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Research Article

Simultaneous versus Nonsimultaneous Blowup for a System of Heat Equations Coupled Boundary Flux

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This paper deals with a semilinear parabolic system in a bounded interval, completely coupled at the boundary with exponential type. We characterize completely the range of parameters for which nonsimultaneous and simultaneous blowup occur.

1. Introduction

In this paper, we consider the positive blowup solution to the following parabolic problem:

$$\begin{gathered} u_t = u_{xx}, \quad v_t = v_{xx}, \quad (x,t) \in (0,L) \times (0,T), \\ -u_x(0,t) = e^{p_{11}u(0,t)+p_{12}v(0,t)}, \quad -v_x(0,t) = e^{p_{21}u(0,t)+p_{22}v(0,t)}, \quad t \in (0,T), \\ u_x(L,t) = 0, \quad v_x(L,t) = 0, \quad t \in (0,T), \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \quad x \in (0,L), \end{gathered} \tag{1.1}$$

where we assume the parameters $p_{ij} \geq 0$ $(i,j = 1,2)$, $p_{11} + p_{22} > 0$ and $p_{21} + p_{12} > 0$ which ensure that (1.1) completely coupled with the nontrivial nonlinear boundary flux. The initial values $u_0(x)$, $v_0(x)$ are positive, nontrivial, bounded, and compatible with the boundary data and smooth enough to guarantee that u, v are regular.

The study of reaction-diffusion systems has received a great deal of interest in recent years and has been used to model, for example, heat transfer, population dynamics, and chemical reactions (see [1] and references therein). The parabolic system like (1.1) can be used to describe, for example, heat propagations in mixed solid nonlinear media with nonlinear boundary flux. The nonlinear Nuemann boundary values in (1.1), coupling the two heat equations, represent some cross-boundary flux. Let T denote the maximal existence time for the solution (u, v). If it is infinite, we say that the solution is global. For appropriate initial data u_0, v_0, there are solutions to (1.1) that blowup in a finite time $T < \infty$ in L^∞-norm, that is,

$$\limsup_{t \to T} \left\{ \|u(\cdot,t)\|_\infty + \|v(\cdot,t)\|_\infty \right\} = \infty. \tag{1.2}$$

However, we note that a priori, there is no reason for both components u and v should go to infinity simultaneously at time T. In this paper, our first purpose is to show that for some certain choice of parameters p_{ij}, there are some initial data for which one of the components remains bounded, while the other blows up (we denote this phenomenon as nonsimultaneous blowup), and for others both components blowup simultaneously. Moreover, we give the complete classification of the simultaneous and nonsimultaneous blowups by the parameters p_{ij}. Nonsimultaneous blowup phenomenon for the heat equations with nonlinear power-like-type boundary conditions was carried out in [2–4].

Let us examine what is known in blowup for the heat equations with nonlinear boundary conditions before presenting our results. In [5], Deng obtained the blowup rate $\max_{\overline{\Omega}} u(\cdot,t) = O(\log(T-t)^{-1/2p_{21}})$, $\max_{\overline{\Omega}} v(\cdot,t) = O(\log(T-t)^{-1/2p_{12}})$ for the following problem (with $p_{11} = 0$ and $p_{22} = 0$):

$$\begin{gathered} u_t = \triangle u, \quad v_t = \triangle v, \quad (x,t) \in \Omega \times (0,T), \\ \frac{\partial u}{\partial \eta} = e^{p_{11}u + p_{12}v}, \quad \frac{\partial v}{\partial \eta} = e^{p_{21}u + p_{22}v}, \quad (x,t) \in \partial\Omega \times (0,T), \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \quad x \in \Omega. \end{gathered} \tag{1.3}$$

In [6], Zhao and Zheng considered the problem (1.3) with $p_{21} > p_{11}$ and $p_{12} > p_{22}$ and obtained the blowup rates. However, whenever there is blowup, both components become unbounded at the same time (see [6, Lemma 2.2]). That is, u blows up in L^∞-norm at time T if and only if v also does so. Nonsimultaneous blowup is therefore not possible in this case.

In order to study the nonsimultaneous blowup phenomena for system (1.1), we need to make further assumptions on the initial data:

$$u_0, v_0 \geq \delta_1 > 0, \quad u_0'(x), v_0'(x) \leq 0, \quad u_0''(x), v_0''(x) \geq \delta_2 > 0 \quad \text{for } x \in [0,L]. \tag{1.4}$$

Firstly, we give a set of parameters for which nonsimultaneous blowup indeed occurs.

Theorem 1.1. *There exists a pair of suitable initial data (u_0, v_0) such that nonsimultaneous blowup occurs if and only if $p_{11} > p_{21}$ or $p_{22} > p_{12}$.*

Corollary 1.2. *If $p_{11} \leq p_{21}$ and $p_{22} \leq p_{12}$, then u and v blowup at the same time for any pairs of initial data.*

However, in this case, we do not exclude the possibility of exceptional solutions with simultaneous blowup. In fact, when $p_{11} > p_{21}$ and $p_{22} > p_{12}$, this implies that each of the components may blowup by itself, then there exists a pair of initial data for which simultaneous blowup indeed occurs.

Theorem 1.3. *If $p_{11} > p_{21}$ and $p_{22} > p_{12}$, both simultaneous and nonsimultaneous blowup may occur, provided that the initial data are chosen properly.*

Theorem 1.4. (i) *If $p_{11} > p_{21}$ and $p_{22} \le p_{12}$, then there exists a finite time T, such that u blows up at T, while v remains bounded up to that time for every pair of initial data.*

(ii) *If $p_{22} > p_{12}$ and $p_{11} \le p_{21}$, then there exists a finite time T, such that v blows up at T, while u remains bounded up to that time for every pair of initial data.*

2. Proof of main results

Without loss of generality, we consider the case $p_{11} > p_{21}$, to show that there exists a pair of initial data such that u blows up at a finite time and v remains bounded up to this time if and only if $p_{11} > p_{21}$. The case $p_{22} > p_{12}$ is handled in a completely analogous form. In this paper, we use c and C to denote positive constants independent of t, which may be different from line to line, even in the same line.

Firstly, we give the estimate of blowup rate for u in the case u blows up while v remains bounded, which plays an important role in the proof of Theorem 1.1. We consider $e^{p_{12}v(0,t)}$ as a frozen coefficient and regard u as a blowup solution to the following auxiliary problem:

$$\begin{gathered} u_t = u_{xx}, \quad (x,t) \in (0,L) \times (0,T), \quad -u_x(0,t) = e^{p_{11}u(0,t)}h(t), \quad t \in (0,T), \\ u_x(L,t) = 0, \quad t \in (0,T), \quad u(x,0) = u_0(x), \quad x \in (0,L), \end{gathered} \tag{2.1}$$

where u_0 satisfies (1.4). The function $h(t) \ge \delta > 0$ is bounded, continuous and $h'(t) \ge 0$. The solutions of problem (2.1) blowup if $p_{11} > 0$ (see [7]). First, we try to establish the upper blowup estimate.

Lemma 2.1. *If $p_{11} > 0$ and u is a solution of (2.1), then there exists $C_0 > 0$ such that*

$$u(0,t) = \max_{x\in[0,L]} u(\cdot,t) \le -\frac{1}{2p_{11}} \log C_0(T-t), \quad \text{for } 0 < t < T. \tag{2.2}$$

Proof. Set $J(x,t) = u_t - \varepsilon u_x^2$, $(x,t) \in (0,L) \times [0,T)$. From the assumptions (1.4) on the initial data, we know that $u_t > 0$, $u_x \ge 0$, so we can choose ε small enough such that

$$\begin{aligned} & J(x,0) = u_t(x,0) - \varepsilon u_x^2(x,0) \ge 0, \quad x \in [0,L], \\ & -J_x(0,t) - (p_{11} - 2\varepsilon)h(t)e^{p_{11}u(0,t)}J(0,t) \\ & \qquad = h'(t)e^{p_{11}u(0,t)} + (p_{11} - 2\varepsilon)h^3(t)e^{3p_{11}u(0,t)} \ge 0, \quad t \in (0,T). \end{aligned} \tag{2.3}$$

For $(x,t) \in (0,L) \times [0,T)$, a simple computation yields $J_t - J_{xx} = 2\varepsilon u_{xx}^2 \ge 0$. Define $J(x,t) = J(2L-x,t)$, $(x,t) \in (L,2L) \times [0,T)$, by comparison principle in $(x,t) \in (0,2L) \times [0,T)$, we have $J \ge 0$. Thus

$$u_t(0,t) \ge \varepsilon u_x^2(0,t) \ge \varepsilon \delta^2 e^{2p_{11}u(0,t)}, \quad t \in [0,t). \tag{2.4}$$

Integrating (2.4) from t to T, we get (2.2). □

In order to obtain that v is bounded when $p_{11} > p_{21}$, we introduce the following lemma, which has been proved in [2, Section 3].

Lemma 2.2. *Consider the following system with $K_1 > 0$:*

$$\begin{aligned} z_t = z_{xx}, \quad (x,t) \in (0,L)\times(0,T), \qquad -z_x(0,t) = K_1(T-t)^{-p_{21}/2p_{11}}, \quad t\in(0,T), \\ z_x(L,t)=0, \quad t\in(0,T), \qquad z(x,0)=v_0(x), \quad x\in(0,L). \end{aligned} \tag{2.5}$$

If $p_{21} < p_{11}$, then there exists T small enough such that the solution of (2.5) verifies

$$z(0,t) = \sup_{0<t<T} \|z(\cdot,t)\|_\infty \le \|v_0(\cdot)\|_\infty + \varepsilon, \tag{2.6}$$

for given $\varepsilon > 0$ and $v_0 > 0$. In particular, z is bounded.

Next, we consider the auxiliary problem

$$\begin{aligned} w_t &= w_{xx}, \quad (x,t)\in(0,L)\times(0,T_0), \\ -w_x(0,t) &= C_0^{-p_{21}/2p_{11}} e^{p_{22}w(0,t)}(T-t)^{-p_{21}/2p_{11}}, \quad t\in(0,T_0), \\ w_x(L,t) &= 0, \quad t\in(0,T_0), \qquad w(x,0)=v_0(x), \quad x\in(0,L), \end{aligned} \tag{2.7}$$

where C_0 is defined in (2.2).

Lemma 2.3. *Assume $p_{11} > p_{21}$, and let w solve (2.7), then for given ε and v_0, w satisfies (2.6) provided that T is sufficiently small. In particular, w is bounded.*

Proof. For given ε and v_0, let z be a solution of (2.5) with $K_1 \ge C_0^{-p_{21}/2p_{11}} e^{p_{22}(\|v_0\|_\infty+\varepsilon)}$. Choose T small enough that (2.6) holds, then z is a supersolution of (2.7). By comparison principle, $w \le z$ in $(0,L)\times[0,T)$, and thus w satisfies (2.6). □

Proof of Theorem 1.1. Assume $p_{11} > p_{21}$, for given ε and v_0, we can choose u_0 large enough to make the blowup time T satisfy (2.2) and (2.6), and we have

$$\begin{aligned} v_t &= v_{xx}, \quad (x,t)\in(0,L)\times(0,T), \\ -v_x(0,t) &\le C_0^{-p_{21}/2p_{11}} e^{p_{22}v(0,t)}(T-t)^{-p_{21}/2p_{11}}, \quad t\in(0,T), \\ v_x(L,t) &= 0, \quad t\in(0,T), \qquad v(x,0)=v_0(x), \quad x\in(0,L). \end{aligned} \tag{2.8}$$

By comparison principle, $v \le w$ in $(0,L)\times(0,T)$. Hence v is bounded.

Next, we assume that u blows up in finite time T, while v remains bounded for $(x,t)\in(0,L)\times(0,T)$. We use [2, Lemma 3.2] to obtain that $p_{11} > p_{21}$, which needs us to establish the lower blowup estimate of problem (2.1) firstly. Let us define $M(t) = \|u(\cdot,t)\|_\infty = u(0,t)$. Using the scaling method from [8], we set

$$\varphi_M(y,s) = e^{u(ay,bs+t)-M(t)}, \quad 0\le y\le \frac{L}{a},\ -\frac{t}{b}\le s\le 0, \tag{2.9}$$

where $a = e^{-p_{11}M}$, $b = e^{-2p_{11}M}$. Since $p_{11} > 0$ and u blows up at T, then $a,b \searrow 0$ as $t \nearrow T$. The function φ_M satisfies $0 \le \varphi_M \le 1$, $(\varphi_M)_s \ge 0$, $\varphi_M(0,0) = 1$, and

$$\begin{gathered}(\varphi_M)_s = (\varphi_M)_{yy} - A\varphi_M, \quad (y,s) \in \left(0, \frac{L}{a}\right) \times \left(-\frac{t}{b}, 0\right], \\ -(\varphi_M)_y(0,s) = \varphi_M^{p_{11}+1}(0,s)h(bs+t), \quad (\varphi_M)_y\left(\frac{L}{a}, s\right) = 0, \quad s \in \left(-\frac{t}{b}, 0\right],\end{gathered} \tag{2.10}$$

where $A = bu_x^2(ay, bs+t) \le bu_x^2(0, bs+t) = h^2(bs+t)$. Noticing that $h(bs+t)$ is bounded, by Schauder estimate, we see that φ_M is uniformly bounded in $C^{2+\alpha,1+\alpha}$ for some $\alpha > 0$ (see [9]). Consequently, $(\varphi_M)_s(0,0) \le C$, which yields

$$u(0,t) = \max_{x\in[0,L]} u(\cdot,t) \ge -\frac{1}{2p_{11}} \log C_1(T-t), \quad \text{for } 0 < t < T, \tag{2.11}$$

where C_1 is a positive constant.

We suppose on the contrary that $p_{11} \le p_{21}$, then from [2, Lemma 3.2], the solution of (2.5) blows up at T. Choose $K_1 \le C_1^{-p_{21}/2p_{11}}$, where C_1 is defined in (2.11), then v is a supersolution of problem (2.5), which contradicts the fact that v remains bounded up to the time T. Therefore, if u blows up while v remains bounded, then $p_{11} > p_{21}$. □

Proof of Theorem 1.3. Its proof is standard and similar to [2, Theorems 1.4 and 1.5], hence we omit it here. □

Finally, we will prove that there are two regions of the parameters where nonsimultaneous blowup occurs for any initial data. Before proving this, we would like to give the blowup set of (1.1) provided that $p_{11}, p_{22} > 0$, which will play an important role in the proof of Theorem 1.4.

Lemma 2.4. *Under the assumptions of (1.4), then the point $x = 0$ is the only blowup point of (1.1) provide that $p_{11}, p_{22} > 0$.*

Proof. From [10], the condition $p_{11}, p_{22} > 0$ ensures the blowup of (1.1). Without loss of generality, we may assume that $\max_{x\in[0,L]} u(\cdot,t) = u(0,t) \to \infty$, as $t \to T$. Assume on the contrary that u blows up at another point $x^* > 0$ as $t \to T$, that is, $\limsup_{t\to T} u(x^*,t) = \infty$. Since $u(x,t)$ is nonincreasing in x, $\limsup_{t\to T} u(x,t) = \infty$ for any $x \in [0,x^*]$. Set $J(x,t) = u_x + \varsigma(L-x)e^{p_{11}u}$, for $(x,t) \in [0,L] \times [0,T)$, where ς is a small constant to be determined.

Noticing that u_0 is nontrivial, from the assumptions on $u_0(x)$ in (1.4), we have $u_0'(x) < 0$ provide that $x \ne L$ and $t \in (0,T)$. We choose ς small enough such that

$$\begin{gathered}J(x,0) \le u_0'(x) + \varsigma(L-x)e^{p_{11}\max_{x\in(0,L)} u_0(x)} \le 0, \quad x \in (0,L), \\ J(0,t) = -e^{p_{11}u(0,t)+p_{12}v(0,t)} + \varsigma L e^{p_{11}u(0,t)} \le e^{p_{11}u(0,t)}(\varsigma L - 1) \le 0, \quad t \in (0,T), \\ J(L,t) = 0, \quad t \in (0,T).\end{gathered} \tag{2.12}$$

On the other hand, a simple computation yields

$$J_t - J_{xx} = 2p_{11}\varsigma e^{p_{11}u}u_x - p_{11}^2\varsigma e^{p_{11}u}u_x^2 \le 0, \quad \text{for } (x,t) \in (0,L) \times (0,T). \tag{2.13}$$

Application of the maximum principle to (2.12)-(2.13) ensures that $J(x,t) \le 0$, for $(x,t) \in (0,L)\times(0,T)$. Namely, $-e^{-p_{11}u}u_x \ge \varsigma(L-x)$.

Integrating from 0 to x^* yields $0 < \int_0^{x^*} \varsigma(L-x)dx \le (1/p_{11})e^{-p_{11}u(x^*,t)}$, $t\in(0,T)$. The fact that $\limsup_{t\to T} u(x^*,t) = \infty$ and $p_{11} > 0$ lead to a contradiction. Therefore, u blows up at a single point $x = 0$, and so does the solution (u,v) of problem (1.1). □

Proof of Theorem 1.4. (i) $p_{11} > p_{21}$ and $p_{22} \le p_{12}$. Clearly, by Theorem 1.1, it is possible that u blows up and v remains bounded in this case. We will show that the simultaneous blowup does not occur in this case. Suppose on the contrary that there exist initial data (u_0,v_0) such that u and v blowup simultaneously. Let us define $M(t) = u(0,t) = \max u(\cdot,t)$ and $N(t) = v(0,t) = \max v(\cdot,t)$. Following the ideas from [8], we set for $t < T$ that

$$\varphi_M(y,s) = e^{u(ay,bs+t)-M(t)}, \quad \psi_N(y,s) = e^{v(cy,ds+t)-N(t)},$$
$$y > 0, \quad \max\left\{-\frac{t}{b}, -\frac{t}{d}\right\} \le s \le 0, \tag{2.14}$$

where $a^2 = b = e^{-(2p_{11}+1)M-2p_{12}N}$, $c^2 = d = e^{-2p_{22}N-(2p_{21}+1)M}$. The pair of function (φ_M,ψ_N) satisfies $0 \le \varphi_M, \psi_N \le 1$, $\varphi_M(0,0) = \psi_N(0,0) = 1$ and $(\varphi_M)_s, (\psi_N)_s \ge 0$, and is the solution of the parabolic problem

$$(\varphi_M)_s = (\varphi_M)_{yy} - A\varphi_M, \qquad (\psi_N)_s = (\psi_N)_{yy} - B\psi_N,$$
$$-(\varphi_M)(0,s) = e^{-M(t)}\varphi_M^{p_{11}+1}(0,s)\psi_N^{p_{12}}(0,s), \qquad -(\psi_N)(0,s) = e^{-M(t)}\psi_N^{p_{22}+1}(0,s)\varphi_M^{p_{21}}(0,s), \tag{2.15}$$

where $A = bu_x^2(ay,bs+t) \le bu_x^2(0,bs+t) \le e^{-2M(t)}$, $B = dv_x^2(ay,bs+t) \le dv_x^2(0,bs+t) \le e^{-2M(t)}$.

With the same idea of the proof of Theorem 1.1, by the well-known Schauder estimates, it is easy to see that there exists a positive constant C such that for sufficiently large M and N,

$$(\varphi_M)_s(0,0) \le C, \qquad (\psi_N)_s(0,0) \le C. \tag{2.16}$$

Next, we claim that there exists a positive constant c such that for every pair of large M, N,

$$(\varphi_M)_s(0,0) \ge c. \tag{2.17}$$

To prove this claim, suppose on the contrary there should be a sequence $\{\varphi_{M_j}\}$ such that $(\varphi_{M_j})_s(0,0) \to 0$ as $M_j, N_j \to \infty$. As φ_{M_j} is uniformly bounded in $C^{2+\alpha,1+\alpha}$ (see [9]), passing to a subsequence if necessary, we obtain a positive function φ such that $\varphi_{M_j} \to \varphi$ in $C^{2+\beta,1+\beta}$ (for some $\beta < \alpha$), and verify $0 \le \varphi \le 1$, $\varphi(0,0) = 1$, $\varphi_s \ge 0$, and $\varphi_s = \varphi_{yy}$, $\varphi_y(0,s) = 0$ in $(0,+\infty)\times(-\infty,0]$. We set $w = \varphi_s$ as w satisfies the heat equation, with the boundary condition $w_y(0,s) = w(0,0) = 0$. We conclude using Hopf's lemma that $w \equiv 0$, that is, $\varphi(y,s)$ does not depend on s and then $\varphi(y) \equiv 1$. Hence, $u(ay,bs+t) \equiv M(t)$ for all $(y,s) \in (0,+\infty)\times(-\infty,0]$ as $t \to T$, which leads to a contradiction with the fact that

u of the system (1.1) possesses a single blowup point at $x = 0$ provided that $p_{11} > 0$ (see Lemma 2.4). Thus we arrive at inequality (2.17).

Inequalities (2.16) and (2.17) imply that $ce^{2p_{12}N} \le e^{-2(p_{11}+1)M}M'(t)$, $e^{-2p_{22}N}N'(t) \le Ce^{2(p_{21}+1)M}$. Noticing that $p_{11} > p_{21}$ and $p_{22} \le p_{12}$, a direct computation yields

$$\frac{1}{2(p_{21}-p_{11})}e^{2(p_{21}-p_{11})M(t)} \ge \begin{cases} \dfrac{C}{2(p_{12}-p_{22})}e^{2(p_{12}-p_{22})N(t)} + C' & \text{for } p_{22} < p_{12}, \\ CN(t) + C' & \text{for } p_{22} = p_{12}, \end{cases} \tag{2.18}$$

where $C > 0$ and C' are constants independent of t. Obviously, they contradict the assumption that u and v blowup simultaneously.

(ii) $p_{22} > p_{12}$ and $p_{11} \le p_{21}$. The proof of this case is parallel to the previous case. □

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