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Research Article

Existence of Solutions for Second-Order Nonlinear Impulsive Differential Equations with Periodic Boundary Value Conditions

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We are concerned with the nonlinear second-order impulsive periodic boundary value problem $u''(t) = f(t,u(t),u'(t))$, $t \in [0,T] \setminus \{t_1\}$, $u(t_1^+) = u(t_1^-) + I(u(t_1))$, $u'(t_1^+) = u'(t_1^-) + J(u(t_1))$, $u(0) = u(T)$, $u'(0) = u'(T)$, new criteria are established based on Schaefer's fixed-point theorem.

1. Introduction

Impulsive differential equations, which arise in physics, population dynamics, economics, and so forth, are important mathematical tools for providing a better understanding of many real-world models, we refer to [1–5] and the references therein. About the applications of the theory of impulsive differential equations to different areas, for example, see [6–15]. Boundary value problems (BVPs) for impulsive differential equations and impulsive difference equations [16–20] have received special attention from many authors in recent years.

Recently, Chen et al. in [21] study the following first-order impulsive nonlinear periodic boundary value problem:

$$\begin{aligned} x'(t) &= f(t,x), \quad t \in [0,N],\ t \neq t_1, \\ x(t_1^+) &= x(t_1^-) + I_1(x(t_1)), \\ x(0) &= x(T), \end{aligned} \tag{1.1}$$

where $N > 0$, $t_1 \in (0,N)$, t_1 is fixed, $f : [0,N] \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous on $(t,u) \in ([0,N] \setminus \{t_1\}) \times \mathbb{R}^n$, and the impulse at $t = t_1$ is given by a continuous function $I_1 : \mathbb{R}^n \to \mathbb{R}^n$. They investigate the existence of solutions to the problem by means of differential inequalities and Schaefer fixed point theorem. Their results complement and extend those of [22, 23] in the sense that they allow superlinear growth of the nonlinearity of $\|f(t,p)\|$ in $\|p\|$.

Inspired by [21, 24, 25], in this paper, we investigate the following second-order impulsive nonlinear differential equations with periodic boundary value conditions problem:

$$\begin{aligned} u''(t) &= f(t,u(t),u'(t)), \quad t\in[0,T],\ t\neq t_1,\\ u(t_1^+) &= u(t_1^-)+I(u(t_1)),\\ u'(t_1^+) &= u'(t_1^-)+J(u(t_1)),\\ u(0) &= u(T), \qquad u'(0)=u'(T), \end{aligned} \tag{1.2}$$

where $T>0$, $t_1\in(0,T)$, t_1 is fixed, $f:[0,T]\times\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}^n$ is continuous on $(t,x,y)\in([0,T]\setminus\{t_1\})\times\mathbb{R}^n\times\mathbb{R}^n$, and the impulse is given at t_1 by two continuous functions $I,J:\mathbb{R}^n\to\mathbb{R}^n$, the notations $u(t_1^-):=\lim_{t\to t_1^-}u(t)$, $u(t_1^+):=\lim_{t\to t_1^+}u(t)$, $u'(t_1^+)=\lim_{t\to t_1^+}u'(t)$, and $u'(t_1^-)=\lim_{t\to t_1^-}u(t)$.

We note that we could consider impulsive BVPs with an arbitrary finite number of impulses. However, for clarity and brevity, we restrict our attention to BVPs with one impulse. In addition, the difference between the theory of one or an arbitrary finite number of impulses is quite minimal.

Our results extend those of [25] from the nonimpulsive case to the impulsive case. Our approach using differential inequalities is based on ideas in [24, 25]. Moreover, our new results complement and extend those of [26–28] in the sense that we allow superlinear growth of $\|f(t,p,q)\|$ in $\|p\|$ and $\|q\|$.

The main purpose is to establish the existence of solutions for the impulsive BVP (1.2) by employing the well-known Schaefer fixed point theorem.

Lemma 1.1 (see [29] (Schaefer)). *Let E be a normed linear space with $H:E\to E$ be a compact operator. If the set*

$$S:=\{x\in E\mid x=\lambda Hx,\ \text{for some}\ \lambda\in(0,1)\} \tag{1.3}$$

is bounded, then H has at least one fixed point.

The paper is formulated as follows. In Section 2, some definitions and lemmas are given. In Section 3, we establish new existence theorems for (1.2). In Section 4, an illustrative example is given to demonstrate the effectiveness of the obtained results.

2. Preliminaries

First, we briefly introduce some appropriate concepts connected with impulsive differential equations. Most of the following notations can be found in [30].

Assume that

$$f(t_1^+,x,y):=\lim_{t\to t_1^+}f(t,x,y), \qquad f(t_1^-,x,y):=\lim_{t\to t_1^-}f(t,x,y) \tag{2.1}$$

both exist with $f(t_1^-,x,y) = f(t_1,x,y)$. We introduce and denote the Banach space $\mathrm{PC}([0,T],\mathbb{R}^n)$ by

$$\begin{aligned}\mathrm{PC}([0,T];\mathbb{R}^n) = \{&u \in C([0,T]\setminus\{t_1\},\mathbb{R}^n),\ u \text{ is left continuous at } t = t_1,\\ &\text{the right-hand limit } u(t_1^+) \text{ exists}\}\end{aligned} \tag{2.2}$$

with the norm

$$\|u\|_{\mathrm{PC}} = \sup_{t\in[0,T]} ||u(t)||, \tag{2.3}$$

where $\|\cdot\|$ is the usual Euclidean norm.

We define and denote the Banach space $\mathrm{PC}^1([0,T];\mathbb{R}^n)$ by

$$\begin{aligned}\mathrm{PC}^1([0,T];\mathbb{R}^n) = \{&u \in C^1([0,T]\setminus\{t_1\},\mathbb{R}^n),\ u \text{ is left continuous at } t = t_1,\\ &\text{the right-hand limit } u(t_1^+) \text{ exists, and the limits } u'(t_1^+),\ u'(t_1^-) \text{ exist}\}\end{aligned} \tag{2.4}$$

with the norm

$$\|u\|_{\mathrm{PC}^1} = \max\{\|u\|_{\mathrm{PC}}, \|u'\|_{\mathrm{PC}}\}. \tag{2.5}$$

A solution to the impulsive BVP (1.2) is a function $u \in \mathrm{PC}^1([0,T],\mathbb{R}^n) \cap C^2([0,T]\setminus\{t_1\},\mathbb{R}^n)$ that satisfies (1.2) for each $t \in [0,T]$.

Consider the following impulsive BVP with $p \geq 0$, $q > 0$:

$$\begin{aligned}u''(t) - pu'(t) - qu(t) + \sigma(t) = 0, \quad t \in [0,T],\ t \neq t_1,\\ u(t_1^+) = u(t_1^-) + I(u(t_1)),\\ u'(t_1^+) = u'(t_1^-) + J(u(t_1)),\\ u(0) = u(T), \qquad u'(0) = u'(T),\end{aligned} \tag{2.6}$$

where $\sigma \in \mathrm{PC}([0,T],\mathbb{R}^n)$ is given, $I,J : \mathbb{R}^n \to \mathbb{R}^n$ are continuous.

For convenience, we set

$$r_1 := \frac{p+\sqrt{p^2+4q}}{2} > 0, \qquad r_2 := \frac{p-\sqrt{p^2+4q}}{2} < 0. \tag{2.7}$$

Lemma 2.1. *$u \in \mathrm{PC}^1([0,T],\mathbb{R}^n) \cap C^2([0,T]\setminus\{t_1\},\mathbb{R}^n)$ is a solution of (2.6) if and only if $u \in \mathrm{PC}^1([0,T],\mathbb{R}^n)$ is a solution of the following linear impulsive integral equation:*

$$u(t) = \int_0^T G(t,s)\sigma(s)ds + G(t,t_1)(-J(u(t_1))) + W(t,t_1)I(u(t_1)), \tag{2.8}$$

where

$$G(t,s) = \frac{1}{r_1 - r_2} \begin{cases} \dfrac{e^{r_1(t-s)}}{e^{r_1 T} - 1} + \dfrac{e^{r_2(t-s)}}{1 - e^{r_2 T}}, & 0 \le s < t \le T, \\ \dfrac{e^{r_1(T+t-s)}}{e^{r_1 T} - 1} + \dfrac{e^{r_2(T+t-s)}}{1 - e^{r_2 T}}, & 0 \le t \le s \le T, \end{cases} \tag{2.9}$$

$$W(t,s) = \frac{1}{r_1 - r_2} \begin{cases} \dfrac{r_2 e^{r_1(t-s)}}{e^{r_1 T} - 1} + \dfrac{r_1 e^{r_2(t-s)}}{1 - e^{r_2 T}}, & 0 \le s < t \le T, \\ \dfrac{r_2 e^{r_1(T+t-s)}}{e^{r_1 T} - 1} + \dfrac{r_1 e^{r_2(T+t-s)}}{1 - e^{r_2 T}}, & 0 \le t \le s \le T. \end{cases} \tag{2.10}$$

Proof. If $u \in \mathrm{PC}^1([0,T];\mathbb{R}^n) \bigcap C^2([0,T] \setminus \{t_1\},\mathbb{R}^n)$ is a solution of (2.6), setting

$$v(t) = u'(t) - r_2 u(t), \tag{2.11}$$

then by the first equation of (2.6), we have

$$v'(t) - r_1 v(t) = -\sigma(t), \quad t \ne t_1. \tag{2.12}$$

Multiplying (2.12) by $e^{-r_1 t}$ and integrating on $[0,t_1)$ and $(t_1,T]$, respectively, we get

$$\begin{aligned} e^{-r_1 t_1} v(t_1^-) - v(0) &= -\int_0^{t_1} \sigma(s) e^{-r_1 s} ds, \quad 0 \le t < t_1, \\ e^{-r_1 t} v(t) - e^{-r_1 t_1} v(t_1^+) &= -\int_{t_1}^{T} \sigma(s) e^{-r_1 s} ds, \quad t_1 < t \le T, \end{aligned} \tag{2.13}$$

then, we have by the second equation and third equation of (2.6) that

$$v(t) = e^{r_1 t}\left[v(0) - \int_0^t e^{-r_1 s}\sigma(s)ds + I^*\right], \quad t \in [0,T], \tag{2.14}$$

where

$$v(0) = u'(0) - r_2 u(0), \quad I^* = \left(J(u(t_1)) - r_2 I(u(t_1))\right)e^{-r_1 t_1}. \tag{2.15}$$

Integrating (2.11), we have

$$u(t) = e^{r_2 t}\left[u(0) + \int_0^t v(s) e^{-r_2 s} ds + I(u(t_1)) e^{-r_2 t_1}\right], \quad t \in [0,T]. \tag{2.16}$$

By some calculation, we get

$$\begin{aligned} &\int_0^t v(s) e^{-r_2 s} ds \\ &= \frac{1}{r_1 - r_2}\left[v(0)\left(e^{(r_1-r_2)t} - 1\right), - \int_0^t \left(e^{(r_1-r_2)t} - e^{(r_1-r_2)s}\right)\sigma(s) e^{-r_1 s} ds + I^*\left(e^{(r_1-r_2)t} - e^{(r_1-r_2)t_1}\right)\right]. \end{aligned} \tag{2.17}$$

Substituting (2.17) into (2.16), we have

$$u(t)=\frac{1}{r_1-r_2}\Big[(u'(0)-r_2u(0))e^{r_1t}+(r_1u(0)-u'(0))e^{r_2t} \\ +\int_0^t\left(e^{r_2(t-s)}-e^{r_1(t-s)}\right)\sigma(s)ds \\ +(J(u(t_1))-r_2I(u(t_1)))e^{r_1(t-t_1)} \\ -(J(u(t_1))-r_1I(u(t_1)))e^{r_2(t-t_1)}\Big],\quad t\in[0,T]. \tag{2.18}$$

By the fourth equation (boundary condition) of (2.6), we have

$$r_1u(0)-u'(0)=\frac{1}{1-e^{r_2T}}\Big[\int_0^T e^{r_2(T-s)}\sigma(s)ds-(J(u(t_1))-r_1I(u(t_1)))e^{r_2(T-t_1)}\Big], \tag{2.19}$$

$$u'(0)-r_2u(0)=\frac{1}{e^{r_1T}-1}\Big[\int_0^T e^{r_1(T-s)}\sigma(s)ds-(J(u(t_1))-r_2I(u(t_1)))e^{r_1(T-t_1)}\Big], \tag{2.20}$$

substituting (2.19) and (2.20) into (2.18), we get (2.8).

Conversely, if u is a solution to (2.8), then direct differentiation of (2.8) gives $u''(t)=-\sigma(t)+pu'(t)+qu(t)$, $t\neq t_1$. Moreover, we have $u(t_1^+)=u(t_1^-)+I(u(t_1))$, $u'(t_1^+)=u'(t_1^-)+J(u(t_1))$, $u(0)=u(T)$, and $u'(0)=u'(T)$.

Note that the linear part of the periodic BVP (1.2) is not necessarily invertible, that is, we may be unable to equivalently rewrite (1.2) in the integral form. However, if we use Lemma 2.1, then impulsive BVP (1.2) may be equivalently reformulated as the impulsive integral equation.

We now introduce a mapping $A:\mathrm{PC}^1([0,T];\mathbb{R}^n)\to\mathrm{PC}([0,T];\mathbb{R}^n)$ defined by

$$Au(t)=\int_0^T G(t,s)[-f(s,u(s),u'(s))+pu'(s)+qu(s)]ds \\ +G(t,t_1)(-J(u(t_1)))+W(t,t_1)I(u(t_1)),\quad t\in[0,T]. \tag{2.21}$$

In view of Lemma 2.1, we easily know that u is a fixed point of operator A if and only if u is a solution to the impulsive boundary value problem (1.2).

It is easy to check that

$$0\leq G(t,s)\leq G(s,s)=\frac{e^{r_1T}-e^{r_2T}}{(r_1-r_2)(e^{r_1T}-1)(1-e^{r_2T})}:=G_1. \tag{2.22}$$

By $p \geq 0$ and $q > 0$, we have $r_1 \geq -r_2 > 0$. Thus we obtain that

$$\begin{aligned}
|W(t,s)| &\leq \frac{1}{r_1 - r_2}\begin{cases} \dfrac{-r_2 e^{r_1(t-s)}}{e^{r_1 T}-1} + \dfrac{r_1 e^{r_2(t-s)}}{1-e^{r_2 T}}, & 0 \leq s < t \leq T, \\ \dfrac{-r_2 e^{r_1(T+t-s)}}{e^{r_1 T}-1} + \dfrac{r_1 e^{r_2(T+t-s)}}{1-e^{r_2 T}}, & 0 \leq t \leq s \leq T, \end{cases} \\
&\leq \frac{r_1}{r_1 - r_2}\begin{cases} \dfrac{e^{r_1(t-s)}}{e^{r_1 T}-1} + \dfrac{e^{r_2(t-s)}}{1-e^{r_2 T}}, & 0 \leq s < t \leq T, \\ \dfrac{e^{r_1(T+t-s)}}{e^{r_1 T}-1} + \dfrac{e^{r_2(T+t-s)}}{1-e^{r_2 T}}, & 0 \leq t \leq s \leq T, \end{cases} \\
&= r_1 G(t,s) \leq r_1 G_1.
\end{aligned} \tag{2.23}$$

Since

$$\begin{aligned}
G_t(t,s) := \frac{\partial}{\partial t} G(t,s) &= \frac{1}{r_1 - r_2}\begin{cases} \dfrac{r_1 e^{r_1(t-s)}}{e^{r_1 T}-1} + \dfrac{r_2 e^{r_2(t-s)}}{1-e^{r_2 T}}, & 0 \leq s < t \leq T, \\ \dfrac{r_1 e^{r_1(T+t-s)}}{e^{r_1 T}-1} + \dfrac{r_2 e^{r_2(T+t-s)}}{1-e^{r_2 T}}, & 0 \leq t \leq s \leq T, \end{cases} \\
W_t(t,s) := \frac{\partial}{\partial t} W(t,s) &= \frac{1}{r_1 - r_2}\begin{cases} \dfrac{r_1 r_2 e^{r_1(t-s)}}{e^{r_1 T}-1} + \dfrac{r_2 r_1 e^{r_2(t-s)}}{1-e^{r_2 T}}, & 0 \leq s < t \leq T, \\ \dfrac{r_1 r_2 e^{r_1(T+t-s)}}{e^{r_1 T}-1} + \dfrac{r_1 r_2 e^{r_2(T+t-s)}}{1-e^{r_2 T}}, & 0 \leq t \leq s \leq T, \end{cases}
\end{aligned} \tag{2.24}$$

we easily get that

$$\begin{aligned}
|G_t(t,s)| &\leq \frac{1}{r_1 - r_2}\begin{cases} \dfrac{r_1 e^{r_1(t-s)}}{e^{r_1 T}-1} + \dfrac{-r_2 e^{r_2(t-s)}}{1-e^{r_2 T}}, & 0 \leq s < t \leq T, \\ \dfrac{r_1 e^{r_1(T+t-s)}}{e^{r_1 T}-1} + \dfrac{-r_2 e^{r_2(T+t-s)}}{1-e^{r_2 T}}, & 0 \leq t \leq s \leq T, \end{cases} \\
&\leq r_1 G(t,s) \leq r_1 G_1, \\
|W_t(t,s)| &\leq \frac{1}{r_1 - r_2}\begin{cases} \dfrac{-r_2 r_1 e^{r_1(t-s)}}{e^{r_1 T}-1} + \dfrac{-r_2 r_1 e^{r_2(t-s)}}{1-e^{r_2 T}}, & 0 \leq s < t \leq T, \\ \dfrac{-r_2 r_1 e^{r_1(T+t-s)}}{e^{r_1 T}-1} + \dfrac{-r_2 r_1 e^{r_2(T+t-s)}}{1-e^{r_2 T}}, & 0 \leq t \leq s \leq T, \end{cases} \\
&\leq r_1^2 G(t,s) \leq r_1^2 G_1.
\end{aligned} \tag{2.25}$$

Let

$$H := \max\{r_1 G_1,\ r_1^2 G_1\}. \tag{2.26}$$

So

$$|G_t(t,s)| \le H, \qquad |W_t(t,s)| \le H. \tag{2.27}$$

□

Lemma 2.2. *Let* $f:[0,T]\times\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}^n$ *and* $I,J:\mathbb{R}^n\to\mathbb{R}^n$ *be continuous. Then* $A:\mathrm{PC}^1([0,T];\mathbb{R}^n)\to\mathrm{PC}^1([0,T];\mathbb{R}^n)$ *is a compact map.*

Proof. This is similar to that of [31, Lemma 3.2]. Define two operators B, F as follows:

$$\begin{aligned} Bu(t) &= \int_0^T G(t,s)\big[-f(s,u(s),u'(s))+pu'(s)+qu(s)\big]ds, \quad t\in[0,T], \\ Fu(t) &= G(t,t_1)\big(-J(u(t_1))\big)+W(t,t_1)I(u(t_1)), \quad t\in[0,T]. \end{aligned} \tag{2.28}$$

From the continuity of f, it is easy to see that B is compact. Since I, J are continuous, we have that F is compact. Thus $A=B+F$ is a compact map. □

3. Main results

Theorem 3.1. *Suppose that* $f:[0,T]\times\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}^n$ *and* $I,J:\mathbb{R}^n\to\mathbb{R}^n$ *are continuous. If there exist nonnegative constants* α, β, γ, L_1, L_2, N, *and* M *such that for each* $\lambda\in(0,1)$,

$$\begin{gathered} \|f(t,x,y)-py-qx\| \le 2\alpha\big[\langle x+y,f(t,x,y)\rangle+\|y\|^2\big]+M, \\ (t,x,y)\in\big([0,T]\setminus\{t_1\}\big)\times\mathbb{R}^n\times\mathbb{R}^n, \text{ where } \langle\cdot\rangle \text{ is the Euclidean inner product,} \end{gathered} \tag{3.1}$$

$$\|I(x)\| \le \beta\|x\|+L_1, \quad \|J(x)\| \le \gamma\|x\|+L_2, \quad \forall x\in\mathbb{R}^n, \tag{3.2}$$

$$\beta+\gamma<\frac{1}{H}, \tag{3.3}$$

where H is as in (2.26), then BVP (1.2) has at least one solution.

Proof. From Lemma 2.2, we know that A is a compact map. In order to show that A has at least one fixed point, we apply Lemma 1.1 (Schaefer's theorem) by showing that all potential solutions to

$$u=\lambda Au, \quad \lambda\in(0,1), \tag{3.4}$$

are bounded a priori, with the bound being independent of λ. Let u be a solution to (3.4), then

$$\begin{gathered} u''(t)-pu'(t)-qu(t)=\lambda\big[f(t,u(t),u'(t))-pu'(t)-qu(t)\big], \quad t\in[0,T], \\ u(t_1^+)=u(t_1^-)+\lambda I(u(t_1)), \\ u'(t_1^+)=u'(t_1^-)+\lambda J(u(t_1)), \\ u(0)=u(T), \qquad u'(0)=u'(T). \end{gathered} \tag{3.5}$$

By (3.1)–(3.3), (2.22) and (2.23), we obtain

$$\begin{aligned}
\|u(t)\| &= \lambda\|Au(t)\| \\
&= \left\| \int_0^T G(t,s)\lambda(f(s,u(s),u'(s)) - pu'(s) - qu(s))\,ds \right. \\
&\qquad \left. + \lambda G(t,t_1)(-J(u(t_1))) + \lambda W(t,t_1)I(u(t_1)) \right\| \\
&\le G_1 \int_0^T \lambda\|f(s,u(s),u'(s)) - pu'(s) - qu(s)\|\,ds \\
&\qquad + \lambda G_1(\|J(u(t_1))\| + \|I(u(t_1))\|) \\
&\le G_1 \left[\int_0^T (2\alpha(\langle u(s)+u'(s), \lambda f(s,u(s),u'(s))\rangle + \|u'\|^2) + M)\,ds \right. \\
&\qquad \left. + \beta\|u(t_1)\| + L_1 + \gamma\|u(t_1)\| + L_2 \right] \\
&= G_1 \left[\int_0^T \Big(2\alpha\Big(\langle u(s)+u'(s), \lambda f(s,u(s),u'(s)) + (1-\lambda)pu'(s) \right. \\
&\qquad + (1-\lambda)qu(s)\rangle + \|u'(s)\|^2\Big) + M\Big)\,ds \\
&\qquad - \int_0^T 2\alpha\langle u(s)+u'(s), (1-\lambda)pu'(s) + (1-\lambda)qu(s)\rangle\,ds \\
&\qquad \left. + (\beta+\gamma)\|u(t_1)\| + L_1 + L_2 \right].
\end{aligned} \tag{3.6}$$

Since

$$\begin{aligned}
&-\int_0^T \langle u(s)+u'(s), (1-\lambda)pu'(s) + (1-\lambda)qu(s)\rangle\,ds \\
&\quad = -(1-\lambda)q\int_0^T \|u(s)\|^2 ds - (1-\lambda)p\|u'(s)\|^2 ds + (1-\lambda)(p+q)\int_0^T \langle u(s),u'(s)\rangle\,ds \\
&\quad \le (1-\lambda)(p+q)\int_0^T \langle u(s),u'(s)\rangle\,ds = \frac{1}{2}(1-\lambda)(p+q)\int_0^T \frac{d}{ds}\left(\|u(s)\|^2\right) \\
&\quad = \frac{1}{2}(1-\lambda)(p+q)\left(\|u(T)\|^2 - \|u(0)\|^2\right) = 0,
\end{aligned} \tag{3.7}$$

we have by (3.6) and (3.7) that

$$\begin{aligned}
\|u(t)\| &= \lambda\|Au(t)\| \\
&\leq G_1\Big[\int_0^T \Big(2\alpha\Big(\langle u(s)+u'(s),\lambda f(s,u(s),u'(s))+(1-\lambda)pu'(s)+(1-\lambda)qu(s)\rangle \\
&\qquad\qquad + \|u'(s)\|^2\Big)+M\Big)ds+(\beta+\gamma)\|u(t_1)\|+L_1+L_2\Big] \\
&= G_1\Big[\int_0^T \Big(2\alpha\langle u(s)+u'(s),u''(s)\rangle+\langle u(s)+u'(s),u'(s)\rangle \\
&\qquad\qquad -\langle u(s),u'(s)\rangle+M\Big)ds+(\beta+\gamma)\|u(t_1)\|+L_1+L_2\Big] \\
&= G_1\Big[\int_0^T (2\alpha\langle u(s)+u'(s),u''(s)+u'(s)\rangle+M)ds+(\beta+\gamma)\|u(t_1)\|+L_1+L_2\Big] \\
&= G_1\Big[\int_0^T \Big(\alpha\frac{d}{ds}\Big(\|u(s)+u'(s)\|^2\Big)+M\Big)ds+(\beta+\gamma)\|u(t_1)\|+L_1+L_2\Big] \\
&= G_1\Big[\alpha\Big(\|u(T)+u'(T)\|^2-\|u(0)+u'(0)\|^2\Big)+TM+(\beta+\gamma)\|u(t_1)\|+L_1+L_2\Big] \\
&= G_1[TM+(\beta+\gamma)\|u(t_1)\|+L_1+L_2].
\end{aligned} \tag{3.8}$$

Thus, taking the supremum and rearranging, we have

$$\sup_{t\in[0,T]}\|u(t)\| \leq \frac{G_1(TM+L_1+L_2)}{1-G_1(\beta+\gamma)}. \tag{3.9}$$

A similar calculation yields an estimate on u': differentiating both sides of the integration equation (3.4) and taking norms yields, by (2.27), for each $t\in[0,T]$ that

$$\sup_{t\in[0,T]}\|u'(t)\| \leq \frac{H(TM+L_1+L_2)}{1-H(\beta+\gamma)}, \tag{3.10}$$

where H is as in (2.26). By (3.9) and (3.10), we conclude that

$$\|u\|_{PC^1} = \max\left\{\frac{G_1(TM+L_1+L_2)}{1-G_1(\beta+\gamma)},\frac{H(TM+L_1+L_2)}{1-H(\beta+\gamma)}\right\} = \frac{H(TM+L_1+L_2)}{1-H(\beta+\gamma)}. \tag{3.11}$$

As a result, we obtain the desired bound. We see that the bound on all possible solutions to (3.4) is independent of λ. Applying Scheafer fixed point theorem, A has at least one fixed point, which means that (1.2) has at least one solution. We complete the proof. □

Theorem 3.1 may be suitably modified to include an alternate class of f as follows.

Theorem 3.2. *Suppose that $f : [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ and $I,J : \mathbb{R}^n \to \mathbb{R}^n$ are continuous. Let the conditions of Theorem 3.1 hold with (3.1) replaced by*

$$\|f(t,x,y) - py - qx\| \le 2\alpha\langle y, f(t,x,y)\rangle + M, \quad (t,x,y) \in ([0,T] \setminus \{t_1\}) \times \mathbb{R}^n \times \mathbb{R}^n. \tag{3.12}$$

Then the impulsive BVP (1.2) has at least one solution.

The proof of Theorem 3.2 is similar to that of Theorem 3.1. It is enough to notce that (3.6) in Theorem 3.1 reduces to

$$\begin{aligned}
\|u(t)\| &= \lambda\|Au(t)\| \\
&\le G_1\int_0^T \lambda\|f(s,u(s),u'(s)) - pu'(s) - qu(s)\|ds + \lambda G_1(\|J(u(t_1))\| + \|I(u(t_1))\|) \\
&\le G_1\Big[\int_0^T (2\alpha\langle u'(s), \lambda f(s,u(s),u'(s))\rangle + M)ds \quad (\text{use (3.12)}) \\
&\qquad + (\beta+\gamma)\|u(t_1)\| + L_1 + L_2\Big] \\
&\le G_1\Big[\int_0^T (2\alpha\langle u'(s), \lambda f(s,u(s),u'(s)) + (1-\lambda)pu'(s)\rangle + M)ds \\
&\qquad + (\beta+\gamma)\|u(t_1)\| + L_1 + L_2\Big] \\
&= G_1\Big[\int_0^T (2\alpha\langle u'(s), \lambda f(s,u(s),u'(s)) + (1-\lambda)pu'(s) + (1-\lambda)qu(s)\rangle + M)ds \\
&\qquad - (1-\lambda)q\int_0^T 2\alpha\langle u'(s), u(s)\rangle ds + (\beta+\gamma)\|u(t_1)\| + L_1 + L_2\Big] \\
&= G_1\Big[\int_0^T (2\alpha\langle u'(s), u''(s)\rangle + M)ds + (\beta+\gamma)\|u(t_1)\| + L_1 + L_2\Big] \\
&= G_1\Big[\int_0^T \Big(\alpha\frac{d}{ds}\big(\|u'(s)\|^2\big) + M\Big)ds + (\beta+\gamma)\|u(t_1)\| + L_1 + L_2\Big] \\
&= G_1\Big[\alpha\big(\|u'(T)\|^2 - \|u'(0)\|^2\big) + TM + (\beta+\gamma)\|u(t_1)\| + L_1 + L_2\Big] \\
&= G_1[TM + (\beta+\gamma)\|u(t_1)\| + L_1 + L_2].
\end{aligned} \tag{3.13}$$

Remark 3.3. If f does not depend on u', let the conditions of Theorem 3.1 hold with (3.1) replaced by

$$\|f(t,x) - qx\| \le 2\alpha\langle x, f(t,x)\rangle + M, \quad (t,x) \in ([0,T] \setminus \{t_1\}) \times \mathbb{R}^n \times \mathbb{R}^n. \tag{3.14}$$

Then the impulsive BVP (1.2) has at least one solution.

4. An example

In this section, we consider an example to illustrate the effectiveness of our new theorems. For brevity, we restrict our attention to scalar-valued impulsive BVPs, although we note that it is not difficult to construct a vector-valued f such that the conditions of Theorems 3.1 and 3.2 are satisfied.

Example 4.1. Consider the scalar impulsive BVP given by

$$\begin{aligned} &u''(t) = \left(u(t)+u'(t)\right)^3 + u(t) + \left(u'(t)\right)^2 + u'(t) + t, \quad t \in [0,1] \setminus \{t_1\}, \\ &u(t_1^+) = u(t_1^-) + \frac{u(t_1)}{5}, \qquad u'(t_1^+) = u'(t_1^-) + \frac{u(t_1)}{7}, \\ &u(0) = u(1), \qquad u'(0) = u'(1), \end{aligned} \tag{4.1}$$

we claim that the above impulsive BVP has at least one solution.

Proof. Let $T = 1$, $f(t,x,y) = (x+y)^5 + x + y^2 + y + t$, and $p = q = 1$. Then $r_1 = (\sqrt{5}+1)/2$ and $r_2 = (1-\sqrt{5})/2$. Obviously, (3.2) holds with $\beta = 1/5$, $\gamma = 1/7$, and $L_1 = L_2 = 0$. We get $1/H = 0.3534$ (H is as in (2.26)). Thus, (3.3) in Theorem 3.1 holds. Moreover, we see that

$$\left| f(t,x,y) - x - y \right| \le |x+y|^5 + y^2 + 1, \quad \forall (t,x,y) \in [0,1] \times \mathbb{R}^2, \tag{4.2}$$

and for $\alpha = 1/2$ and $M = 2$,

$$\begin{aligned} 2\alpha\left[(x+y)f(t,x,y) + y^2\right] + M &= (x+y)^6 + (x+y)^2 + (x+y)t + y^2 + 2 \\ &\ge (x+y)^6 + (x+y)^2 - |x+y| + y^2 + 2 \ge |x+y|^5 + y^2 + 1, \quad \forall (t,x,y) \in [0,1] \times \mathbb{R}^2. \end{aligned} \tag{4.3}$$

Thus (3.1) holds. Therefore, by Theorem 3.1, BVP (4.1) has at least one solution. □

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