(t) - P_2(t)\|^2)dt \\
&\qquad - \alpha_2\delta\int_a^b (\|x_1(t) - x_2(t)\|^2 + \|p_1(t) - p_2(t)\|^2)dt.
\end{aligned} \tag{3.15}$$

Combine (3.9) and (3.15), then we have

$$\begin{aligned}&\int_a^b (||X_1(t)-X_2(t)||^2+||P_1(t)-P_2(t)||^2)\,dt\\&\quad\le \frac{4(C_1+\alpha_2)C_2}{\alpha_2}\delta\int_a^b (||x_1(t)-x_2(t)||^2+||p_1(t)-p_2(t)||^2)\,dt.\end{aligned}\tag{3.16}$$

Let δ be small further that $4(C_1+\alpha_2)C_2\delta/\alpha_2<1/2$. Then J is a contraction.

Case 3 ($\alpha_1>0$ and $\alpha_2=0$). Consider the following differential equation derived from system (3.4):

$$\begin{aligned}\frac{d}{dt}(P_1(t)-P_2(t)) &= -A^*(t)(P_1(t)-P_2(t))-(1-\beta_0)\alpha_1(X_1(t)-X_2(t))\\&\quad+\beta_0(G(X_1(t),P_1(t),t)-G(X_2(t),P_2(t),t))\\&\quad+\alpha_1\delta(x_1(t)-x_2(t))+\delta(G(x_1(t),p_1(t),t)-G(x_2(t),p_2(t),t)),\\P_1(b)-P_2(b)&=0.\end{aligned}\tag{3.17}$$

Apply the variation of constants formula to (3.17), then we have

$$\begin{aligned}&\int_a^b ||P_1(t)-P_2(t)||^2dt\\&\quad\le C_2\int_a^b ||X_1(t)-X_2(t)||^2dt+\delta C_2\int_a^b (||x_1(t)-x_2(t)||^2+||p_1(t)-p_2(t)||^2)\,dt\end{aligned}\tag{3.18}$$

for some constant $C_2\ge 1$ dependent of M, L, and α_1. Choose δ sufficiently small such that $(\alpha_1-\delta C_1)/2>\alpha_1/4C_2$ and $(\alpha_1-\delta C_1)/2C_2-\delta C_1>\alpha_1/4C_2$ and taking into account (3.18), then we have

$$\begin{aligned}&(\alpha_1-\delta C_1)\int_a^b ||X_1(t)-X_2(t)||^2dt-\delta C_1\int_a^b ||P_1(t)-P_2(t)||^2dt\\&\quad\ge \frac{\alpha_1}{4C_2}\int_a^b (||X_1(t)-X_2(t)||^2+||P_1(t)-P_2(t)||^2)\,dt\\&\qquad-\alpha_1\delta\int_a^b (||x_1(t)-x_2(t)||^2+||p_1(t)-p_2(t)||^2)\,dt.\end{aligned}\tag{3.19}$$

Similar to Case 2, we can show that J is a contraction.

Since we assume $\alpha_1+\alpha_2>0$, we can summarize that there exists $\delta_0>0$ independent of β_0 such that J is a contraction whenever $\delta\in(0,\delta_0)$. Hence, J has a unique fixed point $(\overline{x}(\cdot),\overline{p}(\cdot))$ that is a solution of (3.2). Therefore, (3.2) has a solution for any $\beta\in[\beta_0,\beta_0+\delta]$. The proof of the lemma is complete. □

Lemma 3.2. *Assume $\alpha_1 \geq 0$, $\alpha_2 \geq 0$, and $\alpha_1 + \alpha_2 > 0$. The following linear BVP:*

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) - \alpha_2 p(t) + \varphi(t), \quad x(a) = x_0, \\ \dot{p}(t) &= -A^*(t)p(t) - \alpha_1 x(t) + \psi(t), \quad p(b) = \lambda x(b) + \nu \end{aligned} \tag{3.20}$$

has a unique solution on $[a,b]$ for any $\varphi(\cdot),\psi(\cdot) \in L^2(a,b;X)$, $\lambda \geq 0$, and $\nu, x_0 \in X$; that is, system (3.2) has a unique solution on $[a,b]$ for $\beta = 0$.

Proof. We may assume $\nu = 0$ without loss of generality.

Case 1 ($\alpha_1 > 0$ and $\alpha_2 > 0$). Consider the following quadratic linear optimal control system:

$$\begin{aligned} \inf_{u(\cdot)\in L^2(a,b;X)} \Bigg\{ &\frac{1}{2}\lambda\langle x(b), x(b)\rangle \\ &+ \frac{1}{2}\int_a^b \left[\alpha_1 \left\langle x(t) - \frac{1}{\alpha_1}\psi(t), x(t) - \frac{1}{\alpha_1}\psi(t)\right\rangle + \alpha_2\langle u(t), u(t)\rangle\right] dt \Bigg\} \end{aligned} \tag{3.21}$$

subject to the constraints

$$\dot{x}(t) = A(t)x(t) + \alpha_2 u(t) + \varphi(t), \quad x(a) = x_0. \tag{3.22}$$

The corresponding Hamiltonian function is

$$H(x,p,u,t) := \frac{1}{2}\left[\alpha_1 \left\langle x - \frac{1}{\alpha_1}\psi(t), x - \frac{1}{\alpha_1}\psi(t)\right\rangle + \alpha_2\langle u, u\rangle\right] + \langle p, A(t)x + \alpha_2 u + \varphi(t)\rangle. \tag{3.23}$$

Clearly, the related Hamiltonian system is (3.20). By the well-known quadratic linear optimal control theory, the above control problem has a unique optimal control. Therefore, system (3.20) has a unique solution.

Case 2 ($\alpha_1 > 0$ and $\alpha_2 = 0$). Note that

$$\dot{x}(t) = A(t)x(t) + \varphi(t), \quad x(a) = x_0 \tag{3.24}$$

has a unique solution x, then the equation

$$\dot{p}(t) = -A^*(t)p(t) - \alpha_1 x(t) + \psi(t), \qquad p(b) = \lambda x(b) \tag{3.25}$$

has a unique solution p. Therefore, (x,p) is the unique solution of system (3.20).

Case 3 ($\alpha_1 = 0$ and $\alpha_2 > 0$). If $\lambda = 0$, since

$$\dot{p}(t) = -A^*(t)p(t) + \psi(t), \qquad p(b) = 0 \tag{3.26}$$

has a unique solution p, then

$$\dot{x}(t) = A(t)x(t) - \alpha_2 p(t) + \varphi(t), \qquad x(a) = x_0 \tag{3.27}$$

has a unique solution x. Hence, system (3.20) has a unique solution (x, p).

If $\lambda > 0$, we may assume $0 < \lambda < 1/(M^2\alpha_2(b-a))$. Otherwise, choose a sufficient large number N such that $\lambda/N < 1/(M^2\alpha_2(b-a))$ and let $\widetilde{p}(t) = p(t)/N$. Then we reduce to the desired case.

For any $\overline{x}(\cdot) \in C([a,b];X)$,

$$\dot{p}(t) = -A^*(t)p(t) + \psi(t), \qquad p(b) = \lambda\overline{x}(b) \tag{3.28}$$

has a unique solution $\overline{p}$:

$$\overline{p}(t) = \lambda U^*(b,t)\overline{x}(b) + \int_t^b U^*(s,t)\psi(s)ds. \tag{3.29}$$

Note that

$$\dot{x}(t) = A(t)x(t) - \alpha_2\overline{p}(t) + \varphi(t), \qquad x(a) = x_0 \tag{3.30}$$

has a unique solution $x(\cdot) \in C([a,b];X)$. Hence, we can define a mapping $C([a,b];X) \to C([a,b];X)$ by

$$J : \overline{x}(t) \longrightarrow x(t) = U(t,a)x_0 + \int_a^t U(t,s)[\varphi(s) - \alpha_2\overline{p}(s)]ds. \tag{3.31}$$

We will prove that J is a contraction and hence has a unique fixed point that is the unique solution of (3.20).

For any $\overline{x}_1(\cdot), \overline{x}_2(\cdot) \in C([a,b];X)$, taking into account that

$$||\overline{p}_1(t) - \overline{p}_2(t)|| \le \lambda M||\overline{x}_1(b) - \overline{x}_2(b)|| \le \lambda M||\overline{x}_1 - \overline{x}_2||_C, \tag{3.32}$$

we have

$$||(J\overline{x}_1)(t) - (J\overline{x}_2)(t)|| \le M\alpha_2(b-a)||\overline{p}_1 - \overline{p}_2||_C \le \lambda M^2\alpha_2(b-a)||\overline{x}_1 - \overline{x}_2||_C. \tag{3.33}$$

Therefore,

$$||J\overline{x}_1 - J\overline{x}_2||_C \le \lambda M^2\alpha_2(b-a)||\overline{x}_1 - \overline{x}_2||_C, \tag{3.34}$$

where $\|\cdot\|_C$ stands for the maximum norm in space $C([a,b];X)$. It follows that J is a contraction due to $\lambda M^2\alpha_2(b-a) < 1$. Now, we are ready to prove the first existence and uniqueness theorem. □

Theorem 3.3. *System (3.1) has a unique solution on $[a,b]$ under assumptions (A1) and (A2).*

Proof

Existence. By Lemma 3.2, system (3.2) has a solution on $[a,b]$ for $\beta_0 = 0$. Lemma 3.1 implies that there exists $\delta > 0$ independent of β_0 such that (3.2) has a solution on $[a,b]$ for any $\beta \in [0,\delta]$ and $\varphi(\cdot),\psi(\cdot) \in L^2(a,b;X)$. Now let $\beta_0 = \delta$ in Lemma 3.1 and repeat this process. We can prove that system (3.2) has a solution on $[a,b]$ for any $\beta \in [\delta,2\delta]$. Clearly, after finitely many steps, we can prove that system (3.2) has a solution for $\beta = 1$. Therefore, system (3.1) has a solution.

Uniqueness. Let (x_1,p_1) and (x_2,p_2) be any two solutions of system (3.1). Then

$$\begin{aligned}\frac{d}{dt}&\langle x_1(t)-x_2(t),p_1(t)-p_2(t)\rangle\\&=\langle F(x_1(t),p_1(t),t)-F(x_2(t),p_2(t),t),p_1(t)-p_2(t)\rangle\\&\quad+\langle G(x_1(t),p_1(t),t)-G(x_2(t),p_2(t),t),x_1(t)-x_2(t)\rangle\\&\le-\alpha_1\|x_1(t)-x_2(t)\|^2-\alpha_2\|p_1(t)-p_2(t)\|^2.\end{aligned}\tag{3.35}$$

Integrating between a and b yields

$$\begin{aligned}0&=\langle x_1(b)-x_2(b),\ p_1(b)-p_2(b)\rangle-\langle x_1(a)-x_2(a),\ p_1(a)-p_2(a)\rangle\\&\le-\alpha_1\int_a^b\|x_1(t)-x_2(t)\|^2dt-\alpha_2\int_a^b\|p_1(t)-p_2(t)\|^2dt.\end{aligned}\tag{3.36}$$

If $\alpha_1 > 0$ and $\alpha_2 > 0$, obviously, $(x_1,p_1) = (x_2,p_2)$ in $C([a,b];X)\times C([a,b];X)$. If $\alpha_1 > 0$ and $\alpha_2 = 0$, then $x_1 = x_2$. From the differential equation of $p(t)$ in (3.1) we have

$$\begin{aligned}&\frac{d}{dt}[p_1(t)-p_2(t)]=-A^*(t)(p_1(t)-p_2(t))+G(x_1(t),p_1(t),t)-G(x_1(t),p_2(t),t),\\&p_1(b)-p_2(b)=0.\end{aligned}\tag{3.37}$$

It follows that

$$\|p_1(t)-p_2(t)\|\le ML\int_t^b\|p_1(s)-p_2(s)\|ds,\quad a\le t\le b.\tag{3.38}$$

Gronwall's inequality implies that $p_1 = p_2$, and hence $(x_1,p_1) = (x_2,p_2)$. The discussion for the case $\alpha_1 = 0$ and $\alpha_2 > 0$ is similar to the previous case. The proof is complete. □

4. Existence and uniqueness: general function ξ

In this section, we consider the solvability of system (1.1) with general functions ξ. Although the proof of the next lemma follows from that of Lemma 3.1, more technical considerations are needed because $p(b)$ depends on $x(b)$ in this case. In particular, the apriori estimate for solutions of the family of BVPs is more complicated.

Lemma 4.1. *Consider the following BVP:*

$$\dot{x}(t) = A(t)x(t) + F_\beta(x(t), p(t), t) + \varphi(t), \qquad x(a) = x_0,$$
$$\dot{p}(t) = -A^*(t)p(t) + G_\beta(x(t), p(t), t) + \psi(t), \qquad p(b) = \beta\xi(x(b)) + (1-\beta)x(b) + \nu, \tag{4.1}$$

where $\varphi(\cdot)$, $\psi(\cdot) \in L^2(a,b;X)$ and x_0, $\nu \in X$. Assume that for a number $\beta = \beta_0 \in [0,1)$, system (4.1) has a solution in space $L^2(a,b;X) \times L^2(a,b;X)$ for any φ,ψ,x_0, and ν. In addition, assumptions (A1)–(A3) hold. Then there exists $\delta > 0$ independent of β_0 such that system (4.1) has a solution for any φ,ψ,ν,x_0, and $\beta \in [\beta_0, \beta_0+\delta]$.

Proof. For any $\varphi(\cdot)$, $\psi(\cdot)$, $x(\cdot)$, $p(\cdot) \in L^2(a,b;X)$, $\nu \in X$, and $\delta > 0$, we consider the following BVP:

$$\begin{aligned}
&\dot{X}(t) = A(t)X(t) + F_{\beta_0}(X(t), P(t), t) + \alpha_2\delta p(t) + \delta F(x(t), p(t), t) + \varphi(t),\\
&X(a) = x_0,\\
&\dot{P}(t) = -A^*(t)P(t) + G_{\beta_0}(X(t), P(t), t) + \alpha_1\delta x(t) + \delta G(x(t), p(t), t) + \psi(t),\\
&P(b) = \beta_0\xi(X(b)) + (1-\beta_0)X(b) + \delta(\xi(x(b)) - x(b)) + \nu.
\end{aligned} \tag{4.2}$$

Similar to the proof of Lemma 3.1, we know that system (4.2) has a solution $(X(\cdot), P(\cdot), X(b)) \in L^2(a,b;X) \times L^2(a,b;X) \times X$ for each triple $(x(\cdot), p(\cdot), x(b)) \in L^2(a,b;X) \times L^2(a,b;X) \times X$. Therefore, the mapping $J : L^2(a,b;X) \times L^2(a,b;X) \times X \to L^2(a,b;X) \times L^2(a,b;X) \times X$ defined by $J(x(\cdot), p(\cdot), x(b)) := (X(\cdot), P(\cdot), X(b))$ is well defined.

Take into account (A3), we have from (4.2) that

$$\begin{aligned}
&\langle X_1(b) - X_2(b),\, P_1(b) - P_2(b)\rangle\\
&\qquad \geq (1-\beta_0)||X_1(b) - X_2(b)||^2 + \delta\langle \xi(x_1(b)) - \xi(x_2(b)), X_1(b) - X_2(b)\rangle\\
&\qquad\quad - \langle X_1(b) - X_2(b), x_1(b) - x_2(b)\rangle\\
&\qquad \geq \frac{2 - 2\beta_0 - \delta - \delta c}{2}||X_1(b) - X_2(b)||^2 - \frac{(c+1)\delta}{2}||x_1(b) - x_2(b)||^2\\
&\qquad \geq \gamma||X_1(b) - X_2(b)||^2 - \delta c_1||x_1(b) - x_2(b)||^2.
\end{aligned} \tag{4.3}$$

Here, $\gamma > 0$ is a constant for small δ and the constant $c_1 = (c+1)/2$.

Combine (3.8) and the above discussion, then we have

$$\begin{aligned}
&(\alpha_1 - \delta C_1)\int_a^b ||X_1(t) - X_2(t)||^2 dt + (\alpha_2 - \delta C_1)\int_a^b ||P_1(t) - P_2(t)||^2 dt + \gamma||X_1(b) - X_2(b)||^2\\
&\qquad \leq \delta C_1 \int_a^b (||x_1(t) - x_2(t)||^2 + ||p_1(t) - p_2(t)||^2)dt + \delta c_1||x_1(b) - x_2(b)||^2.
\end{aligned} \tag{4.4}$$

Case 1 ($\alpha_1 > 0$ and $\alpha_2 > 0$). Let $\underline{\alpha} = \min\{\alpha_1, \alpha_2, \gamma\}$. Inequality (4.4) implies

$$\begin{aligned}&\int_a^b (\|X_1(t) - X_2(t)\|^2 + \|P_1(t) - P_2(t)\|^2)dt + \|X_1(b) - X_2(b)\|^2 \\ &\quad \le \frac{\delta C_1}{\underline{\alpha} - \delta C_1} \int_a^b (\|x_1(t) - x_2(t)\|^2 + \|p_1(t) - p_2(t)\|^2)dt + \frac{\delta c_1}{\underline{\alpha} - \delta C_1} \|x_1(b) - x_2(b)\|^2.\end{aligned} \tag{4.5}$$

Choose δ further small that $\delta C_1/(\underline{\alpha} - \delta C_1) < 1/2$ and $\delta c_1/(\underline{\alpha} - \delta C_1) < 1/2$, J is a contraction.

Case 2 ($\alpha_1 = 0$ and $\alpha_2 > 0$). Similar to the proof in case 1 of Lemma 3.1, there exists a $C_2 \ge 1$ dependent of M, L, and α_2 such that

$$\begin{aligned}&\int_a^b \|X_1(t) - X_2(t)\|^2 dt \\ &\quad \le C_2 \int_a^b \|P_1(t) - P_2(t)\|^2 dt + C_2\delta \int_a^b (\|x_1(t) - x_2(t)\|^2 + \|p_1(t) - p_2(t)\|^2) dt.\end{aligned} \tag{4.6}$$

Choose a sufficiently small number $\delta > 0$ such that $(\alpha_2 - \delta C_1)/2 > \alpha_2/4C_2$ and $(\alpha_2 - \delta C_1)/2C_2 - \delta C_1 > \alpha_2/4C_2$. From (4.6), we have

$$\begin{aligned}&-\delta C_1 \int_a^b \|X_1(t) - X_2(t)\|^2 dt + (\alpha_2 - \delta C_1) \int_a^b \|P_1(t) - P_2(t)\|^2 dt + \gamma\|X_1(b) - X_2(b)\|^2 \\ &\quad \ge \frac{\alpha_2}{4C_2} \int_a^b (\|X_1(t) - X_2(t)\|^2 + \|P_1(t) - P_2(t)\|^2)dt \\ &\qquad - \alpha_2\delta \int_a^b (\|x_1(t) - x_2(t)\|^2 + \|p_1(t) - p_2(t)\|^2)dt + \gamma\|X_1(b) - X_2(b)\|^2.\end{aligned} \tag{4.7}$$

By (4.4) and (4.7), we have

$$\begin{aligned}&\frac{\alpha_2}{4C_2} \int_a^b (\|X_1(t) - X_2(t)\|^2 + \|P_1(t) - P_2(t)\|^2)dt + \gamma\|X_1(b) - X_2(b)\|^2 \\ &\quad \le \delta(C_1 + \alpha_2) \int_a^b (\|x_1(t) - x_2(t)\|^2 + \|p_1(t) - p_2(t)\|^2)dt + \delta c_1\|x_1(b) - x_2(b)\|^2.\end{aligned} \tag{4.8}$$

Let $\rho = \min\{\alpha_2/4C_2, \gamma\}$. Then we have from (4.8) that

$$\int_a^b (\|X_1(t) - X_2(t)\|^2 + \|P_1(t) - P_2(t)\|^2)dt + \|X_1(b) - X_2(b)\|^2$$
$$\leq \frac{C_1 + \alpha_2}{\rho}\delta \int_a^b (\|x_1(t) - x_2(t)\|^2 + \|p_1(t) - p_2(t)\|^2)dt + \frac{c_1}{\rho}\delta\|x_1(b) - x_2(b)\|^2. \tag{4.9}$$

Let δ be small further that $(C_1 + \alpha_2)\delta/\rho < 1/2$ and $c_1\delta/\rho < 1/2$. Then J is a contraction.

Case 3 ($\alpha_1 > 0$ and $\alpha_2 = 0$). To prove this case, we need to carefully deal with the terminal condition. From system (4.2), we have

$$\begin{aligned}
\frac{d}{dt}(P_1(t) - P_2(t)) = {} & -A^*(t)(P_1(t) - P_2(t)) - (1 - \beta_0)\alpha_1(X_1(t) - X_2(t)) \\
& + \beta_0(G(X_1(t), P_1(t), t) - G(X_2(t), P_2(t), t)) \\
& + \alpha_1\delta(x_1(t) - x_2(t)) + \delta(G(x_1(t), p_1(t), t) - G(x_2(t), p_2(t), t)), \\
P_1(b) - P_2(b) = {} & \beta_0(\xi(X_1(b)) - \xi(X_2(b))) + (1 - \beta_0)(X_1(b) - X_2(b)) \\
& + \delta(\xi(x_1(b)) - \xi(x_2(b))) - \delta(x_1(b) - x_2(b)).
\end{aligned} \tag{4.10}$$

Apply the variation of constants formula to (4.10) and use Gronwall's inequality, then we have

$$\begin{aligned}
&\|P_1(t) - P_2(t)\| \\
&\quad \leq e^{ML(b-a)}\Big(M(1 - \beta_0 + \beta_0 c)\|X_1(b) - X_2(b)\| + M(1 + c)\delta\|x_1(b) - x_2(b)\| \\
&\qquad + M(\alpha_1\delta + \delta L)\int_a^b (\|x_1(t) - x_2(t)\| + \|p_1(t) - p_2(t)\|)dt \\
&\qquad + M(\alpha_1 + L)\int_a^b \|X_1(t) - X_2(t)\|dt\Big).
\end{aligned} \tag{4.11}$$

Therefore, there exists a number $C_2 > 1$ dependent of M, L, and α_1 such that

$$\begin{aligned}
\int_a^b \|P_1(t) - P_2(t)\|^2 dt \leq {} & C_2\int_a^b \|X_1(t) - X_2(t)\|^2 dt + C_2\|X_1(b) - X_2(b)\|^2 \\
& + \delta C_2\int_a^b (\|x_1(t) - x_2(t)\|^2 + \|p_1(t) - p_2(t)\|^2)dt \\
& + \delta C_2\|x_1(b) - x_2(b)\|^2.
\end{aligned} \tag{4.12}$$

Choose a natural number N large enough such that $\gamma - \alpha_1/N > \gamma/2$ and a small number $\delta > 0$ such that $(\alpha_1 - \delta C_1)(N-1)/N > \alpha_1/(2NC_2)$ and $(\alpha_1 - \delta C_1)/NC_2 - \delta C_1 > \alpha_1/(2NC_2)$. It follows from (4.12) that

$$\begin{aligned}
&(\alpha_1 - \delta C_1)\int_a^b ||X_1(t) - X_2(t)||^2 dt - \delta C_1 \int_a^b ||P_1(t) - P_2(t)||^2 dt \\
&\quad \geq \frac{\alpha_1}{2NC_2}\int_a^b \left(||X_1(t) - X_2(t)||^2 + ||P_1(t) - P_2(t)||^2\right) dt \\
&\quad - \frac{\alpha_1 \delta}{N}\int_a^b \left(||x_1(t) - x_2(t)||^2 + ||p_1(t) - p_2(t)||^2\right) dt \\
&\quad - \frac{\alpha_1}{N}||X_1(b) - X_2(b)||^2 - \frac{\alpha_1 \delta}{N}||x_1(b) - x_2(b)||^2.
\end{aligned} \tag{4.13}$$

We have by combining (4.4) and (4.13) that

$$\begin{aligned}
&\frac{\alpha_1}{2NC_2}\int_a^b \left(||X_1(t) - X_2(t)||^2 + ||P_1(t) - P_2(t)||^2\right) dt + \frac{\gamma}{2}||X_1(b) - X_2(b)||^2 \\
&\quad \leq \frac{\alpha_1 + NC_1}{N}\delta \int_a^b \left(||x_1(t) - x_2(t)||^2 + ||p_1(t) - p_2(t)||^2\right) dt \\
&\quad + \frac{\alpha_1 + Nc_1}{N}\delta ||x_1(b) - x_2(b)||^2.
\end{aligned} \tag{4.14}$$

Let $h = \min\{\alpha_1/(2NC_2), \gamma/2\}$. Then

$$\begin{aligned}
&\int_a^b \left(||X_1(t) - X_2(t)||^2 + ||P_1(t) - P_2(t)||^2\right) dt + ||X_1(b) - X_2(b)||^2 \\
&\quad \leq \frac{\alpha_1 + NC_1}{Nh}\delta \int_a^b \left(||x_1(t) - x_2(t)||^2 + ||p_1(t) - p_2(t)||^2\right) dt \\
&\quad + \frac{\alpha_1 + Nc_1}{Nh}\delta ||x_1(b) - x_2(b)||^2.
\end{aligned} \tag{4.15}$$

Let δ be small further that $(\alpha_1 + NC_1)\delta/(Nh) < 1/2$ and $(\alpha_1 + Nc_1)\delta/(Nh) < 1/2$. Then J is a contraction.

Altogether, J is a contraction, and hence it has a unique fixed point $(\overline{x}(\cdot), \overline{p}(\cdot))$ in $L^2(a,b;X) \times L^2(a,b;X)$. Clearly, the pair is a solution of (4.1) on $[a,b]$. Therefore, (4.1) has a solution on $[a,b]$ for any $\beta \in [\beta_0, \beta_0 + \delta]$. The proof of the lemma is complete. □

Theorem 4.2. *System (1.1) has a unique solution on* $[a,b]$ *under assumptions (A1), (A2), and (A3).*

Existence. The same argument as the proof of Theorem 3.3.

Uniqueness. Assume (x_1, p_1) and (x_2, p_2) are any two solutions of system (1.1). Note that $x_1(\cdot), x_2(\cdot), p_1(\cdot), p_2(\cdot) \in C([a,b];X)$, and

$$\begin{aligned} 0 &\leq \langle x_1(b) - x_2(b),\ p_1(b) - p_2(b)\rangle \\ &\leq -\alpha_1 \int_a^b \|x_1(t) - x_2(t)\|^2 dt - \alpha_2 \int_a^b \|p_1(t) - p_2(t)\|^2 dt. \end{aligned} \tag{4.16}$$

Obviously, $(x_1, p_1) = (x_2, p_2)$ in the case $\alpha_1 > 0$ and $\alpha_2 > 0$. If $\alpha_1 > 0$ and $\alpha_2 = 0$, then $x_1 = x_2$. In particular, $x_1(b) = x_2(b)$. From the differential equation of $p(t)$ in (1.1), we have

$$\begin{aligned} &\frac{d}{dt}[p_1(t) - p_2(t)] = -A^*(t)(p_1(t) - p_2(t)) + G(x_1(t), p_1(t), t) - G(x_1(t), p_2(t), t), \\ &\quad p_1(b) - p_2(b) = 0. \end{aligned} \tag{4.17}$$

Similar to the proof of Theorem 3.3, we conclude that $p_1 = p_2$. Therefore, $(x_1, p_1) = (x_2, p_2)$. The proof for the case $\alpha_1 = 0$ and $\alpha_2 > 0$ is similar. The proof of the theorem is complete. □

Remark 4.3. Theorem 4.2 extends the results of [4] and the results of [2] in the deterministic case to infinite dimensional spaces.

Consider a special case of (1.1) which is a linear BVP in the form

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + B(t)p(t) + \varphi(t), \qquad x(a) = x_0, \\ \dot{p}(t) &= -A^*(t)p(t) + C(t)x(t) + \psi(t), \qquad p(b) = Dx(b). \end{aligned} \tag{4.18}$$

Here, $B(t)$, $C(t) : [a,b] \to \mathcal{L}[X]$ are self-adjoint operators for each $t \in [a,b]$, $D \in \mathcal{L}[X]$ is also self-adjoint, X is a Hilbert space. The operator D is nonnegative, $B(t)$ and $C(t)$ are nonpositive for all $t \in [a,b]$, that is, $\langle B(t)x, x\rangle \leq 0$ and $\langle C(t)x, x\rangle \leq 0$ for all $x \in X$ and $t \in [a,b]$.

Corollary 4.4. *System (4.18) has a unique solution on $[a,b]$ if either $B(t)$ or $C(t)$ is negative uniformly on $[a,b]$, that is, there exists a number $\sigma > 0$ such that $\langle B(t)x, x\rangle \leq -\sigma\|x\|^2$ or $\langle C(t)x, x\rangle \leq -\sigma\|x\|^2$ for any $x \in X$ and $t \in [a,b]$.*

Proof. Indeed, we have

$$\begin{aligned} &\langle F(x_1, p_1, t) - F(x_2, p_2, t), p_1 - p_2\rangle + \langle G(x_1, p_1, t) - G(x_2, p_2, t), x_1 - x_2\rangle \\ &\quad = \langle B(t)(p_1 - p_2), p_1 - p_2\rangle + \langle C(t)(x_1 - x_2), x_1 - x_2\rangle \\ &\quad \leq -\sigma\|x_1 - x_2\|^2 \quad \text{or} \quad \leq -\sigma\|p_1 - p_2\|^2, \\ &\qquad \langle \xi(x_1) - \xi(x_2), x_1 - x_2\rangle = \langle D(x_1 - x_2), x_1 - x_2\rangle \geq 0. \end{aligned} \tag{4.19}$$

Therefore, all assumptions (A1)–(A3) hold and the conclusion follows from Theorem 4.2. □

Remark 4.5. Corollary 4.4 improves the result [1, page 133] which covers the case $D = 0$ only.

5. Examples

Example 5.1. Consider the linear control system

$$\dot{x}(t) = A(t)x(t) + B(t)u(t), \qquad x(0) = x_0, \tag{5.1}$$

with the quadratic cost index

$$\inf_{u(\cdot)\in L^2(0,b;U)} J[u(\cdot)] = \langle Q_1 x(b), x(b)\rangle + \int_0^b [\langle Q(t)x(t), x(t)\rangle + \langle R(t)u(t), u(t)\rangle]dt. \tag{5.2}$$

Here, $B(\cdot) : [0,b]\to\mathscr{L}[U,X]$, $Q(\cdot) : [0,b]\to\mathscr{L}[X]$, $R(\cdot) : [0,b]\to\mathscr{L}[U]$, $Q_1 \in \mathscr{L}[X]$, both U and X are Hilbert spaces. Moreover, Q_1 is self-adjoint and nonpositive, $Q(t)$ and $R(t)$ are self-adjoint for every $t \in [0,b]$.

Based on the theory of optimal control, the corresponding Hamiltonian system of this control system is

$$\begin{aligned} &\dot{x}(t) = A(t)x(t) - B(t)R^{-1}(t)B^*(t)p(t), \qquad x(0) = x_0, \\ &\dot{p}(t) = -A^*(t)p(t) - Q(t)x(t), \qquad p(b) = -Q_1 x(b). \end{aligned} \tag{5.3}$$

By Corollary 4.4, (5.3) has a unique solution on $[0,b]$ if either $Q(t)$ or $R(t)$ is positive uniformly in $[0,b]$, that is, there exists a real number $\sigma > 0$ such that $\langle Q(t)x, x\rangle \geq \sigma\|x\|^2$ for all $x \in X$ and $t \in [0,b]$ or $\langle R(t)u, u\rangle \geq \sigma\|u\|^2$ for all $u \in U$ and $t \in [0,b]$.

In the following, we provide another example which is a nonlinear system.

Example 5.2. Let $X = L^2(0;\pi)$. Let $e_n(x) = \sqrt{2/\pi}\sin(nx)$ for $n = 1,2,\ldots$. Then the set $\{e_n : n = 1,2,\ldots\}$ is an orthogonal base for X. Define $A : X\to X$ by $Ax = x''$ with the domain $D(A) = \{x \in H^2(0,\pi) : x(0) = x(\pi) = 0\}$. It is well known that operator A is self-adjoint and generates a compact semigroup on $[0,b]$ with the form

$$T(t)x = \sum_{n=1}^{\infty} e^{-n^2 t} x_n e_n, \qquad x = \sum_{n=1}^{\infty} x_n e_n \in X. \tag{5.4}$$

Define a nonlinear function $F : X\to X$ as

$$F(p) = \sum_{n=1}^{\infty} (-\sin p_n - 2p_n)e_n, \qquad p = \sum_{n=1}^{\infty} p_n e_n. \tag{5.5}$$

Note that for any $p_1 = \sum_{n=1}^{\infty} p_n^1 e_n$, $p_2 = \sum_{n=1}^{\infty} p_n^2 e_n \in X$, we have

$$\begin{aligned}
\|F(p_1) - F(p_2)\|^2 &= \sum_{n=1}^{\infty} \left(\sin p_n^1 - \sin p_n^2 + 2p_n^1 - 2p_n^2\right)^2 \\
&\le 2\sum_{n=1}^{\infty} \left[\left(\sin p_n^1 - \sin p_n^2\right)^2 + 4\left(p_n^1 - p_n^2\right)^2\right] \le 10\|p_1 - p_2\|^2, \\
\langle F(p_1) - F(p_2), p_1 - p_2 \rangle &= -\sum_{n=1}^{\infty} \left(\sin p_n^1 - \sin p_n^2 + 2p_n^1 - 2p_n^2\right)\left(p_n^1 - p_n^2\right) \le -\|p_1 - p_2\|^2.
\end{aligned} \tag{5.6}$$

Then, F is Lipschitz continuous with $L = \sqrt{10}$ and satisfies (A2) with $\alpha_1 = 0$ and $\alpha_2 = 1$.

Theorem 4.2 implies that the following homogenous BVP:

$$\begin{aligned}
\dot{x}(t) &= Ax(t) + F(p(t)), \qquad x(0) = x_0, \\
\dot{p}(t) &= -A^* p(t) + G(x(t)), \qquad p(b) = \xi(x(b))
\end{aligned} \tag{5.7}$$

has a unique solution on $[0,b]$ for any nonincreasing function G and any nondecreasing function ξ, that is, one has $\langle G(x_1) - G(x_2), x_1 - x_2 \rangle \le 0$ and $\langle \xi(x_1) - \xi(x_2), x_1 - x_2 \rangle \ge 0$ for all $t \in [0,b]$, $x_1, x_2 \in X$.

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