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Research Article

Reaction-Diffusion in Nonsmooth and Closed Domains

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We investigate the Dirichlet problem for the parabolic equation $u_t = \Delta u^m - bu^\beta$, $m > 0$, $\beta > 0$, $b \in \mathbb{R}$, in a nonsmooth and closed domain $\Omega \subset \mathbb{R}^{N+1}$, $N \geq 2$, possibly formed with irregular surfaces and having a characteristic vertex point. Existence, boundary regularity, uniqueness, and comparison results are established. The main objective of the paper is to express the criteria for the well-posedness in terms of the local modulus of lower semicontinuity of the boundary manifold. The two key problems in that context are the boundary regularity of the weak solution and the question whether any weak solution is at the same time a viscosity solution.

1. Introduction

Consider the equation

$$u_t = \Delta u^m - bu^\beta, \tag{1.1}$$

where $u = u(x,t)$, $x = (x_1,\dots,x_N) \in \mathbb{R}^N$, $N \geq 2$, $t \in \mathbb{R}_+$, $\Delta = \sum_{i=1}^N \partial^2/\partial x_i^2$, $m > 0$, $\beta > 0$, $b \in \mathbb{R}$. Equation (1.1) is usually called a reaction-diffusion equation. It is a simple model for various physical, chemical, and biological problems involving diffusion with a source ($b < 0$) or absorption ($b > 0$) of energy (see [1]). In this paper, we study the Dirichlet problem (DP) for (1.1) in a general domain $\Omega \subset \mathbb{R}^{N+1}$ with $\partial\Omega$ being a closed N-dimensional manifold. It can be stated as follows: given any continuous function on the boundary $\partial\Omega$ of Ω, to find a continuous extension of this function to the closure of Ω which satisfies (1.1) in Ω. The main objective of the paper is to express the criteria for the well-posedness in terms of the local modulus of lower semicontinuity of the boundary manifold.

Let Ω be bounded open subset of $\mathbb{R}^{N+1}$, $N \geq 2$, lying in the strip $0 < t < T$, $T \in (0,\infty)$. Denote

$$\Omega(\tau) = \{(x,t) \in \Omega : t = \tau\} \tag{1.2}$$

and assume that $\Omega(t) \neq \emptyset$ for $t \in (0,T)$, but $\Omega(0) = \emptyset$, $\Omega(T) = \emptyset$. Moreover, assume that $\partial\Omega \cap \{t = 0\}$ and $\partial\Omega \cap \{t = T\}$ are single points. This situation arises in applications when a nonlinear reaction-difusion process is going on in a time-dependent region which originates from a point source and shrinks back to a single point at the end of the time interval. We will use the standard notation: $z = (x,t) = (x_1,\dots,x_N,t) \in \mathbb{R}^{N+1}$, $N \geq 2$, $x = (x_1,\overline{x}) \in \mathbb{R}^N$, $\overline{x} = (x_2,\dots,x_N) \in \mathbb{R}^{N-1}$, $|x|^2 = \sum_{i=1}^N |x_i|^2$, $|\overline{x}|^2 = \sum_{i=2}^N |x_i|^2$. For a point $z = (x,t) \in \mathbb{R}^{N+1}$ we denote by $B(z;\delta)$ an open ball in $\mathbb{R}^{N+1}$ of radius $\delta > 0$ and with center being in z.

Assume that for arbitrary point $z_0 = (x^0,t_0) \in \partial\Omega$ with $0 < t_0 < T$ there exists $\delta > 0$ and a continuous function ϕ such that, after a suitable rotation of x-axes, we have

$$\begin{gathered}\partial\Omega \cap B(z_0,\delta) = \{z \in B(z_0,\delta) : x_1 = \phi(\overline{x},t)\}, \\ \operatorname{sign}(x_1 - \phi(\overline{x},t)) = 1 \quad \text{for } z \in B(z_0,\delta) \cap \Omega.\end{gathered} \tag{1.3}$$

Concerning the vertex boundary point $z_0 = (x_1^0,\overline{x}^0,T) \in \partial\Omega$ assume that there exists $\delta > 0$ and a continuous function ϕ such that, after a suitable rotation of x-axes, we have

$$\Omega \cap \{T - \delta < t < T\} \subset \{z : x_1 > \phi(\overline{x},t),\ (\overline{x},t) \in R(\delta)\}, \tag{1.4}$$

where

$$R(\delta) \subset \{z : x_1 = 0,\ T - \delta < t < T\}, \qquad \partial R(\delta) \cap \{t = T\} = (0,\overline{x}^0,T), \qquad x_1^0 = \phi(\overline{x}^0,T). \tag{1.5}$$

The simplest example of the domain Ω satisfying imposed conditions is a space-time ball in $\mathbb{R}^{N+1}$ lying in the strip $0 < t < T$. In general, the structure of $\partial\Omega$ near the vertex point may be very complicated. For example, $\partial\Omega$ may be a unification of infinitely many conical hypersurfaces with common vertex point on the top of Ω.

The restriction (1.4) on the vertex boundary point is not a technical one and is dictated by the nature of the diffusion process. Basically, the regularity of the vertex boundary point does not depend on the smoothness of the boundary manifold, but significantly depends on its "flatness" with respect to the characteristic hyperplane $t = T$. In fact, for the regularity of the vertex point the boundary manifold should not be too flat in at least one space direction. Otherwise speaking, "nonthinness" of the exterior set near the vertex point and below the hyperplane $t = T$ defines the regularity of the top boundary point. The main novelty of this paper is to characterize the critical "flatness" or "thinness" through one-side Hölder condition on the function ϕ from (1.4). The techniques developed in earlier papers [2, 3] are not applicable to present situation. Surprisingly, the critical Hölder exponent is 1/2, which is dictated by the second-order parabolicity, but not by the nonlinearities. Another important novelty of this paper is that the uniqueness of weak solutions to nonlinear degenerate and singular parabolic problem is expressed in terms of similar local "flatness" of the boundary manifold with respect to the characteristic hyperplanes. The developed techniques are applicable to general second-order nonlinear degenerate and singular parabolic problems.

We make now precise meaning of the solution to DP. Let ψ be an arbitrary continuous nonnegative function defined on $\partial\Omega$. DP consists in finding a solution to (1.1) in Ω satisfying initial-boundary condition

$$u = \psi \quad \text{on } \partial\Omega. \tag{1.6}$$

Obviously, in view of degeneration of the (1.1) and/or non-Lipschitzness of the reaction term we cannot expect the considered problem to have a classical solution near the points (x,t), where $u = 0$. Before giving the definition of weak solution, let us remind the definition of the class of domains $\mathcal{D}_{t_1,t_2}$ introduced in [2]. Let Ω_1 be a bounded subset of $\mathbb{R}^{N+1}$, $N \geq 2$. Let the boundary $\partial\Omega_1$ of Ω_1 consist of the closure of a domain $B\Omega_1$ lying on $t = t_1$, a domain $D\Omega_1$ lying on $t = t_2$ and a (not necessarily connected) manifold $S\Omega_1$ lying in the strip $t_1 < t \leq t_2$. Assume that $\Omega_1(t) \neq \emptyset$ for $t \in [t_1, t_2]$ and for all points $z_0 = (x^0, t_0) \in S\Omega_1$ (or $z_0 = (x^0, 0) \in \overline{S\Omega_1}$) there exists $\delta > 0$ and a continuous function ϕ such that, after a suitable rotation of x-axes, the representation (1.3) is valid. Following the notation of [2], the class of domains Ω_1 with described structure is denoted as $\mathcal{D}_{t_1,t_2}$. The set $\mathcal{P}\Omega_1 = \overline{B\Omega_1} \cup S\Omega_1$ is called a parabolic boundary of Ω_1.

Obviously $\Omega \cap \{z : t_0 < t < t_1\} \in \mathcal{D}_{t_0,t_1}$ for arbitrary t_0, t_1 satisfying $0 < t_0 < t_1 < T$. However, note that $\Omega \notin \mathcal{D}_{0,T}$, since $\partial\Omega$ consists of, possibly characteristic, single points at $t = 0$ and $t = T$. We will follow the following notion of weak solutions (super- or subsolutions).

Definition 1.1. The function $u(x,t)$ is said to be a solution (resp., super- or subsolution) of DP (1.1), (1.6), if

(a) u is nonnegative and continuous in $\overline{\Omega}$, locally Hölder continuous in Ω, satisfying (1.6) (resp., satisfying (1.6) with $=$ replaced by $\geq$ or $\leq$),

(b) for any t_0, t_1 such that $0 < t_0 < t_1 < T$ and for any domain $\Omega_1 \in \mathcal{D}_{t_0,t_1}$ such that $\overline{\Omega}_1 \subset \Omega$ and $\partial B\Omega_1, \partial D\Omega_1, S\Omega_1$ being sufficiently smooth manifolds, the following integral identity holds:

$$\int_{D\Omega_1} uf\,dx = \int_{B\Omega_1} uf\,dx + \int_{\Omega_1} \left(uf_t + u^m \Delta f - bu^\beta f\right) dx\,dt - \int_{S\Omega_1} u^m \frac{\partial f}{\partial \nu}\,dx\,dt, \tag{1.7}$$

(resp., (1.7) holds with $=$ replaced by $\geq$ or $\leq$), where $f \in C^{2,1}_{x,t}(\overline{\Omega}_1)$ is an arbitrary function (resp., nonnegative function) that equals to zero on $S\Omega_1$ and ν is the outward-directed normal vector to $\Omega_1(t)$ at $(x,t) \in S\Omega_1$.

Concerning the theory of the boundary value problems in smooth cylindrical domains and interior regularity results for general second-order nonlinear degenerate and singular parabolic equations, we refer to [4–6] and to the review article [1]. The well-posedness of the DP to nonlinear diffusion equation ((1.1) with $b = 0$, $m \neq 1$) in a domain $\Omega \in \mathcal{D}_{0,T}$

is accomplished in [2, 3]. Existence and boundary regularity result for the reaction-diffusion (1.1) in a domain $\Omega \in \mathscr{D}_{0,T}$ is proved in [7]. For the precise result concerning the solvability of the classical DP for the heat/diffusion equation we refer to [8]. Necessary and sufficient condition for the regularity of a characteristic top boundary point of an arbitrary open subset of $\mathbb{R}^{N+1}$ for the classical heat equation is proved in [9, 10]. Investigation of the DP for (1.1) in a domain possibly with a characteristic vertex point, in particular, is motivated by the problem about the structure of interface near the possible extinction time $T_0 = \inf(\tau : u(x,t) = 0 \text{ for } t \geq \tau)$. If we consider the Cauchy problem for (1.1) with $b > 0$ and $0 < \beta < \min(1; m)$ and with compactly supported initial data, then the solution is compactly supported for all $t > 0$ and from the comparison principle it follows that $T_0 < \infty$. In order to find the structure and asymptotics of interface near $t = T_0$, it is important at the first stage to develop the general theory of boundary value problems in non cylindrical domain with boundary surface which has the same kind of behavior as the interface near extinction time. In many cases this may be a characteristic single point. It should be mentioned that in the one-dimensional case Dirichlet and Cauchy-Dirichlet problems for the reaction-diffusion equations in irregular domains were studied in papers by the author [11, 12]. Primarily applying this theory a complete description of the evolution of interfaces were presented in other papers [13, 14].

Furthermore, we assume that $0 < T < +\infty$ if $b \geq 0$ or $b < 0$ and $0 < \beta \leq 1$, and $T \in (0, T^*)$ if $b < 0$ and $\beta > 1$, where $T^* = M^{1-\beta}/(b(1-\beta))$ and $M > \sup \psi$. In fact, T^* is a lower bound for the possible blow-up time.

Our general strategy for the existence result coincides with the classical strategy for the DP to Laplace equation [15]. As pointed out by Lebesgue and independently by Wiener, "the Dirichlet problem divides itself into two parts, the first of which is the determination of a harmonic function corresponding to certain boundary conditions, while the second is the investigation of the behavior of this function in the neighborhood of the boundary." By using an approximation of both Ω and ψ, as well as regularization of (1.1), we also construct a solution to (1.1) as a limit of a sequence of classical solutions of regularized equation in smooth domains. We then prove a boundary regularity by using barriers and a limiting process. In particular, we prove the regularity of the vertex point under Assumption $\mathscr{A}$ (see Section 2). Geometrically it means that locally below the vertex point our domain is situated on one side of the N-dimensional exterior touching surface, which is slightly "less flat" than paraboloid with axes in $-t$-direction and with the same vertex point. Otherwise speaking, at the vertex point the function ϕ from (1.4) should satisfy one-side Hölder condition with critical value of the Hölder exponent being 1/2. In the case when the constructed solution is positive in Ω (accordingly, it is a classical one), from the classical maximum principle it follows that the solution is unique (see Corollaries 2.3 and 2.4 in Section 2). The next question which we clear in this paper is whether arbitrary weak solution is unique. We are interested in cases when weak solution may vanish in Ω, having one or several interfaces. Mostly, solution is nonsmooth near the interfaces and classical maximum principle is not applicable. Accordingly, we prove the uniqueness of the weak solution (Theorem 2.6, Section 2) assuming that either $m > 0$, $0 < \beta < 1$, $b > 0$ or $m > 1$, $\beta \geq 1$, and b is arbitrary. Our strategy for the uniqueness result is very similar to the one which applies to the existence result. Given arbitrary two weak solutions, the proof of uniqueness divides itself into two parts, the first of which is the determination of a limit solution whose integral difference from both given solutions may be estimated via boundary gradient bound of the solution to the linearized adjoint problem, while the second part is the investigation of the gradient of the solution to the linearized adjoint problem in the neigborhood of the boundary. In fact, the second step is of local nature and related auxiliary question is the following one: what is the minimal restriction on the lateral boundary manifold in order to get boundary gradient boundedness for the solution to the second-order linear parabolic equation? We introduce in the next section Assumption $\mathcal{M}$, which imposes pointwise geometric restriction to the boundary manifold $\partial\Omega$ in a small neigborhood of its point $z_0 = (x^0, t_0)$, $0 < t_0 < T$, which is situated upper the hyperplane $t = t_0$. Assumption $\mathcal{M}$ plays a crucial role within the second step of the uniqueness proof, allowing us to prove boundary gradient estimate for the solution to the linearized adjoint problem, which is a backward-parabolic one. At this point it should be mentioned that one can "avoid" the consideration of the uniqueness question by adapting the well-known notion of viscosity solution to the case of (1.1). For example, in the paper [16] this approach is applied to the DP for the porous-medium kind equations in smooth and cylindrical domain and under the zero boundary condition. In the mentioned paper [16] the notion of admissible solution, which is the adaptation of the notion of viscosity solution, was introduced. Roughly speaking, admissible solutions are solutions which satisfy a comparison principle. Accordingly, admissible solution of the DP will be unique in view of its definition. By using a simple analysis one can show that the limit solution of the DP (1.1), (1.6) which we construct in this paper is an admissible solution. However, this does not solve the problem about the uniqueness of the weak solution to DP. The question must be whether every weak solution in the sense of Definition 1.1 is an admissible solution. It is not possible to answer this question staying in the "admissible framework" and one should take as a starting point the integral identity (1.7). In fact, the uniqueness Theorem 2.6 addresses exactly this question and one can express its proof as follows: if there are two weak solutions of the DP, then we can construct a limit solution (or admissible solution) which coincides with both of them, provided that Assumption $\mathcal{M}$ is satisfied as it is required in Theorem 2.6. Under the same conditions we prove also a comparison theorem (see Theorem 2.7. Section 2), as well as continuous dependence on the boundary data (see Corollary 2.8, Section 2).

Although we consider in this paper the case $N \geq 2$, analogous results may be proved (with simplification of proofs) for the case $N = 1$ as well. Since the uniqueness and comparison results of this paper significantly improve the one-dimensional results from [11, 12], we describe the one-dimensional results separately in Section 3. We prove Theorems 2.2, 2.6, and 2.7 in Sections 4–6, respectively.

2. Statement of main results

Let $z_0 = (x^0, t_0) \in \partial\Omega$ be a given boundary point with $t_0 > 0$. If $t_0 < T$, then for an arbitrary sufficiently small $\delta > 0$ consider a domain

$$P(\delta) = \{(\overline{x}, t) : |\overline{x} - \overline{x}^0| < (\delta + t - t_0)^{1/2},\ t_0 - \delta < t < t_0\}. \tag{2.1}$$

Definition 2.1. Let

$$\begin{aligned} \omega(\delta) &= \max\left(\phi(\overline{x}^0, t_0) - \phi(\overline{x}, t) : (\overline{x}, t) \in \overline{P(\delta)}\right) \quad \text{if } t_0 < T, \\ \omega(\delta) &= \max\left(\phi(\overline{x}^0, T) - \phi(\overline{x}, t) : (\overline{x}, t) \in \overline{R(\delta)}\right) \quad \text{if } t_0 = T. \end{aligned} \tag{2.2}$$

For sufficiently small $\delta > 0$ these functions are well-defined and converge to zero as $\delta \downarrow 0$.

Assumption $\mathcal{A}$. There exists a function $F(\delta)$ which is defined for all positive sufficiently small δ; F is positive with $F(\delta) \to 0+$ as $\delta \downarrow 0$ and

$$\omega(\delta) \leq \delta^{1/2} F(\delta). \tag{2.3}$$

It is proved in [2] that Assumption $\mathcal{A}$ is sufficient for the regularity of the boundary point $z_0 = (x^0, t_0) \in \partial\Omega$ with $0 < t_0 < T$. Namely, the constructed limit solution takes the boundary value $\psi(z_0)$ at the point $z = z_0$ continuously in $\overline{\Omega}$. We prove in Section 4 that Assumption $\mathcal{A}$ is sufficient for the regularity of the vertex boundary point. Thus our existence theorem reads.

THEOREM 2.2. *DP (1.1), (1.6) is solvable in a domain Ω which satisfies Assumption $\mathcal{A}$ at every point $z_0 \in \partial\Omega$ with $t_0 > 0$.*

The following corollary is an easy consequence of Theorem 2.2.

COROLLARY 2.3. *If the constructed solution $u = u(x,t)$ to DP (1.1), (1.6) is positive in Ω, then under the conditions of Theorem 2.2, $u \in C(\overline{\Omega}) \cap C^\infty(\Omega)$ and it is a unique classical solution.*

In particular, we have the following corollary.

COROLLARY 2.4. *Let $\beta \geq 1$ and $\inf_{\partial\Omega} \psi > 0$. Then under the conditions of Theorem 2.2, there exists a unique classical solution $u \in C(\overline{\Omega}) \cap C^\infty(\Omega)$ of the DP (1.1), (1.6).*

Furthermore, *we always suppose in this paper that the condition of Theorem 2.2 is satisfied.* Let us now formulate another pointwise restriction at the point $z_0 = (x^0, t_0) \in \partial\Omega$, $0 < t_0 < T$, which plays a crucial role in the proof of uniqueness of the constructed solution. For an arbitrary sufficiently small $\delta > 0$ consider a domain

$$Q(\delta) = \left\{(\overline{x}, t) : \left|\overline{x} - \overline{x}^0\right| < (\delta + t_0 - t)^{1/2},\ t_0 < t < t_0 + \delta\right\}. \tag{2.4}$$

Our restriction on the behavior of the funtion ϕ in $Q(\delta)$ for small δ is as follows.

Assumption $\mathcal{M}$. Assume that for all sufficiently small positive δ we have

$$\phi(\overline{x}^0, t_0) - \phi(\overline{x}, t) \leq \left[t - t_0 + \left|\overline{x} - \overline{x}^0\right|^2\right]^\mu \quad \text{for } (\overline{x}, t) \in \overline{Q(\delta)}, \tag{2.5}$$

where $\mu > 1/2$ if $0 < m < 1$, and $\mu > m/(m+1)$ if $m > 1$.

Assumption $\mathcal{M}$ is of geometric nature. We explained its geometric meaning in [3, Section 3]. Assumption $\mathcal{M}$ is pointwise and related number μ in (2.5) depends on $z_0 \in \partial\Omega$ and may vary for different points $z_0 \in \partial\Omega$. For our purposes we need to define "the uniform Assumption $\mathcal{M}$" for certain subsets of $\partial\Omega$.

Definition 2.5. Assumption $\mathcal{M}$ is said to be satisfied uniformly in $[c,d] \subset (0,T)$ if there exists $\delta_0 > 0$ and $\mu > 0$ as in (2.5) such that for $0 < \delta \le \delta_0$, (2.5) is satisfied for all $z_0 \in \partial\Omega \cap \{(x,t) : c \le t \le d\}$ with the same μ.

Our next theorems read.

Theorem 2.6 (uniqueness). *Let either $m > 0$, $0 < \beta < 1$, $b \ge 0$ or $m > 1$, $\beta \ge 1$, and b is arbitrary. Assume that there exists a finite number of points t_i, $i = 1,\dots,k$ such that $t_1 = 0 < t_2 < \cdots < t_k < t_{k+1} = T$ and for the arbitrary compact subsegment $[\delta_1,\delta_2] \subset (t_i,t_{i+1})$, $i = 1,\dots,k$, Assumption $\mathcal{M}$ is uniformly satisfied in $[\delta_1,\delta_2]$. Then the solution of the DP is unique.*

Theorem 2.7 (comparison). *Let u be a solution of DP and g be a supersolution (resp., subsolution) of DP. Assume that the assumption of Theorem 2.6 is satisfied. Then $u \le$ (resp., $\ge$) g in $\overline{\Omega}$.*

Corollary 2.8. *Assume that the assumption of Theorem 2.6 is satisfied. Let u be a solution of DP. Assume that $\{\psi_n\}$ be a sequence of nonnegative continuous functions defined on $\partial\Omega$ and $\lim_{n\to\infty}\psi_n(z) = \psi(z)$, uniformly for $z \in \partial\Omega$. Let u_n be a solution of DP (1.1), (1.6) with $\psi = \psi_n$. Then $u = \lim_{n\to\infty} u_n$ in $\overline{\Omega}$ and convergence is uniform on compact subsets of Ω.*

Remark 2.9. It should be mentioned that we might have supposed that $\Omega(0)$ is nonempty, bounded, and open domain lying on the hyperplane $\{t = 0\}$. In this case the condition (1.6) includes also initial condition imposed on $\Omega(0)$. The existence Theorem 2.2 is true in this case as well if we assume additionally that the boundary points $z \in \partial\Omega(0)$ on the bottom of the lateral boundary of Ω satisfy the Assumption $\mathcal{B}$ from [7, 2]. In [7] it is proved that under the Assumption $\mathcal{B}$ the boundary point $z \in \partial\Omega(0)$ is a regular point. Assumption $\mathcal{B}$ is just the restriction of Assumption $\mathcal{A}$ to the part of the lateral boundary which lies on the hyperplane $t =$ const. Moreover, Assumptions $\mathcal{A}$ and $\mathcal{B}$ coincide in the case of cylindrical domain. Assertions of the Theorems 2.6, 2.7 and Corollaries 2.3, 2.4, and 2.8 are also true in this case. The proofs are similar to the proofs given in this paper.

3. The one-dimensional theory

Let $E = \{(x,t) : \phi_1(t) < x < \phi_2(t),\ 0 < t < T\}$, where $0 < T < +\infty$, $\phi_i \in C[0,T]$, $i = 1,2$: $\phi_1(t) < \phi_2(t)$ for $t \in (0,T)$ and $\phi_1(0) \le \phi_2(0)$, $\phi_1(T) = \phi_2(T)$.

Consider the problem

$$u_t - (u^m)_{xx} + bu^\beta = 0 \quad \text{in } E, \tag{3.1}$$

$$u(\phi_i(t),t) = \psi_i(t), \quad 0 \le t \le T, \tag{3.2}$$

where $u = u(x,t)$, $m > 0$, $b \in \mathbb{R}^1$, $\beta > 0$, $\psi_i \in C[0,T]$, $\psi_i \ge 0$, $i = 1,2$; $\psi_1(T) = \psi_2(T)$. If $\phi_1(0) = \phi_2(0)$, then we assume that $\psi_1(0) = \psi_2(0)$. If $\phi_1(0) < \phi_2(0)$, then we impose additionally the initial condition

$$u(x,0) = u_0(x), \quad \phi_1(0) \le x \le \phi_2(0), \tag{3.3}$$

where $u_0 \in C[\phi_1(0),\phi_2(0)]$, $u_0 \ge 0$ and $u_0(\phi_i(0)) = \psi_i(0)$, $i = 1,2$.

Definition 3.1. The function $u(x,t)$ is said to be a solution (resp., super- or subsolution) of problem (3.1), (3.2) (or (3.1)–(3.3)) if

(a) u is nonnegative and continuous in $\overline{E}$, satisfying (3.2) (or (3.2) and (3.3)) (resp., satisfying (3.2), (3.3) with $=$ replaced by $\ge$ or $\le$),

(b) for any t_0, t_1 such that $0 < t_0 < t_1 < T$ and for any C^∞ functions $\mu_i(t)$, $t_0 \le t \le t_1$, $i = 1,2$, such that $\phi_1(t) < \mu_1(t) < \mu_2(t) < \phi_2(t)$ for $t \in [t_0,t_1]$, the following integral identity holds:

$$\int_{t_0}^{t_1}\int_{\mu_1(t)}^{\mu_2(t)} \left(uf_t + u^m f_{xx} - bu^\beta f\right)dx\,dt - \int_{\mu_1(t)}^{\mu_2(t)} uf\,\Big|_{t=t_0}^{t=t_1} dx - \int_{t_0}^{t_1} u^m f_x\,\Big|_{x=\mu_1(t)}^{x=\mu_2(t)} dt = 0, \tag{3.4}$$

(resp., (3.4) holds with $=$ replaced by $\le$ or $\ge$) where $D_1 = \{(x,t):\ \mu_1(t) < x < \mu_2(t),\ t_0 < t < t_1\}$ and $f \in C^{2,1}_{x,t}(\overline{D}_1)$ is an arbitrary function (resp., nonnegative function) that equals zero when $x = \mu_i(t)$, $t_0 \le t \le t_1$, $i = 1,2$.

Furthermore, we assume that $0 < T < +\infty$ if $b \ge 0$ or $b < 0$ and $0 < \beta \le 1$, and $T \in (0,T^*)$ if $b < 0$ and $\beta > 1$, where $T^* = M^{1-\beta}/b(1-\beta)$ and $M = \max(\max\psi_1,\max\psi_2) + \epsilon$ (or $M = \max(\max\psi_1,\max\psi_2,\max u_0) + \epsilon$), and $\epsilon > 0$ is an arbitrary sufficiently small number.

For any $\phi \in C[0,T]$ and for any fixed $t_0 > 0$ define the functions

$$\begin{aligned} \omega^-_{t_0}(\phi;\delta) &= \max\left(\phi(t_0) - \phi(t):\ t_0 - \delta \le t \le t_0\right), \\ \omega^+_{t_0}(\phi;\delta) &= \min\left(\phi(t_0) - \phi(t):\ t_0 - \delta \le t \le t_0\right). \end{aligned} \tag{3.5}$$

The function $\omega^-_{t_0}(\phi;\cdot)$ (resp., $\omega^+_{t_0}(\phi;\cdot)$) is called a left modulus of lower (resp., upper) semicontinuity of the function ϕ at the point t_0.

The following theorem is the one-dimensional case of Theorem 2.2.

Theorem 3.2 (existence) (see [11, 12]). *For each $t_0 \in (0,T)$ let there exist a function $F(\delta)$ which is defined for all positive sufficiently small δ; F is positive with $F(\delta) \to 0+$ as $\delta \to 0+$ and*

$$\omega^-_{t_0}(\phi_1;\delta) \le \delta^{1/2}F(\delta), \tag{3.6}$$

$$\omega^+_{t_0}(\phi_2;\delta) \ge -\delta^{1/2}F(\delta). \tag{3.7}$$

Assume also that for $t = T$ there exists a function $F(\delta)$, defined as before, such that either $\omega^-_T(\phi_1;\delta)$ satisfies (3.6) or $\omega^+_T(\phi_2;\delta)$ satisfies (3.7) for sufficiently small positive δ. Then there exists a solution of the problem (3.1), (3.2) (or (3.1)–(3.3)).

Assume that $t_0 \in (0,T)$ is fixed. The following is the one-dimensional case of Assumption $\mathcal{M}$.

Assumption $\mathcal{M}_1$. Assume that for all sufficiently small positive δ we have

$$\begin{aligned} \phi_1(t_0) - \phi_1(t) &\le (t - t_0)^\mu \quad \text{for } t_0 \le t \le t_0 + \delta, \\ \phi_2(t_0) - \phi_2(t) &\ge -(t - t_0)^\mu \quad \text{for } t_0 \le t \le t_0 + \delta, \end{aligned} \tag{3.8}$$

where $\mu > 1/2$ if $0 < m < 1$, and $\mu > m/(m+1)$ if $m > 1$.

Otherwise speaking, Assumption $\mathcal{M}_1$ means that at each point $t_0 \in (0,T)$ the left boundary curve (resp., the right boundary curve) is right-lower-Hölder continuous (resp., right-upper-Hölder continuous) with Hölder exponent μ.

Definition 3.3. Let $[c,d] \subset (0,T)$ be a given segment. Assumption $\mathcal{M}_1$ is said to be satisfied uniformly in $[c,d]$ if there exists $\delta_0 > 0$ and $\mu > 0$ as in (3.8) such that for $0 < \delta \le \delta_0$, (3.8) is satisfied for all $t_0 \in [c,d]$ with the same μ.

If we replace Assumption $\mathcal{M}$ with Assumption $\mathcal{M}_1$, then Theorems 2.6, 2.7 and Corollary 2.8 apply to the one-dimensional problem (3.1), (3.2) (or (3.1)–(3.3)) as well.

4. Proof of Theorem 2.2

Step 1 (construction of the limit solution). Consider a sequence of domains $\Omega_n \in \mathcal{D}_{0,T}$, $n = 1,2,\ldots$ with $S\Omega_n$, $\partial B\Omega_n$ and $\partial D\Omega_n$ being sufficiently smooth manifolds. Assume that $\{S\Omega_n\}$ approximate $\partial\Omega$, while $\{B\Omega_n\}$ and $\{D\Omega_n\}$ approximate single points $\partial\Omega \cap \{t = 0\}$ and $\partial\Omega \cap \{t = T\}$, respectively. The latter means that for arbitrary $\epsilon > 0$ there exists $N(\epsilon)$ such that $B\Omega_n$ (resp., $D\Omega_n$), for all $n \ge N(\epsilon)$, lies in the ϵ-neigborhood of the point $\partial\Omega \cap \{t = 0\}$ (resp., $\partial\Omega \cap \{t = T\}$) on the hyperplane $\{t = 0\}$ (resp., $\{t = T\}$). Moreover, let $\overline{S\Omega_n}$ at some neigborhood of its every point after suitable rotation of x-axes has a representation via the sufficiently smooth function $x_1 = \phi_n(\overline{x},t)$. More precisely, assume that $\partial\Omega$ in some neigborhood of its point $z_0 = (x_1^0, \overline{x}^0, t_0)$, $0 < t_0 < T$, after suitable rotation of x-axes, is represented by the function $x_1 = \phi(\overline{x},t)$, $(\overline{x},t) \in P(\delta_0)$ with some $\delta_0 > 0$, where ϕ satisfies Assumption $\mathcal{A}$ from Section 2. Then we also assume that $S\Omega_n$ in some neigborhood of its point $z_n = (x_1^{(n)}, \overline{x}^{(0)}, t_0)$, after the same rotation, is represented by the function $x_1 = \phi_n(\overline{x},t), (\overline{x},t) \in P(\delta_0)$, where $\{\phi_n\}$ is a sequence of sufficiently smooth functions and $\phi_n \to \phi$ as $n \to \infty$, uniformly in $P(\delta_0)$. We can also assume that ϕ_n satisfies Assumption $\mathcal{A}$ uniformly with respect to n.

Concerning approximation near the vertex boundary point assume that after the same rotation of x-axes which provides (1.4), we have

$$\begin{gathered} \Omega_n \cap \{T - \delta_0 < t < T\} \subset \{z : x_1 > \phi_n(\overline{x},t),\ (\overline{x},t) \in R_n(\delta_0)\}, \\ R_n(\delta_0) \subset \{z : x_1 = 0,\ T - \delta_0 < t < T\} \cap \widetilde{O}_{\gamma_n}(R(\delta_0)), \\ (\overline{x}^0, T) \in \partial R_n(\delta_0), \quad x_1^0 = \phi_n(\overline{x}^0, T) = \phi(\overline{x}^0, T), \end{gathered} \tag{4.1}$$

where $\delta_0 > 0$, $\{\phi_n\}$ is a sequence of sufficiently smooth functions in $\overline{R_n(\delta_0)}$ and $\phi_n \to \phi$ as $n \to \infty$ uniformly in $\overline{R(\delta_0)}$; $\{\gamma_n\}$ is a positive sequence of real numbers satisfying $\gamma_n \downarrow 0$ as $n \to \infty$; $\widetilde{O}_\rho(R(\delta))$ denotes ρ-neigborhood of $R(\delta)$ in N-dimensional subspace $\{x_1 = 0\}$.

We can also assume that as an implication of Assumption $\mathscr{A}$, ϕ_n satisfies

$$\phi_n(\overline{x}^0, T) - \phi_n(\overline{x}, t) \leq \omega(\delta) \quad \text{for } (\overline{x}, t) \in \overline{R_n(\delta)}. \tag{4.2}$$

Assume also that for arbitrary compact subset $\Omega^{(0)}$ of Ω there exists a number n_0 which depends on the distance between $\Omega^{(0)}$ and $\partial\Omega$ such that $\Omega^{(0)} \subset \Omega_n$ for $n \geq n_0$.

Let Ψ be a nonnegative and continuous function in $\mathbb{R}^{N+1}$ which coincides with ψ on $\partial\Omega$ and let M be an upper bound for $\psi_n = \Psi + n^{-1}$, $n \geq N_0$, in some compact which contains $\overline{\Omega}$ and $\overline{\Omega_n}$, $n \geq N_0$, where N_0 is a large positive integer. Introduce the following regularized equation:

$$u_t = \Delta u^m - bu^\beta + b\theta_b n^{-\beta}, \tag{4.3}$$

where $\theta_b = (1 \text{ if } b > 0;\ 0 \text{ if } b \leq 0)$. We then consider the DP in Ω_n for (4.3) with the initial-boundary data ψ_n. This nondegenerate parabolic problem and classical theory (see [17–19]) implies the existence of a unique classical solution u_n which satisfies

$$n^{-1} \leq u_n(x,t) \leq \psi^1(t) \quad \text{in } \overline{\Omega_n}, \tag{4.4}$$

where

$$\psi^1(t) = \begin{cases} \left[M^{1-\beta} - b(1-\theta_b)(1-\beta)t\right]^{1/(1-\beta)} & \text{if } \beta \neq 1, \\ M\exp\left(-b(1-\theta_b)t\right) & \text{if } \beta = 1. \end{cases} \tag{4.5}$$

Next we take a sequence of compact subsets $\Omega^{(k)}$ of Ω such that

$$\Omega = \bigcup_{k=1}^{\infty} \Omega^{(k)}, \quad \Omega^{(k)} \subseteq \Omega^{(k+1)}, \quad k = 1, 2, \ldots. \tag{4.6}$$

By our construction, for each fixed k there exists a number n_k such that $\Omega^{(k)} \subseteq \Omega_n$ for $n \geq n_k$. Since the sequence of uniformly bounded solutions u_n, $n \geq n_k$, to (4.3) is uniformly equicontinuous in a fixed compact $\Omega^{(k)}$ (see, e.g., [5, Theorem 1, Proposition 1, and Theorem 7.1]), from (4.6) by diagonalization argument and Arzela-Ascoli theorem, it follows that there exists a subsequence n' and a limit function $\widetilde{u}$ such that $u_{n'} \to \widetilde{u}$ as $n' \to +\infty$, pointwise in Ω and the convergence is uniform on compact subsets of Ω. Now consider a function $u(x,t)$ such that $u(x,t) = \widetilde{u}(x,t)$ for $(x,t) \in \Omega$, $u(x,t) = \psi$ for $(x,t) \in \partial\Omega$. Obviously, the function u satisfies the integral identity (1.7). Hence, the constructed function u is a solution of the DP (1.1), (1.6) if it is continuous on $\partial\Omega$.

Step 2 (boundary regularity). Let $z_0 = (x_1^0, \overline{x}^0, t_0) \in \partial\Omega$. We will prove that z_0 is regular, namely, that

$$\lim u(z) = \psi(z_0) \quad \text{as } z \longrightarrow z_0,\ z \in \Omega. \tag{4.7}$$

If $0 < t_0 < T$, then (4.7) is proved in [7]. Consider the case $t_0 = T$. In order to make the role of Assumption $\mathscr{A}$ clear for the reader, we keep the function $\omega(\delta)$ from Definition 2.1 free, just assuming without loss of generality that $\omega(\delta)$ is some positive function defined for positive small δ and $\omega(\delta) \to 0$ as $\delta \downarrow 0$. It will be clear at the end of the proof that in the framework of our method the optimal upper bound for $\omega(\delta)$ is given via (2.3).

If $\psi(z_0) > 0$, we will prove that for arbitrary sufficiently small $\epsilon > 0$ the following two inequalities are valid:

$$\liminf u(z) \geq \psi(z_0) - \epsilon \quad \text{as } z \longrightarrow z_0,\ z \in \Omega, \tag{4.8}$$

$$\limsup u(z) \leq \psi(z_0) + \epsilon \quad \text{as } z \longrightarrow z_0,\ z \in \Omega. \tag{4.9}$$

Since $\epsilon > 0$ is arbitrary, from (4.8) and (4.9), (4.7) follows. If $\psi(z_0) = 0$, however, then it is sufficient to prove (4.9), since (4.8) follows directly from the fact that $u \geq 0$ in $\overline{\Omega}$. Let $\psi(z_0) > 0$. Take an arbitrary $\epsilon \in (0, \psi(z_0))$ and prove (4.8). For arbitrary $\delta > 0$ consider a function

$$w_n(x,t) = f(\xi) \equiv M_1 \left(\frac{\xi}{h(\delta)} \right)^{\alpha}, \tag{4.10}$$

where

$$\xi = h(\delta) + \phi_n(\overline{x}^0, T) - x_1 - g(\delta)(T - t), \qquad M_1 = \psi(z_0) - \epsilon, \tag{4.11}$$

and $h(\delta)$, $g(\delta)$ are some positive functions at our disposal. Then if $b \leq 0$, we take the following two cases:

(a) $\alpha > m^{-1}$ if $0 < m \leq 1$ and,

(b) $m^{-1} < \alpha \leq (m-1)^{-1}$ if $m > 1$.

If $b > 0$, we take four different cases:

(I) $m^{-1} < \alpha \leq \min((m-1)^{-1}; (1-\beta)^{-1})$ if $m > 1$, $0 < \beta < 1$;

(II) $m^{-1} < \alpha \leq (m-1)^{-1}$ if $m > 1$, $\beta \geq 1$;

(III) $\alpha > m^{-1}$ if $0 < m \leq 1$, $\beta \geq m$;

(IV) $m^{-1} < \alpha \leq (m-\beta)^{-1}$ if $0 < m \leq 1$, $0 < \beta < m$.

Then we set

$$\begin{gathered} V_n = \Omega_n \cap \{z : x_1 < \xi_n(t),\ T - \delta < t < T\}, \\ \xi_n = h(\delta) + g(\delta)(t - T) + \phi_n(\overline{x}^0, T) - \eta_n, \quad \eta_n = h(\delta)(2M_1 n)^{-1/\alpha}. \end{gathered} \tag{4.12}$$

In the next lemma we clear the structure of V_n. We denote the parabolic boundary of V_n as $\mathcal{P}V_n$.

Lemma 4.1. *Let $h(\delta) \leq C\omega(\delta)$, $C > 0$, and*

$$\frac{\omega(\delta)}{\delta g(\delta)} = o(1), \quad \text{as } \delta \downarrow 0. \tag{4.13}$$

Then for all sufficiently small positive δ at the points $z = (x_1, \overline{x}, t) \in \mathcal{P}V_n$ either $z \in \partial\Omega_n$ or $x_1 = \xi_n(t)$ holds.

Proof. By using (4.2), we have

$$\xi_n(t) - \phi_n(\overline{x},t) \le (C+1)\omega(\delta) - \delta g(\delta) \le 0, \quad \text{for } t = T-\delta,\ \overline{x} \in \overline{R_n(\delta)} \cap \{t = T-\delta\} \tag{4.14}$$

if $h(\delta)$, δ and $\omega(\delta)$ are chosen as in Lemma 4.1. This together with the structural assumption on Ω_n immediately implies the assertion of lemma. Lemma is proved. □

Furthermore, we will take $h(\delta) = C\omega(\delta)$, assuming that $\omega(\delta)$ satisfies (4.13). Note that the constant C is still at our disposal.

Our purpose is to estimate u_n in V_n via the barrier function w_n. In the next lemma, we estimate u_n via w_n on $\mathcal{P}V_n$. For that the special structure of V_n due to Lemma 4.1 plays an important role. Namely, our barrier function takes the value $(2n)^{-1}$, which is less than a minimal value of u_n, on the part of the parabolic boundary of V_n which lies in Ω_n. Hence it is enough to compare u_n and w_n on the part of the boundary of Ω_n, which may be easily done in view of boundary condition for u_n. In particular, Lemma 4.2 makes the choice of the constant C precise.

Lemma 4.2. *Let (4.13) be satisfied and*

$$C = \left[\left(\frac{M_2}{M_1}\right)^{1/\alpha} - 1\right]^{-1}, \quad \textit{where } M_2 = \psi(z_0) - \frac{\epsilon}{2}. \tag{4.15}$$

If $\delta > 0$ is chosen small enough, then

$$u_n > w_n \quad \textit{on } \mathcal{P}V_n \textit{ for } n \ge n_1, \tag{4.16}$$

where $n_1 = n_1(\epsilon)$ is some number depending on ϵ.

Proof. If $\delta > 0$ is chosen as in Lemma 4.1, then at the points of $\mathcal{P}V_n$ with $x_1 = \xi_n(t)$ we have

$$w_n = (2n)^{-1} \le u_n. \tag{4.17}$$

From (4.1) it follows that if δ is chosen small enough, then at the points $z = (x_1,\overline{x},t) \in \mathcal{P}V_n \cap \partial\Omega_n$ we have $x_1 \ge \phi_n(\overline{x},t)$. Hence, from (4.2) it follows that

$$\begin{aligned} w_n &= f\left(h(\delta) + \phi_n(\overline{x}^0,T) - x_1 - g(\delta)(T-t)\right) \le f\left(h(\delta) + \phi_n(\overline{x}^0,T) - \phi_n(\overline{x},t)\right) \\ &\le f\left((C^{-1}+1)h(\delta)\right) = M_2 \quad \text{for } z \in \mathcal{P}V_n \cap \partial\Omega_n. \end{aligned} \tag{4.18}$$

We can also easily estimate u_n on $\mathcal{P}V_n \cap \partial\Omega_n$. First, we choose $n_1 = n_1(\epsilon)$ so large that for $n \ge n_1$,

$$\min_{\partial\Omega_n \cap \{T-\delta_0 \le t \le T\}} \Psi > \min_{\partial\Omega \cap \{T-\delta_0 \le t \le T\}} \Psi - \frac{\epsilon}{8}. \tag{4.19}$$

Then we choose $\delta > 0$ small enough in order that

$$\min_{\partial\Omega \cap \{T-\delta \le t \le T\}} \Psi > \psi(z_0) - \frac{\epsilon}{8}. \tag{4.20}$$

If δ and n are chosen like this, then we have

$$u_n(z) > \psi(z_0) - \frac{\epsilon}{4}, \quad \text{for } z \in \mathcal{P}V_n \cap \partial\Omega_n. \tag{4.21}$$

Thus from (4.17)–(4.21), (4.16) follows. Lemma is proved. □

LEMMA 4.3. *Let the conditions of Lemma 4.2 be satisfied and assume that*

$$\omega(\delta)g(\delta) = o(1), \quad \text{as } \delta \downarrow 0. \tag{4.22}$$

If $\delta > 0$ is chosen small enough, then

$$Lw_n \equiv w_{n_t} - \Delta w_n^m + bw_n^\beta - b\theta_b n^{-\beta} < 0 \quad \text{in } V_n. \tag{4.23}$$

Proof. We have

$$\begin{aligned} Lw_n = {} & g(\delta)C^{-1}\omega^{-1}(\delta)\alpha M_1^{1/\alpha} f^{(\alpha-1)/\alpha} \\ & - C^{-2}\omega^{-2}(\delta)\alpha m(\alpha m - 1)M_1^{2/\alpha} f^{(\alpha m-2)/\alpha} + bf^\beta - b\theta_b n^{-\beta}. \end{aligned} \tag{4.24}$$

In view of our construction of V_n, we have $w_n \le M_2$ in $\overline{V_n}$ (see (4.18)). Hence, if either $b \le 0$ or $b > 0$, $m > 1$ and m, β belong to one of the regions I, II, then from (4.24) it follows that

$$\begin{aligned} Lw_n \le C^{-2}\omega^{-2}(\delta)\alpha M_1^{1/\alpha} f^{(\alpha-1)/\alpha} \{ & Cg(\delta)\omega(\delta) - m(\alpha m - 1)M_1^{1/\alpha} M_2^{m-1-1/\alpha} \\ & + b\theta_b M_2^{\beta-1+1/\alpha} \alpha^{-1} M_1^{-1/\alpha} C^2\omega^2(\delta) \}. \end{aligned} \tag{4.25}$$

Hence, if δ is chosen small enough, from (4.25) and (4.22), (4.23) follows. If $b > 0$, $0 < m \le 1$ and m, β belong to one of the regions III, IV, then from (4.24) we similarly derive

$$\begin{aligned} Lw_n \le C^{-2}\omega^{-2}(\delta)\alpha M_1^{1/\alpha} f^{(\alpha m-2)/\alpha} \{ & Cg(\delta)\omega(\delta)M_2^{1-m+1/\alpha} - m(\alpha m - 1)M_1^{1/\alpha} \\ & + bC^2\omega^2(\delta)\alpha^{-1} M_1^{-1/\alpha} M_2^{\beta-m+2/\alpha} \}. \end{aligned} \tag{4.26}$$

If δ is chosen small enough, from (4.26) and (4.22), (4.23) follows. Lemma is proved. □

If the conditions of Lemmas 4.1–4.3 are satisfied, then by the standard maximum principle, from (4.16) and (4.23) we easily derive that

$$u_n \ge w_n \quad \text{in } \overline{V}_n, \text{ for } n \ge n_1. \tag{4.27}$$

In the limit as $n' \to +\infty$, we have

$$u \ge w \quad \text{in } \overline{V}, \tag{4.28}$$

where

$$w = f\left(h(\delta) + \phi(\overline{x}^0, T) - x_1 - g(\delta)(T - t)\right),$$
$$V = \Omega \cap \{z : x_1 < h(\delta) + g(\delta)(t - T) + \phi(\overline{x}^0, T),\ T - \delta < t < T\}. \tag{4.29}$$

We have

$$\lim_{z \to z_0, z \in \overline{V}} w = \lim_{z \to z_0, z \in \overline{\Omega}} w = \psi(z_0) - \varepsilon. \tag{4.30}$$

Obviously, from (4.28), (4.8) follows. Hence if $\omega(\delta)$ satisfies (4.13) and (4.22) (for some positive function $g(\delta)$), then for arbitrary $\epsilon > 0$ (4.8) is valid. Next we prove that (4.9) is true under the same conditions.

Let us prove (4.9) for an arbitrary $\varepsilon > 0$ such that $\psi(z_0) + \varepsilon < M$. For arbitrary $\delta > 0$ consider a function

$$w_n(x,t) = f_1(\xi) \equiv \left[\overline{M}^{1/\alpha} + \xi h^{-1}(\delta)\left(M_4^{1/\alpha} - \overline{M}^{1/\alpha}\right)\right]^{\alpha}, \tag{4.31}$$

where ξ is defined as before, $h(\delta) = C\omega(\delta)$ with $C > 0$ being at our disposal, $M_4 = \psi(z_0) + \varepsilon$, $\overline{M} = \psi^1(T)$ and α is an arbitrary number such that $0 < \alpha < \min(1; m^{-1})$. Similarly, consider the domains V_n by replacing η_n with 0 in the expression of $\xi_n(t)$. Obviously, Lemma 4.1 is true. Next we prove an analog of Lemma 4.2.

Lemma 4.4. *Let (4.13) be satisfied and*

$$C = \left(\overline{M}^{1/\alpha} - M_4^{1/\alpha}\right)\left(M_4^{1/\alpha} - M_5^{1/\alpha}\right)^{-1}, \quad \text{where } M_5 = \psi(z_0) + \frac{\epsilon}{2}. \tag{4.32}$$

If $\delta > 0$ is chosen small enough, then

$$u_n \le w_n \quad \text{on } \mathcal{P}V_n, \text{ for } n \ge n_1, \tag{4.33}$$

where $n_1 = n_1(\epsilon)$ is some number depending on ϵ.

Proof. If $\delta > 0$ is chosen according to Lemma 4.1, then at the points of $\mathcal{P}V_n$ with $x_1 = \xi_n(t)$ we have

$$w_n = \overline{M} > u_n. \tag{4.34}$$

From (4.1) it follows that if μ is chosen large enough, then at the points $z = (x_1, \overline{x}, t) \in \mathcal{P}V_n \cap \partial\Omega_n$ we have $x_1 \ge \phi_n(\overline{x}, t)$. Hence, from (4.2) it follows that

$$w_n = f_1\left(h(\delta) + \phi_n(\overline{x}^0, T) - x_1 - g(\delta)(T - t)\right) \ge f_1\left(h(\delta) + \phi_n(\overline{x}^0, T) - \phi_n(\overline{x}, t)\right)$$
$$\ge f_1\left((C^{-1} + 1)h(\delta)\right) = M_5 \quad \text{for } z \in \mathcal{P}V_n \cap \partial\Omega_n. \tag{4.35}$$

We can also easily estimate u_n on $\mathcal{P}V_n \cap \partial\Omega_n$. First, we choose $n_1 = n_1(\epsilon)$ so large that for $n \ge n_1$,

$$\max_{\partial\Omega_n \cap \{T - \delta_0 \le t \le T\}} \Psi < \max_{\partial\Omega \cap \{T - \delta_0 \le t \le T\}} \Psi + \frac{\epsilon}{8}. \tag{4.36}$$

Then we choose $\delta > 0$ small enough in order that

$$\max_{\partial\Omega\cap\{T-\delta\le t\le T\}} \Psi < \psi(z_0) + \frac{\epsilon}{8}. \tag{4.37}$$

If δ is chosen like this and n_1 additionally satisfies $n_1^{-1} < \epsilon/4$, then for $n \ge n_1$ we have

$$u_n(z) < \psi(z_0) + \frac{\epsilon}{2} \quad \text{for } z \in \mathcal{P}V_n \cap \partial\Omega_n. \tag{4.38}$$

Thus from (4.34)–(4.38), (4.33) follows. Lemma is proved. □

The next lemma is an analog of Lemma 4.3.

Lemma 4.5. *Let (4.22) and the conditions of Lemma 4.4 be satisfied. If $\delta > 0$ is chosen small enough, then*

$$Lw_n > 0 \quad \text{in } V_n. \tag{4.39}$$

Proof. In view of our construction of V_n, we have $w_n \ge M_5$ in $\overline{V}_n$ (see (4.35)). Hence, we have (taking into account that $n^{-1} < \overline{M}$)

$$\begin{aligned} Lw_n &= -g(\delta)C^{-1}\omega^{-1}(\delta)\alpha\big(\overline{M}^{1/\alpha} - M_4^{1/\alpha}\big) f_1^{(\alpha-1)/\alpha} \\ &\quad + m\alpha(1-\alpha m)C^{-2}\omega^{-2}(\delta)\big(\overline{M}^{1/\alpha} - M_4^{1/\alpha}\big)^2 f_1^{(\alpha m-2)/\alpha} + bf^\beta - b\theta_b n^{-\beta} \\ &\ge C^{-2}\omega^{-2}(\delta)\alpha\big(\overline{M}^{1/\alpha} - M_4^{1/\alpha}\big)\big\{-CM_5^{(\alpha-1)/\alpha}g(\delta)\omega(\delta) \\ &\qquad + m(1-\alpha m)\big(\overline{M}^{1/\alpha} - M_4^{1/\alpha}\big)\overline{M}^{(\alpha m-2)/\alpha} \\ &\qquad - |b|\overline{M}^\beta\alpha^{-1}\big(\overline{M}^{1/\alpha} - M_4^{1/\alpha}\big)^{-1}C^2\omega^2(\delta)\big\}. \end{aligned} \tag{4.40}$$

Hence, if δ is chosen small enough, from (4.40) and (4.22), (4.39) follows. The lemma is proved. □

If the conditions of Lemmas 4.1, 4.4, and 4.5 are satisfied, then by the standard maximum principle, from (4.33) and (4.39) it follows that

$$u_n \le w_n \quad \text{in } \overline{V_n}, \text{ for } n \ge n_1. \tag{4.41}$$

In the limit as $n' \to \infty$, we have

$$u \le w \quad \text{in } \overline{V}, \tag{4.42}$$

where

$$w(x,t) = f_1\big(h(\delta) + \phi(\overline{x}^0, T) - x_1 - g(\delta)(T-t)\big) \tag{4.43}$$

and the domain V being defined as in (4.28). We have

$$\lim_{z\to z_0, z\in\overline{V}} w = \lim_{z\to z_0, z\in\overline{\Omega}} w = \psi(z_0) + \varepsilon. \tag{4.44}$$

Obviously, from (4.42), (4.9) follows. Hence if $\omega(\delta)$ satisfies (4.13) and (4.22) (for some positive function $g(\delta)$), then for arbitrary $\epsilon > 0$ both (4.8) and (4.9) are valid. This proves (4.7) for the vertex boundary point $z_0 = (x_1^0, \bar{x}^0, T) \in \partial\Omega$. Let us now consider the conditions (4.13) and (4.22). One can easily show that if $\omega(\delta)$ satisfies both (4.13) and (4.22) then it necessarily satisfies the following condition:

$$\frac{\omega(\delta)}{\delta^{1/2}} = o(1), \quad \text{as } \delta \downarrow 0. \tag{4.45}$$

But since our purpose is to make the function $\omega(\delta)$ as large as possible, it is clear that the optimal choice of $\omega(\delta)$ is given like in the right-hand side of (2.3) and in order to justify (4.13) and (4.22) we are forced to choose $g(\delta) = \delta^{-1/2}$, which reduces both (4.13) and (4.22) to (4.45).

It remains only to prove the continuity of u at the bottom boundary point $z_0 = (x_1^0, \bar{x}^0, 0) \in \partial\Omega$. The proof is similar (and much simpler) to that given for the vertex boundary point. As before, we need to prove (4.8) (if $\psi(z_0) > 0$) and (4.9). To prove (4.8), we set $V_n = \Omega_n \cap \{0 < t < \delta\}$. First of all there is no need to prove analog of Lemma 4.1 and there is no function $\omega(\delta)$ to be controlled in this case. As in Lemma 4.2, it may be proved that if $\delta = \delta(\epsilon) > 0$ is small enough and $n = n(\epsilon)$ is large enough, then $w_n \le u_n$ on $\mathcal{P}V_n$ for $n \ge n(\epsilon)$, where w_n, f are chosen as in (4.10) with $\phi_n(\bar{x}^0, T)$, T, $g(\delta)$, and $h(\delta)$ replaced by x_1^0, 0, 1, and δ, respectively. We then prove (4.23) as in Lemma 4.3. The maximum principle implies $w_n \le u_n$ in $\overline{V}_n$. In the limit $n' \to \infty$ we obtain (4.28), where $V = \Omega \cap \{0 < t < \delta\}$ and w is defined as in (4.28) with $\phi(\bar{x}^0, T)$ and T replaced by x_1^0 and 0, respectively. From (4.28), (4.8) follows. The proof of (4.9) is similar; the only difference is that we choose w_n, f_1 as in (4.31) with $\phi_n(\bar{x}^0, T)$ and T replaced by x_1^0 and 0, respectively. Thus, we have completed the proof of the boundary continuity of the constructed solution. Theorem 2.2 is proved.

Similarly, as in [3], Corollary 2.3 follows from Theorem 2.2. It may be easily shown that if $\beta \ge 1$ and $\inf_{\partial\Omega} \psi > 0$, then constructed solution satisfies $\inf_{\partial\Omega} u > 0$. Hence, Corollary 2.4 is immediate.

5. Proof of Theorem 2.6

In order to make the role of Assumption $\mathcal{M}$ clear for the reader, we keep free the exponent μ from (2.5), just assuming that $\mu \in (0,1)$. The choice of the critical exponent μ will be clear at the end of the proof.

Suppose that g_1 and g_2 are two solutions of DP. We will prove uniqueness by proving that

$$g_1 \equiv g_2 \quad \text{in } \overline{\Omega} \cap \{(x,\tau) : t_j \le \tau \le t_{j+1}\},\ j = 1, \dots, k. \tag{5.1}$$

First, we present the proof of (5.1) for the case $j = 1$. The proof for cases $j = 2, \dots, k$ is similar to the proof for the case $j = 1$. We prove (5.1) with $j = 1$ by proving that for some limit solution $u = \lim u_n$ the following inequalities are valid:

$$\int_{\Omega(t)} (u(x,t) - g_i(x,t))\omega(x)dx \le 0, \quad i = 1, 2, \tag{5.2}$$

for every $t \in (0,t_2)$ and for every $\omega \in C_0^\infty(\Omega(t))$ such that $|\omega| \le 1$. Obviously, from (5.2) it follows that

$$g_1 = u = g_2 \quad \text{in } \overline{\Omega} \cap \{(x,\tau) : t_1 \le \tau < t_2\}, \tag{5.3}$$

which implies (5.1) with $j = 1$ in view of continuity of u, g_1, and g_2 in $\overline{\Omega}$. Since the proof of (5.2) is similar for each i, we will henceforth let $g = g_i$. Let $t \in (0,t_2)$ be fixed and let $\omega \in C_0^\infty(\Omega(t))$ be an arbitrary function such that $|\omega| \le 1$. We divide the proof of (5.2) into two steps.

Step 1 (estimation of the integral difference in (5.2) for the solution to the regularized problem via the boundary gradient bound of the solution to the linearized adjoint problem). To construct the required limit solution, as in the proof of Theorem 2.2, we approximate Ω and ψ with a sequence of smooth domains $\Omega_n \in \mathcal{D}_{0,T}$ and smooth positive functions ψ_n. We make a slight modification to the construction of Ω_n and ψ_n. As before, Ψ be a nonnegative and continuous function in $\mathbb{R}^{N+1}$, which coincides with ψ on $\partial\Omega$. Let ψ_n be a sequence of smooth functions such that

$$\max\left(\Psi; n^{-1}\right) \le \psi_n \le \left(\Psi^m + Cn^{-m}\right)^{1/m}, \quad n = 1,2,\ldots, \tag{5.4}$$

where $C > 1$ is a fixed constant. For arbitrary subset $G \subset \mathbb{R}^{N+1}$ and $\rho > 0$ we define

$$O_\rho(G) = \bigcup_{z\in G} B(z,\rho). \tag{5.5}$$

Since g and Ψ are continuous functions in $\overline{\Omega}$ and $g = \psi$ on $\partial\Omega$, for arbitrary n there exists $\rho_n > 0$ such that

$$\left| g^m(z) - \Psi^m(z) \right| \le n^{-m} \quad \text{for } z \in O_{\rho_n}(\overline{\partial\Omega}) \cap \overline{\Omega}. \tag{5.6}$$

We then assume that Ω_n satisfies the following:

$$\Omega_n \in \mathcal{D}_{0,T}, \qquad S\Omega_n \subseteq \overline{O_{\rho_n}(\partial\Omega)}, \tag{5.7a}$$

and for arbitrary $\epsilon > 0$ there exists $N(\epsilon)$ such that

$$\overline{\Omega_n \cap \{\epsilon < t < T - \epsilon\}} \subset \Omega \quad \text{for } n \ge N(\epsilon). \tag{5.7b}$$

We now formulate assumptions on $S\Omega_n$ near its point z_n, which are direct implications of Assumption $\mathcal{M}$ at the point $z_0 = (x_1^0, \overline{x}^0, t_0) \in \partial\Omega$. Assume that $S\Omega_n$ in some neighborhood of its point $z_n = (x_1^{(n)}, \overline{x}^0, t_0)$ is represented by the function $x_1 = \phi_n(\overline{x},t)$, where $\{\phi_n\}$ is a sequence of sufficiently smooth functions and $\phi_n \to \phi$ as $n \to +\infty$, uniformly in $\overline{Q(\delta_0)}$, where $\delta_0 > 0$ be a sufficiently small fixed number, which does not depend on n. Obviously, we can assume that ϕ_n satisfies Assumption $\mathcal{M}$ (namely, (2.5)) at the point $(\overline{x}^0, t_0)$, uniformly with respect to n and with the same exponent μ. Let $\{\delta_n\}$ be some sequence of positive real numbers such that $\delta_n \to 0$ as $n \to +\infty$. Assume also that the sequence $\{\phi_n\}$ is chosen such that, for n being large enough, the following inequality is satisfied:

$$\phi_n(\bar{x}^0,t_0)-\phi_n(\bar{x},t)\le\delta_n^{\mu-1}\left[t-t_0+\left|\bar{x}-\bar{x}^0\right|^2\right]\quad\text{for }(\bar{x},t)\in\overline{Q(\delta_n)}.\tag{5.8}$$

Obviously, this is possible in view of uniform convergence of ϕ_n to ϕ. For example, if $\phi(\bar{x},t)$ coincides with its lower bound $\tilde{\phi}(\bar{x},t)=\phi(\bar{x}^0,t_0)-[t-t_0+|\bar{x}-\bar{x}^0|^2]^\mu$ for $(\bar{x},t)\in\overline{Q(\delta_0)}$ (namely, (2.5) is satisfied with $=$ instead of $\le$), then for all large n such that $\delta_n<\delta_0$ we first choose $\tilde{\phi}_n$ as follows:

$$\tilde{\phi}_n(\bar{x},t)=\begin{cases}\phi(\bar{x}^0,t_0)-\delta_n^{\mu-1}\left[t-t_0+\left|\bar{x}-\bar{x}^0\right|^2\right] & \text{for }(\bar{x},t)\in\overline{Q(\delta_n)},\\ \tilde{\phi}(\bar{x},t) & \text{for }(\bar{x},t)\in\overline{Q(\delta_0)}\setminus\overline{Q(\delta_n)}.\end{cases}\tag{5.9}$$

Obviously, $\tilde{\phi}_n$ satisfies (5.8) and converges to ϕ uniformly in $Q(\delta_0)$. Then we easily construct ϕ_n by smoothing $\tilde{\phi}_n$ at the boundary points of $Q(\delta_n)$ satisfying $t-t_0+|\bar{x}-\bar{x}^0|^2=\delta_n$. In general, we can do similar construction by taking instead of $\tilde{\phi}_n(\bar{x},t)$ the function $\tilde{\tilde{\phi}}_n(\bar{x},t)=\max(\tilde{\phi}_n(\bar{x},t);\phi(\bar{x},t))$, which satisfies (5.8) and converges to $\phi(\bar{x},t)$ as $n\to+\infty$, uniformly in $\overline{Q(\delta_0)}$.

Let u_n be a classical solution to DP in Ω_n for (4.3) with the initial boundary data ψ_n. As before, (4.4) is valid. As in the proof of Theorem 2.2, we then prove that for some subsequence n', $u=\lim_{n'\to\infty}u'_n$ is a solution of DP (1.1), (1.6). Furthermore, without loss of generality we write n instead of n'. Take an arbitrary sequence of real numbers $\{\alpha_l\}$ such that

$$0<\alpha_{l+1}<\alpha_l<t,\quad \alpha_l\downarrow 0\quad \text{as } l\longrightarrow+\infty.\tag{5.10}$$

Let

$$\Omega_n^l=\Omega_n\cap\{(x,\tau):\alpha_l<\tau<t\},\qquad S\Omega_n^l=S\Omega_n\cap\{(x,\tau):\alpha_l<\tau<t\}.\tag{5.11}$$

From (5.7) it follows that for arbitrary fixed l there exists $N=N(l)$ such that $\overline{\Omega}_n^l\subset\Omega$ for $n\ge N(l)$. Furthermore, we will assume that $n\ge N(l)$ provided that l is fixed. Since u_n is a classical solution of (4.3), it satisfies

$$\begin{aligned}&\int_{\Omega_n(t)}u_nf\,dx\\&\quad=\int_{\Omega_n(\alpha_l)}u_nf\,dx+\int_{\Omega_n^l}\left(u_nf_\tau+u_n^m\Delta f-bu_n^\beta f+b\theta_bn^{-\beta}f\right)dx\,d\tau-\int_{S\Omega_n^l}u_n^m\frac{\partial f}{\partial\nu}dx\,d\tau\end{aligned}\tag{5.12}$$

for arbitrary $f\in C_{x,t}^{2,1}(\overline{\Omega}_n^l)$ that equals to zero on $S\Omega_n^l$, and $\nu=\nu(x,\tau)$ is the outward-directed normal vector to $\Omega_n(\tau)$ at $(x,\tau)\in S\Omega_n^l$. Since g is the weak solution of the DP

(1.1), (1.6), we also have

$$\int_{\Omega_n(t)} gf\,dx = \int_{\Omega_n(\alpha_l)} gf\,dx + \int_{\Omega_n^l} (gf_\tau + g^m\Delta f - bg^\beta f)\,dx\,d\tau - \int_{S\Omega_n^l} g^m \frac{\partial f}{\partial \nu}\,dx\,d\tau. \tag{5.13}$$

Subtracting (5.13) from (5.12), we derive

$$\begin{aligned}\int_{\Omega_n(t)} (u_n - g) f\,dx &= \int_{\Omega_n(\alpha_l)} (u_n - g) f\,dx - \int_{S\Omega_n^l} (u_n^m - g^m)\frac{\partial f}{\partial \nu}\,dx\,d\tau \\ &\quad + \int_{\Omega_n^l} \{(u_n^{1/\gamma} - g^{1/\gamma})[C_n f_\tau + A_n\Delta f - B_n f] + b\theta_b n^{-\beta} f\}\,dx\,d\tau,\end{aligned} \tag{5.14}$$

where $\gamma = 1$ if $m > 1$, and $\gamma > 1/m$ if $0 < m < 1$; $C_n = 1$ if $m > 1$ (accordingly $\gamma = 1$) and $C_n = C_n'$ if $0 < m < 1$, and

$$\begin{gathered} C_n' = \gamma\int_0^1 (\theta u_n^{1/\gamma} + (1-\theta)g^{1/\gamma})^{\gamma-1}\,d\theta, \qquad B_n = b\beta\gamma\int_0^1 (\theta u_n^{1/\gamma} + (1-\theta)g^{1/\gamma})^{\beta\gamma-1}\,d\theta, \\ A_n = m\gamma\int_0^1 (\theta u_n^{1/\gamma} + (1-\theta)g^{1/\gamma})^{m\gamma-1}\,d\theta. \end{gathered} \tag{5.15}$$

The functions A_n, B_n, and C_n are Hölder continuous in $\overline{\Omega}_n^l$. From (4.4) and Definition 1.1 it follows that

$$\begin{aligned} n^{(1-m\gamma)/\gamma} \le A_n \le \overline{A}, &\qquad n^{(1-\gamma)/\gamma} \le C_n' \le \overline{C}, \\ -\overline{B} \le B_n \le bn^{(1-\beta\gamma)/\gamma} &\quad \text{for } b < 0,\ (x,\tau) \in \overline{\Omega}_n^l, \end{aligned} \tag{5.16}$$

where $\overline{A}$, $\overline{B}$, $\overline{C}$ are some positive constants which do not depend on n. To choose the test function $f = f(x,\tau)$ in (5.14), consider the following problem:

$$C_n f_\tau + A_n\Delta f - B_n f = 0 \quad \text{in } \Omega_n^l \cup B\Omega_n^l, \tag{5.17a}$$

$$f = 0 \quad \text{on } S\Omega_n^l, \tag{5.17b}$$

$$f = \omega(x) \quad \text{on } \Omega_n(t). \tag{5.17c}$$

This is the linear nondegenerate backward-parabolic problem. From the classical parabolic theory (see [17–19]) it follows that there exists a unique classical solution $f_n \in C_{x,\tau}^{2+\upsilon,1+\upsilon/2}(\overline{\Omega}_n^l)$ with some $\upsilon > 0$. From the maximum principle it follows that

$$|f_n| \le \exp(\sigma_b\overline{B}(t-\tau)) \quad \text{in } \overline{\Omega}_n^l, \tag{5.18}$$

where $\sigma_b = (1 \text{ if } b < 0;\ 0 \text{ if } b \geq 0)$. Taking $f = f_n(x,\tau)$ in (5.14), we have

$$\begin{aligned}
&\int_{\Omega_n(t)} (u_n - g)\omega(x)dx \\
&\quad = \int_{\Omega_n(\alpha_l)} (u_n - g) f\,dx - \int_{S\Omega_n^l} (u_n^m - g^m)\frac{\partial f}{\partial \nu}dx\,d\tau + b\theta_b n^{-\beta}\int_{\Omega_n^l} f\,dx\,d\tau \\
&\quad \equiv \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3.
\end{aligned} \tag{5.19}$$

By using (5.4)–(5.7), we have

$$\begin{aligned}
|\mathcal{I}_2| &\leq \sup_{z\in S\Omega_n^l} |\nabla f(z)| \int_{S\Omega_n^l} (|\psi_n^m - \Psi^m| + |\Psi^m - g^m|)dx\,d\tau \\
&\leq (C+1)n^{-m} \sup_{z\in S\Omega_n^l} |\nabla f(z)|.
\end{aligned} \tag{5.20}$$

Applying (5.18), we have

$$|\mathcal{I}_1| \leq \exp(\sigma_b \overline{B}T) \int_{\Omega_n(\alpha_l)} |u_n - g|\,dx. \tag{5.21}$$

To estimate the right-hand side, introduce a function

$$u_n^l(x) = \begin{cases} u_n(x,\alpha_l), & x \in \overline{\Omega_n(\alpha_l)}, \\ \psi_n(x,\alpha_l), & x \in \overline{\Omega(\alpha_l)} \backslash \overline{\Omega_n(\alpha_l)}. \end{cases} \tag{5.22}$$

Obviously, $u_n^l(x)$, $x \in \overline{\Omega(\alpha_l)}$ is bounded uniformly with respect to n, l. From (5.21), we have

$$|\mathcal{I}_1| \leq \exp(\sigma_b \overline{B}T) \int_{\Omega(\alpha_l)} |u_n^l - g|\,dx. \tag{5.23}$$

Since

$$\lim_{n\to+\infty} u_n^l(x) = u(x,\alpha_l) \quad \text{for } x \in \overline{\Omega(\alpha_l)}, \tag{5.24}$$

from Lebesgue's theorem it follows that

$$\lim_{n\to+\infty} \int_{\Omega(\alpha_l)} |u_n^l - g|\,dx = \int_{\Omega(\alpha_l)} |u(x,\alpha_l) - g(x,\alpha_l)|\,dx. \tag{5.25}$$

From (5.18) it also follows that

$$\lim_{n\to+\infty} \mathcal{I}_3 = 0. \tag{5.26}$$

Assume that the following condition is satisfied:

$$\sup_{z\in S\Omega_n^l} |\nabla f(z)| = o(n^m). \tag{5.27}$$

From (5.20) and (5.27) it follows that

$$\lim_{n\to+\infty} \mathcal{I}_2 = 0. \tag{5.28}$$

Hence, by using (5.20)–(5.28) in (5.19) and passing to the limit $n \to +\infty$, we have

$$\int_{\Omega(t)} (u-g)\omega(x)dx \le \exp\left(\sigma_b \overline{B}T\right) \int_{\Omega(\alpha_l)} |u-g|dx. \tag{5.29}$$

Passing to the limit $l \to \infty$, from (5.29), (5.2) follows. As it is explained earlier, from (5.2), (5.1) with $j = 1$ follows. Similarly, we can prove (5.1) (step by step) for each $j = 2,\dots,k$. The only difference consists in the handling of the right-hand side of (5.29), where now $\{\alpha_l\}$ is a sequence of real numbers satisfying $\alpha_l \downarrow t_j$ as $l \to +\infty$. We need to introduce a function

$$U_l(x) = \begin{cases} u(x,\alpha_l) - g(x,\alpha_l), & x \in \overline{\Omega(\alpha_l)}, \\ 0, & x \notin \overline{\Omega(\alpha_l)}. \end{cases} \tag{5.30}$$

Obviously, U_l is uniformly bounded with respect to l. Hence, from (5.29) we derive that

$$\int_{\Omega(t)} (u-g)\omega(x)dx \le \exp\left(\sigma_b \overline{B}T\right) \int_{\Omega(t_j)} |\,U_l(x)\,|\,dx + C_2 \cdot \operatorname{meas}\left(\Omega(\alpha_l)\backslash\Omega(t_j)\right), \tag{5.31}$$

where the constant C_2 does not depend on l. Since $u(x,t_j) \equiv g(x,t_j)$ by the previous step, from Lebesgue's theorem it follows that

$$\lim_{l\to+\infty} \int_{\Omega(t_j)} |\,U_l(x)\,|\,dx = 0. \tag{5.32}$$

Hence, passing to the limit $l \to +\infty$, from (5.31), (5.2) follows.

Thus, Step 1 would accomplish the proof of Theorem 2.6 if the condition (5.27) is satisfied. Our only resource to achieve (5.27) is the choice of the sequence $\{\delta_n\}$ with $\delta_n \downarrow 0$, from (5.8). But first, we need precise estimation of $\sup_{z\in S\Omega_n^l} |\nabla f(z)|$ via δ_n. For that reason we proceed to Step 2.

Step 2 (boundary gradient estimates for the linearized adjoint problem (5.17)). In this step we prove the following result: let Assumption $\mathcal{M}$ be uniformly satisfied on every compact subsegment of $(0,t]$. Then for every fixed l (see (5.10)) there exists a positive constant $C(l)$, which does not depend on n such that

$$\sup_{z\in S\Omega_n^l} |\,\nabla f_n(z)\,| \le C(l)\delta_n^{-\mu}. \tag{5.33}$$

First, we prove the estimation (5.33) pointwise. Consider a point $z_0 = (x_1^0, \overline{x}^0, t_0) \in \partial\Omega$, $0 < t_0 \le t$ and let $z_n = (x_1^{(n)}, \overline{x}^0, t_0) = (x^n, t_0) \in S\Omega_n^l$, $n = 1,2,\dots$, be such a sequence that $z_n \to z_0$ as $n \to \infty$. We formulated within Step 1 implications of Assumption $\mathcal{M}$ for the boundary $S\Omega_n^l$ near z_n (see (5.8)). Obviously, it is enough to consider the case $t_0 < t$, since if $t_0 = t$ then $\nabla f_n(z_n) = \nabla\omega(z_n) = 0$.

Let us now estimate $|\nabla f_n(z_n)|$. Denote $x^n = (x_1^{(n)}, \overline{x}^0) \equiv (\phi_n(\overline{x}^0, t_0), \overline{x}^0)$. Instead of estimating directly $|\nabla f_n(z_n)|$, we are going to estimate

$$[f_n(z_n)] = \sup_{x \in F_n} \frac{|f_n(x,t_0) - f_n(x^n,t_0)|}{|x - x^n|} = \sup_{x \in F_n} \frac{|f_n(x,t_0)|}{|x - x^n|}, \tag{5.34}$$

where F_n is some neighborhood of z_n in $\Omega_n(t_0)$. Since $\nabla f_n \in C(\overline{\Omega}_n^l)$, we have

$$|\nabla f_n(z_n)| \leq [f_n(z_n)]. \tag{5.35}$$

To estimate $[f_n(z_n)]$, we establish a suitable upper estimation for f_n in some neighborhood of the point z_n. To estimate f_n, we use a modified version of the method used within the proof of Theorem 2.2 for the boundary regularity of the solution to the DP (1.1), (1.6).

Consider a function

$$\omega_n(x,\tau) = g(\xi) \equiv C\log\left[e - (e-1)\delta_n^{-\mu}\xi\right], \tag{5.36}$$

where

$$\begin{gathered} C = \{1, \text{ if } b \geq 0;\ \exp(\overline{B}t), \text{ if } b < 0\}, \\ \xi = \phi_n(\overline{x}^0, t_0) + \delta_n^{\mu} - x_1 - 2\delta_n^{\mu-1}\left[\tau - t_0 + |\overline{x} - \overline{x}^0|^2\right]. \end{gathered} \tag{5.37}$$

Then we set

$$\begin{gathered} V_n = \{z = (x,\tau) : \phi_n(\overline{x},\tau) < x_1 < \phi_{1n}(\overline{x},\tau),\ (\overline{x},\tau) \in Q(\delta_n)\}, \\ \phi_{1n}(\overline{x},\tau) = \phi_n(\overline{x}^0, t_0) + \delta_n^{\mu} - 2\delta_n^{\mu-1}\left[\tau - t_0 + |\overline{x} - \overline{x}^0|^2\right]. \end{gathered} \tag{5.38}$$

In the next lemma we clear the structure of ∂V_n.

Lemma 5.1. *The closure of the set*

$$\partial_0 V_n = \partial V_n \cap \{(x,\tau) : \tau > t_0\} \tag{5.39}$$

consists of two boundary surfaces $x_1 = \phi_n(\overline{x},\tau)$ *and* $x_1 = \phi_{1n}(\overline{x},\tau)$.

Proof. From (5.8) for ϕ_n it follows that

$$\begin{aligned} \phi_{1n}(\overline{x},\tau) - \phi_n(\overline{x},\tau) &= \phi_n(\overline{x}_0, t_0) - \phi_n(\overline{x},\tau) - \delta_n^{\mu} \\ &\leq 0 \quad \text{for } |\overline{x} - \overline{x}^0| = (\delta_n + t_0 - \tau)^{1/2},\ t_0 \leq \tau \leq t_0 + \delta_n, \end{aligned} \tag{5.40}$$

and the assertion of lemma immediately follows. Lemma is proved. □

It is natural to call $\overline{\partial_0 V_n}$ the backward-parabolic boundary of V_n. The latter means that $\overline{\partial_0 V_n}$ is a parabolic boundary of the transformed domain V_n after change of the variable τ with $-\tau$. In the next lemma, we estimate f_n via the barrier function ω_n on the backward-parabolic boundary $\overline{\partial_0 V_n}$ of V_n. A special structure of V_n, established in Lemma 5.1, plays a crucial role in the proof of this lemma.

Lemma 5.2. *If n is large enough, then*

$$f_n(x,\tau) \le \omega_n(x,\tau) \quad on\ \overline{\partial_0 V_n}. \tag{5.41}$$

Proof. We have

$$\omega_n|_{x_1=\phi_{1n}(\bar{x},\tau)} = g(0) = C. \tag{5.42}$$

Hence, from (5.18) it follows that (5.41) is valid on the part of $\overline{\partial_0 V_n}$ with $x_1 = \phi_{1n}(\bar{x},t)$. Then we observe that

$$\begin{gathered} \omega_n|_{x_1=\mathcal{H}_n(\bar{x},\tau)} = 0, \\ \omega_n \ge 0 \quad \text{for } x_1 \ge \mathcal{H}_n(\bar{x},\tau), \end{gathered} \tag{5.43}$$

where

$$\mathcal{H}_n(\bar{x},\tau) = \phi_n(\bar{x}^0,t_0) - 2\delta_n^{\mu-1}\left[\tau - t_0 + |\bar{x}-\bar{x}^0|^2\right]. \tag{5.44}$$

Hence, from (5.8) it follows that

$$\mathcal{H}_n(\bar{x},\tau) \le \phi_n(\bar{x},\tau) \quad \text{for } (\bar{x},\tau) \in \overline{Q(\delta_n)}. \tag{5.45}$$

From (5.43), (5.45), it follows that

$$\omega_n \ge 0 \quad \text{for } (x,\tau) \in \overline{\partial V_n} \cap \{(x,\tau) : x_1 = \phi_n(\bar{x},\tau)\}, \tag{5.46}$$

and hence (5.41) is also valid on the part of $\overline{\partial_0 V_n}$ with $x_1 = \phi_n(\bar{x},\tau)$. Lemma is proved. □

Lemma 5.3. *If for large n,*

$$\delta_n^{2\mu-1} = o\left(n^{(1-m\gamma)/\gamma}\right), \tag{5.47}$$

then

$$L\omega_n \equiv -C_n(x,\tau)\omega_{n_\tau} - A_n(x,\tau)\Delta\omega_n + B_n(x,\tau)\omega_n > 0 \quad \text{for } (x,\tau) \in V_n. \tag{5.48}$$

Proof. First, we easily derive that

$$0 \le \xi \le \delta_n^{\mu} \quad \text{for } (x,\tau) \in \overline{V}_n. \tag{5.49}$$

The right-hand side of (5.49) follows from (5.43)–(5.46), while the left-hand side is a consequence of the inequality $x_1 \le \phi_{1n}(\bar{x},\tau)$. Let us transform $L\omega_n$,

$$L\omega_n = \left(2C_n + 4A_n(N-1)\right)\delta_n^{\mu-1}g'(\xi) - A_n\left(1 + 16\delta_n^{2\mu-2}|\bar{x}-\bar{x}^0|^2\right)g''(\xi) + B_n g. \tag{5.50}$$

Obviously, we have

$$1-e \le \delta_n^{\mu}C^{-1}g'(\xi) \le \frac{1-e}{e}, \quad -(e-1)^2 \le \delta_n^{2\mu}C^{-1}g'' \le -\frac{(e-1)^2}{e^2} \quad \text{for } 0 \le \xi \le \delta_n^{\mu}. \tag{5.51}$$

Thus, from (5.50), (5.16), and (5.51) it follows that

$$L\omega_n \geq -(2\widetilde{C}+4\overline{A}(N-1))C(e-1)\delta_n^{-1}+n^{(1-m\gamma)/\gamma}Ce^{-2}(e-1)^2\delta_n^{-2\mu}-\sigma_b\overline{B}C \quad \text{in } V_n, \tag{5.52}$$

where $\widetilde{C}=1$ if $m>1$, and $\widetilde{C}=\overline{C}$ if $0<m<1$. Hence, from (5.47), (5.48) follows. Lemma is proved. □

By the standard maximum principle from Lemma 5.1, (5.41), and (5.48) it follows that

$$f_n \leq \omega_n \quad \text{in } \overline{V}_n. \tag{5.53}$$

Since (5.17a) is linear, we also derive that $f_n \geq -\omega_n$ in $\overline{V}_n$ and hence,

$$|f_n| \leq \omega_n \quad \text{in } \overline{V}_n. \tag{5.54}$$

Now by using (5.54) we can estimate $[f_n(z_n)]$ from (5.34) letting $F_n=\overline{V}_n\cap\{(x,\tau):\tau=t_0\}$ and keeping in mind that $f_n(x^n,t_0)=\omega_n(x^n,t_0)=0$,

$$[f_n(z_n)] \leq \sup_{x\in F_n}\frac{\omega_n(x,t_0)}{|x-x^n|}=\sup_{x\in F_n}\frac{\omega_n(x,t_0)-\omega_n(x^n,t_0)}{|x-x^n|}\leq \sup_{x\in F_n}|\nabla\omega_n(x,t_0)|. \tag{5.55}$$

We have

$$|\omega_{n_{x_1}}| \leq C(e-1)\delta_n^{-\mu}, \qquad |\omega_{n_{x_i}}| \leq 4C(e-1)\delta_n^{-1/2}, \quad i=2,\dots,N \text{ in } \overline{F}_n. \tag{5.56}$$

Since the sequence δ_n must converge to zero, the condition $\mu>1/2$ is necessary for (5.47). Accordingly, from (5.35), (5.55), it follows that

$$|\nabla f_n(z_n)| = O(\delta_n^{-\mu}) \quad \text{as } n\longrightarrow+\infty. \tag{5.57}$$

The estimation (5.33) follows from (5.57) by using Definition 2.5 from Section 2. Indeed, for each fixed l (or $\alpha_l\in(0,t)$ from (5.10)) Assumption $\mathcal{M}$ is satisfied uniformly in $[\delta_l,t]$. The related numbers μ and δ_0 (see Definition 2.5) may depend on l and t, but do not depend on the points $z\in S\Omega\bigcap\{(x,\tau):\delta_l\leq\tau\leq t\}$. It may be easily seen that under this condition neither the largeness of n which is required in the proof of Lemmas 5.1–5.3, nor the right-hand sides of (5.55), (5.57) vary for different points $z_n\in S\Omega_n^l$. Hence, boundary gradient estimate (5.33) is true, provided that the sequence $\{\delta_n\}$ satisfies (5.47). Step 2 is completed. From another side, in order to accomplish Step 1 and accordingly the whole proof we need just to use (5.33) in (5.27), which gives the following second relation between δ_n and n for large n:

$$\delta_n^{-\mu} = o(n^m). \tag{5.58}$$

We are ready now to complete the proof and at the same time to explain the choice of the critical exponent μ in the inequality (2.5) of Assumption $\mathcal{M}$. Since our purpose is to make the exponent $\mu>0$ in Assumption $\mathcal{M}$ as small as possible, we reduced the whole problem about the uniqueness of the solution to DP (under the minimal restriction on the lateral boundary) to the following one: *find* $\mu_* = \inf_{\mu_0 \in S} \mu_0$, *where* S *is the set of real numbers* $\mu_0 \in (0,1)$ *with the property that for arbitrary* $\mu > \mu_0$ *there exists a sequence* δ_n *with* $\delta_n \downarrow 0$ *as* $n \to \infty$ *and satisfying (5.47), (5.58).* Obviously, μ_* would be a critical exponent in (2.5).

Simple calculation shows that if $m > 1$ (accordingly $\gamma = 1$), then $\mu_* = m/(m+1)$ and for each $\mu > m/(m+1)$ we can choose

$$\delta_n = n^{-(m-\epsilon)/\mu} \tag{5.59}$$

with

$$0 < \epsilon < \frac{\mu(1+m) - m}{2\mu - 1}. \tag{5.60}$$

While if $0 < m < 1$, then $\mu_* = 1/2$ and for each $\mu > 1/2$ we can again choose δ_n as in (5.59), with ϵ and γ satisfying

$$0 < \epsilon < \frac{\mu(1+m\gamma) - \gamma m}{\gamma(2\mu - 1)}, \qquad \frac{1}{m} < \gamma < \frac{\mu}{(1-\mu)m}. \tag{5.61}$$

Theorem is proved.

6. Proof of Theorem 2.7

Let us prove the theorem for supersolutions. The proof is similar to the proof of uniqueness. We prove (step by step) that

$$u \le g \quad \text{in } \overline{\Omega} \cap \{(x,\tau) : t_j \le \tau \le t_{j+1}\},\ j = 1,\dots,k. \tag{6.1}$$

First, we present the proof of (6.1) for the case $j = 1$. The proof for cases $j = 2,\dots,k$ is similar to the proof for the case $j = 1$. Obviously, to prove (6.1) with $j = 1$ it is enough to prove that for each fixed $t \in (0,t_2)$ the following inequality is valid:

$$u \le g \quad \text{in } \overline{\Omega(t)}. \tag{6.2}$$

Our goal will be achieved if we prove the inequality

$$\int_{\Omega(t)} (u(x,t) - g(x,t))\,\omega(x)\,dx \le 0 \tag{6.3}$$

for every $\omega \in C_0^\infty(\Omega(t))$ with $0 \le \omega \le 1$. Let us prove (6.3). First, we construct a sequence $\{u_n\}$ as in the proof of Theorem 2.6. A slight modification is made concerning the choice of the number $\rho_n > 0$ via (5.6). Consider the function $G = \max(\Psi; g)$. Since $\Psi = \psi \le g$ on $\partial\Omega$, it may easily be observed that $G = g$ on $\partial\Omega$. Obviously, G is a continuous function satisfying

$$\Psi \le G \quad \text{in } \overline{\Omega}. \tag{6.4}$$

Since g and G are continuous functions in $\overline{\Omega}$ and $g = G$ on $\partial\Omega$, for arbitrary n there exists $\rho_n > 0$ such that

$$\left|g^m(x,\tau) - G^m(x,\tau)\right| \le n^{-m} \quad \text{for } (x,\tau) \in O_{\rho_n}(\overline{\partial\Omega}) \cap \overline{\Omega}. \tag{6.5}$$

We then assume that Ω_n satisfies (5.7) with ρ_n defined via (6.5). As before, there exists a subsequence n' such that $u_{n'}$ converges to the solution of DP (without loss of generality we write n instead of n'). Since u is a unique solution of DP, we have $u = \lim u_n$. We then take a sequence of real numbers $\{\alpha_l\}$ as in (5.10). Since g is a supersolution of DP, it satisfies (5.13) with $\ge$ instead of $=$. Subtracting this inequality from (5.12), we derive instead of (5.14)

$$\begin{aligned}\int_{\Omega_n(t)} (u_n - g) f\,dx \le \int_{\Omega_n(\alpha_l)} (u_n - g) f\,dx - \int_{S\Omega_n^l} (u_n^m - g^m)\frac{\partial f}{\partial \nu}dx\,d\tau \\ + \int_{\Omega_n^l} \left[(u_n^{1/\gamma} - g^{1/\gamma})(C_n f_\tau + A_n \Delta f - B_n f) + b\theta_b n^{-\beta} f\right]dx\,d\tau.\end{aligned} \tag{6.6}$$

Instead of f in (6.6) we take the classical solution f_n of the problem (5.17). Since $0 \le \omega \le 1$, from the maximum principle it follows that $0 \le f_n \le \exp(\sigma_b \overline{B}(t-\tau))$ in $\overline{\Omega}_n^l$, and hence

$$\frac{\partial f_n}{\partial \nu} \le 0 \quad \text{in } S\Omega_n^l. \tag{6.7}$$

From (5.17), (6.4), (6.6), and (6.7), we have

$$\begin{aligned}\int_{\Omega_n(t)} (u_n - g)\omega(x)dx \le \int_{\Omega_n(\alpha_l)} (u_n - g) f\,dx - \int_{S\Omega_n^l} (\psi_n^m - \Psi^m + G^m - g^m)\frac{\partial f}{\partial \nu}dx\,d\tau \\ + b\theta_b n^{-\beta}\int_{\Omega_n^l} f\,dx\,d\tau \equiv I_1 + I_2 + I_3.\end{aligned} \tag{6.8}$$

As in the proof of Theorem 2.6, we estimate I_1 as follows:

$$\begin{gathered}I_1 \le \exp\left(\sigma_b \overline{B}T\right)\int_{\Omega_n(\alpha_l)} (u_n - g)_+ dx \le \exp\left(\sigma_b \overline{B}T\right)\int_{\Omega(\alpha_l)} (u_n^l - g)_+ dx, \\ (x)_+ = \max(x;0), \qquad \lim_{n\to+\infty}\int_{\Omega(\alpha_l)} (u_n^l - g)_+ dx = \int_{\Omega(\alpha_l)} (u-g)_+ dx.\end{gathered} \tag{6.9}$$

Obviously, the right-hand side converges to zero when $l \to \infty$. Similarly, by using (5.18) we get (5.26) for I_3. Finally, the estimation of I_2 coincides with the estimation of $\mathcal{I}_2$ from the proof of Theorem 2.6. First, by using (5.4), (5.7), and (6.5) we derive (5.20) for I_2. This implies (5.28), provided that (5.27) is true. To prove (5.27), we repeat Step 2 of the proof of Theorem 2.6. The only difference is that in the expressions of $A_n(x,\tau)$, $B_n(x,\tau)$, and $C_n(x,\tau)$ from the linearized adjoint problem (5.17) the function g means the supersolution of DP instead of solution. Hence, by using (5.26) and (5.28) (for I_2 and I_3) and passing to the limit $n \to \infty$ from (6.8) we derive

$$\int_{\Omega(t)} (u-g)\omega(x)dx \le \exp\left(\sigma_b \overline{B} T\right) \int_{\Omega(\alpha_l)} (u-g)_+ dx. \tag{6.10}$$

Passing to the limit $l \to \infty$, from (6.10), (6.3) follows. As it is explained earlier, from (6.3), (6.1) with $j = 1$ follows. Similarly, we can prove (6.1) (step by step) for each $j = 2, \ldots, k$. The only difference consists in the handling of the right-hand side of (6.10), where now $\{\alpha_l\}$ is a sequence of real numbers satisfying $\alpha_l \downarrow t_j$ as $l \to +\infty$. By introducing a function $U_l(x)$ as in (5.30), we derive instead of (6.10)

$$\int_{\Omega(t)} (u-g)\omega(x)dx \le \exp\left(\sigma_b \overline{B} T\right) \int_{\Omega(\alpha_l)} \left(U_l(x)\right)_+ dx. \tag{6.11}$$

Since $U_l(x)$, $x \in \Omega(\alpha_l)$ is uniformly bounded with respect to l, we have

$$\int_{\Omega(\alpha_l)} \left(U_l(x)\right)_+ dx \le \int_{\Omega(t_j)} \left(U_l(x)\right)_+ dx + C_3 \operatorname{meas}\left(\Omega(\alpha_l) \setminus \Omega(t_j)\right), \tag{6.12}$$

where the constant C_3 does not depend on l. Since $u(x,t_j) \le g(x,t_j)$ by the previous step, from Lebesgue's theorem it follows that

$$\lim_{l \to +\infty} \int_{\Omega(t_j)} \left(U_l(x)\right)_+ dx = \int_{\Omega(t_j)} \left(u(x,t_j) - g(x,t_j)\right)_+ dx = 0, \tag{6.13}$$

and hence

$$\lim_{l \to +\infty} \int_{\Omega(\alpha_l)} \left(U_l(x)\right)_+ dx = 0. \tag{6.14}$$

Hence, passing to the limit $l \to +\infty$, from (6.11), (6.3) follows. As before, from (6.3), (6.1) with $j > 1$ follows. The proof for supersolutions is completed. The proof for subsolutions is similar. Theorem 2.7 is proved.

By using Theorems 2.2, 2.6, and 2.7 we can prove Corollary 2.8 as in [3].

References

[1] A. S. Kalashnikov, "Some problems of the qualitative theory of nonlinear degenerate second-order parabolic equations," *Russian Mathematical Surveys*, vol. 42, no. 2, pp. 169–222, 1987.

[2] U. G. Abdulla, "On the Dirichlet problem for the nonlinear diffusion equation in non-smooth domains," *Journal of Mathematical Analysis and Applications*, vol. 260, no. 2, pp. 384–403, 2001.

[3] U. G. Abdulla, "Well-posedness of the Dirichlet problem for the non-linear diffusion equation in non-smooth domains," *Transactions of the American Mathematical Society*, vol. 357, no. 1, pp. 247–265, 2005.

[4] E. DiBenedetto, "Continuity of weak solutions to certain singular parabolic equations," *Annali di Matematica Pura ed Applicata. Serie Quarta*, vol. 130, no. 1, pp. 131–176, 1982.

[5] E. DiBenedetto, "Continuity of weak solutions to a general porous medium equation," *Indiana University Mathematics Journal*, vol. 32, no. 1, pp. 83–118, 1983.

[6] W. P. Ziemer, "Interior and boundary continuity of weak solutions of degenerate parabolic equations," *Transactions of the American Mathematical Society*, vol. 271, no. 2, pp. 733–748, 1982.

[7] U. G. Abdulla, "On the Dirichlet problem for reaction-diffusion equations in non-smooth domains," *Nonlinear Analysis*, vol. 47, no. 2, pp. 765–776, 2001.
[8] U. G. Abdulla, "First boundary value problem for the diffusion equation. I. Iterated logarithm test for the boundary regularity and solvability," *SIAM Journal on Mathematical Analysis*, vol. 34, no. 6, pp. 1422–1434, 2003.
[9] U. G. Abdulla, "Multidimensional Kolmogorov-Petrovsky test for the boundary regularity and irregularity of solutions to the heat equation," *Boundary Value Problems*, vol. 2005, no. 2, pp. 181–199, 2005.
[10] I. G. Petrovsky, "Zur ersten Randwertaufgabe der Wärmeleitungsgleichung," *Compositio Mathematica*, vol. 1, pp. 383–419, 1935.
[11] U. G. Abdulla, "Reaction-diffusion in a closed domain formed by irregular curves," *Journal of Mathematical Analysis and Applications*, vol. 246, no. 2, pp. 480–492, 2000.
[12] U. G. Abdulla, "Reaction-diffusion in irregular domains," *Journal of Differential Equations*, vol. 164, no. 2, pp. 321–354, 2000.
[13] U. G. Abdulla, "Evolution of interfaces and explicit asymptotics at infinity for the fast diffusion equation with absorption," *Nonlinear Analysis*, vol. 50, no. 4, pp. 541–560, 2002.
[14] U. G. Abdulla and J. R. King, "Interface development and local solutions to reaction-diffusion equations," *SIAM Journal on Mathematical Analysis*, vol. 32, no. 2, pp. 235–260, 2000.
[15] N. Wiener, "The Dirichlet problem," *Journal of Mathematics and Physics*, vol. 3, pp. 127–146, 1924.
[16] E. Feireisl, H. Petzeltová, and F. Simondon, "Admissible solutions for a class of nonlinear parabolic problems with non-negative data," *Proceedings of the Royal Society of Edinburgh. Section A. Mathematics*, vol. 131, no. 4, pp. 857–883, 2001.
[17] A. Friedman, *Partial Differential Equations of Parabolic Type*, Prentice-Hall, New Jersey, 1964.
[18] O. A. Ladyzhenskaya, V. A. Solonnikov, and N. N. Uralceva, *Linear and Quasilinear Equations of Parabolic Type*, American Mathematical Society, Rhode Island, 1968.
[19] G. M. Lieberman, *Second Order Parabolic Differential Equations*, World Scientific, New Jersey, 1996.

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