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Research Article

On Comparison Principles for Parabolic Equations with Nonlocal Boundary Conditions

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A generalization of the comparison principle for a semilinear and a quasilinear parabolic equations with nonlocal boundary conditions including changing sign kernels is obtained. This generalization uses a positivity result obtained here for a parabolic problem with nonlocal boundary conditions.

1. Introduction

The positivity of solutions for parabolic problems is the base of comparison principle which is important in monotonic methods used for these problems. Recently, Yin [1] developed several results in applications of the comparison principle, especially on nonlocal problems. Earlier works on problems with nonlocal boundary conditions can be found in [2], and some of references can be found in [1, 3]. In the literature, for example [2, 4–6], a restriction on the boundary condition (see (2.1)) of the kind

$$\int_{\Omega} |k(x,y)|\,dy < 1, \quad k(x,y) \geq 0, \tag{AK}$$

where k represents the kernel of the nonlocal boundary condition, is sufficient to obtain the comparison principles. Recent results show that this restriction is not necessary for problems with lower regularity (see [3, Theorem 3.11] for problem with Dirichlet-type nonlocal boundary value). Moreover, in [7], an existence result for classical solutions of a parabolic problem with nonlocal boundary condition was obtained. In [8] we find an illustration of how the boundary kernel influences some results such as those on the eigenvalues problem and on the decay of solutions for evolution equation with a special kernel. In this paper, we give some general comparison results without the restriction

(AK). Then, we use these results to discuss nonlocal boundary problems for a semilinear and a fully nonlinear equations.

2. Case of a semilinear equation

In this section, we are interested in the positivity of solution of the following problem:

$$\begin{gathered} u_t + A(t,x)u \geq 0, \quad t > 0,\ x \in \Omega, \\ (\beta(t,x)\partial_\nu u + \alpha(t,x)u) \geq \int_\Omega k(t,x;y)u(t,y)dy, \quad t > 0,\ x \in \Gamma, \\ u(0,x) = u_0(x), \quad x \in \overline{\Omega}, \end{gathered} \tag{2.1}$$

where

$$A(t,x)u := -\mathbf{a}\nabla^2 u + \vec{b}\,\nabla u + cu \tag{2.2}$$

with $\mathbf{a} := (a_{ij})_{n\times n}$, $\vec{b} := \{b_1,\dots,b_n\}^{\mathrm{T}}$, $((\mathbf{a},\vec{b},c),(\alpha,\beta),k,u_0) \in C([0,T],\mathbb{E})$, $\mathbb{E} := C(\overline{\Omega}, \mathbb{R}^{n^2+n+1}) \times C(\Gamma,\mathbb{R}^2)\times C(\Gamma\times\overline{\Omega},\mathbb{R})\times C^2(\overline{\Omega},\mathbb{R})$,

$$\mathbf{a}\nabla^2 u = \sum_{i,j=1}^{n} a_{ij}\frac{\partial^2 u}{\partial x_i \partial x_j}, \qquad \vec{b}\nabla u = \sum_{i=1}^{n} b_i \frac{\partial u}{\partial x_i}, \tag{2.3}$$

and the elliptic operator A satisfies the following: there exists a $\delta_0 > 0$ such that

$$\xi^{\mathrm{T}}\mathbf{a}\xi \geq \delta_0|\xi|^2, \quad \forall \xi \in \mathbb{R}^n. \tag{2.4}$$

The boundary $\Gamma = \partial\Omega$ of the bounded domain $\Omega \subset \mathbb{R}^n$ is a smooth $(n-1)$-dimensional manifold and ν is the outward unit normal vector to Γ.

We also assume the following hypotheses.

(H*) $\alpha(t,x) \geq 1$, $\beta(t,x) \geq 0$, $k(t,x,y)$, and $u_0(x)$ satisfy the compatibility condition

$$\beta(0,x)\partial_\nu u + \alpha(0,x)u \geq \int_\Omega k(0,x;y)u_0(y)dy \quad \text{on } \Gamma. \tag{2.5}$$

Let $Q_T = (0,T] \times \Omega$. *A (classical) solution* $u(t,x)$ of (2.1) should be in $C^{1,2}(Q_T) \cap C^{0,1}(\overline{Q}_T)$. We have the following result.

Theorem 2.1. *If u_0 is nonnegative, then the solution $u(t,x)$ of problem (2.1) is nonnegative.*

Proof. We can find a positive function $\phi(x) \in C^2(\overline{\Omega})$ such that

$$\begin{gathered} \phi(x) \equiv 1, \qquad \partial_\nu\phi(x) \geq 0 \quad \text{on } \Gamma, \\ \min_{\overline{\Omega}} \phi(x) \geq \varepsilon > 0, \\ \int_\Omega |k(t,x,y)\phi(y)|\,dy < 1, \quad t \in [0,T],\ x \in \Gamma. \end{gathered} \tag{2.6}$$

Let us consider the function $v := u/\phi$. We have

$$\begin{aligned} v_t + \widetilde{A}(t,x)v \geq 0, \quad & t>0,\ x\in\Omega, \\ (\beta\partial_\nu v + \widetilde{\alpha}v) \geq \int_\Omega \widetilde{k}(t,x;y)v(t,y)dy, \quad & t>0,\ x\in\Gamma, \\ v(0,x) = v_0(x) := u_0(x)/\phi(x), \quad & x\in\overline{\Omega}, \end{aligned} \tag{2.7}$$

where

$$\begin{aligned} \widetilde{A}(t,x)v &:= -\mathbf{a}\nabla^2 v + \vec{\widetilde{b}}\nabla v + \widetilde{c}v, \\ \widetilde{\alpha} &:= \beta\partial_\nu\phi + \alpha, \\ \widetilde{k}(t,x;y) &:= k(t,x;y)\phi(y), \end{aligned} \tag{2.8}$$

with

$$\vec{\widetilde{b}} := -\frac{2}{\phi}(\nabla\phi)^{\mathrm{T}}\mathbf{a} + \vec{b}, \qquad \widetilde{c} := -\frac{1}{\phi}\left[\mathbf{a}\nabla^2\phi - \vec{b}\,\nabla\phi\right] + c. \tag{2.9}$$

Without loss of generality, we can suppose that $\widetilde{c} > 0$, otherwise, we replace v by $e^{\lambda t}v$ with a $\lambda > 0$ large enough to have $\lambda + \widetilde{c} > 0$. Following the same approach in [2] and using (2.6) we show that $v(t,x) \geq 0$. In fact, suppose there exists a $(t^*,x^*) \in (0,T]\times\overline{\Omega}$ such that $v(t^*,x^*) < 0$. If $x^*\in\Gamma$ and $v(t^*,x^*) = \min\{v(t,x) : (t,x)\in Q_{t^*}\} < 0$, then using (2.6) we get

$$\begin{aligned} 0 > v(t^*,x^*) &\geq (\widetilde{\alpha}v)|_{x^*} \geq (\beta\partial_\nu v + \widetilde{\alpha}v)|_{x^*} \geq \int_\Omega \widetilde{k}(t^*,x^*;y)v(t^*,y)dy \\ &\geq \int_\Omega |\widetilde{k}(t^*,x^*;y)|\,dy\, v(t^*,x^*) > v(t^*,x^*), \end{aligned} \tag{2.10}$$

which is impossible. And if $x^*\in\Omega$, then using the first inequality in (2.7) we get

$$0 \leq (v_t + \widetilde{A}v)\,|_{(t^*,x^*)} \leq \widetilde{c}(t^*,x^*)v(t^*,x^*) < 0, \tag{2.11}$$

which is also impossible.

Therefore, we conclude that $v(t,x)\geq 0$ on $\overline{Q}_T$ and thus $u\geq 0$ in $\overline{Q}_T$. □

Remark 2.2. The existence of the function ϕ can be obtained by means of the function

$$\phi_{\varepsilon,\vartheta} = \begin{cases} 1, & x\in\Omega,\ \operatorname{dist}(x,\Gamma) < \vartheta, \\ \varepsilon, & x\in\Omega,\ \operatorname{dist}(x,\Gamma) > \vartheta. \end{cases} \quad \text{for small positive numbers } \varepsilon,\ \vartheta. \tag{2.12}$$

We define ϕ by

$$\phi(x) = r^{-n}\int_\Omega \rho\left(\frac{x-y}{r}\right)\phi_{\varepsilon,\vartheta}(y)dy, \tag{2.13}$$

where the constants ε and ϑ are small enough so that (2.6) holds. Here $r = \vartheta/4$ and

$$\rho(x) = \begin{cases} \left[\int_{|y|\le 1} e^{1/(|y|^2-1)}dy\right]^{-1} \cdot e^{1/(|x|^2-1)}, & |x| < 1, \\ 0, & |x| \ge 1. \end{cases} \tag{2.14}$$

It is obvious that

$$\varepsilon \le \phi(x) \le 1, \quad \text{for } x \in \Omega, \qquad \partial_\nu \phi|_\Gamma \equiv 0. \tag{2.15}$$

Let $M = \sup\{|k(t,x,y)| : (t,x,y) \in [0,T] \times \partial\Omega \times \overline{\Omega}\}$. If θ and ε satisfy $M(|\Gamma|(5\theta/4) + \varepsilon|\Omega|) < 1$, where $|\Omega|$ denotes the measure of Ω, then (2.6) holds.

More generally, if $\alpha \ge \alpha_0 > 0$, we can get a similar result replacing k by $k/(\alpha_0)$.

In addition, for some special domains Ω, we can construct ϕ according to the geometry of Ω as in the following example.

Example 2.3. Let us consider the following problem on $B_R := \{x \in \mathbb{R}^n, |x| < R\}$:

$$\begin{gathered} u_t - \Delta u = 0, \quad x \in B_R,\ t > 0, \\ \partial_\nu u + \alpha u = k\int_{B_R} u(t,y)dy, \quad |x| = R,\ t > 0, \\ u(0,x) = u_0(x), \quad x \in \overline{B}_R, \end{gathered} \tag{2.16}$$

with the corresponding compatibility condition. In (2.16), α and k are constants. Then, ϕ can be chosen as the following:

$$\phi(x) = \begin{cases} \varepsilon + (1-\varepsilon)(R^2 - \vartheta^2)^{-4}(|x|^2 - \vartheta^2)^4, & R - \vartheta \le |x| \le R, \\ \varepsilon, & |x| \le R - \vartheta \end{cases} \tag{2.17}$$

with ε and ϑ verifying

$$\partial_\nu \phi = \frac{8R(1-\varepsilon)}{R^2 - \vartheta^2} \ge 0, \qquad |k|((\varepsilon - 1)|B_{R-\vartheta}| + |B_R|) < 1. \tag{2.18}$$

Remark 2.4. The condition $\alpha(t,x) \ge 1$ in (H*) is not necessary. We can just assume that $\alpha > 0$ on $[0,T] \times \Gamma$ and we replace β and k, respectively, by β/α and k/α. This means that we can prove Theorem 2.1 without assuming $\alpha(t,x) \ge 1$.

Let us now consider the decay behavior of the following control problem:

$$\begin{gathered} u_t + A(x)u + \omega(x)u = 0, \quad t > 0,\ x \in \Omega, \\ \beta(x)\partial_\nu u + \alpha(x)u = \int_\Omega k(x;y)u(t,y)dy, \quad t > 0,\ x \in \Gamma, \\ u(0,x) = u_0(x), \quad x \in \overline{\Omega}, \end{gathered} \tag{P_ω}$$

where A is an elliptic operator defined as in (2.2) with $((\mathbf{a},\vec{b},c),(\alpha,\beta),k,u_0) \in \mathbb{E}$. Following the same approach as in [4], we obtain that the C-norm $U(t) := \max_{\overline{\Omega}}|u(t,x)|$, u being the classical solution of problem (P_0) ($\omega \equiv 0$ in (P_ω) decays to zero exponentially provided that $\int_\Omega |k(x;y)|dy < 1$).

For any $k(x,y) \in C(\Gamma \times \overline{\Omega})$, we can find ω and ϕ such that

$$\tilde{c} + \omega \geq 0, \qquad \int_\Omega |k(x;y)\phi(y)|\,dy < 1, \tag{2.19}$$

where $\tilde{c}$ and ϕ are defined in (2.6) and (2.9), and the functions β, α, and k also satisfy some corresponding conditions as in (H*). Hence, by using the same method as in [4], we have the following theorem.

Theorem 2.5. *For any fixed $k(x,y)$, there exist a function ω and positive constants M and λ such that the solution u of problem (P_ω) satisfies*

$$\|u(t,\cdot)\|_{C(\overline{\Omega})} \leq Me^{-\lambda t}, \quad \forall t \geq 0. \tag{2.20}$$

We can look at the following one-dimensional example.

Example 2.6. Let $\Omega = [a, 3\pi - a]$ with $a \in (0,\pi/2)$. The following problem

$$\begin{gathered} u_t - u_{xx} - u + \omega u = 0, \quad \text{in } Q_T, \\ u(t,a) = u(t,3\pi - a) = \frac{1}{2}\tan a \int_a^{3\pi-a} u(t,y)\,dy, \\ u(0,x) = \sin x \end{gathered} \tag{E_ω}$$

has a solution $u(t,x) \equiv \sin x$ when $\omega = 0$. But when $\omega = 1$, (E_1) has a decay solution $u = e^{-t}\sin x$. We can see that $\int_\Omega k\,dy = ((3\pi - 2a)/2)\tan a > 1$ when $a \in (\arctan 1/\pi, \pi/2)$.

We propose to use a positivity result of Theorem 2.1 in order to establish a comparison principle for a semilinear parabolic equation with nonlinear nonlocal boundary condition. Let us consider the following problem:

$$\begin{gathered} u_t - \mathbf{a}\nabla^2 u = f(t,x,u,\nabla u) \quad \text{in } Q_T, \\ \beta\partial_\nu u + u = \int_\Omega k(t,x,y;u(t,y))\,dy \quad \text{on } (0,T)\times\Gamma, \\ u(0,x) = u_0(x), \quad x \in \Omega, \end{gathered} \tag{SP}$$

where $\mathbf{a}$, β, and u_0 satisfy the hypotheses above, and f and k satisfying the following hypotheses:

(i) $k(\cdot;u) \in C([0,T]\times\Gamma\times\overline{\Omega})$ and $k(t,x,y;\cdot) \in C^1(\mathbb{R})$;

(ii) f satisfies the following Lipschitz condition: there exists $L_1, L_2 > 0$ such that

$$\begin{gathered} f(t,x,u,P) - f(t,x,v,P) \leq L_1(u - v), \quad \text{if } u \geq v; \\ |f(t,x,u,P) - f(t,x,u,Q)| \leq L_2|P - Q|. \end{gathered} \tag{2.21}$$

A function $u(t,x) \in C^{1,2}(Q_T) \cap C^{0,1}(\overline{Q}_T)$ is called an *upper solution* of (SP) on $\overline{Q}_T$ if it satisfies

$$\begin{gathered} u_t - \mathbf{a}\nabla^2 u \geq f(t,x,u,\nabla u) \quad \text{in } Q_T, \\ \beta\partial_\nu u + u \geq \int_\Omega k(t,x,y;u(t,y))\,dy \quad \text{on } (0,T)\times\Gamma, \\ u(0,x) \geq u_0(x), \quad x \in \Omega. \end{gathered} \tag{2.22}$$

A *lower solution* is defined analogously by reversing the inequalities in (2.22). A *solution* u of problem (SP) means that u is both an upper and a lower solutions.

Theorem 2.7. *If u,v are, respectively, an upper and a lower solutions of the problem (SP), then $u \geq v$ for all $(t,x) \in \overline{Q}_T$.*

Proof. Let us consider the function $w(t,x) = u(t,x) - v(t,x)$. This function verifies

$$\begin{gathered} w_t - \mathbf{a}\nabla^2 w \geq f(t,x,u,\nabla u) - f(t,x,v,\nabla v) \quad \text{in } Q_T, \\ \beta\partial_\nu w + w \geq \int_\Omega k_u(t,x,y;\xi(t,y))w(t,y)\,dy \quad \text{on } (0,T)\times\Gamma, \\ w(0,x) = u_0(x) - v_0(x) \geq 0, \quad x \in \Omega \end{gathered} \tag{2.23}$$

with ξ situated between u and v.

We note that the right-hand side of the first inequality in (2.23) depends on u and ∇u, thus, Theorem 2.1 cannot be applied directly. We introduce

$$w(t,x) = V(t,x)\phi(x)e^{\lambda t}, \tag{2.24}$$

where $\phi(x)$ satisfies (2.6) with $k(t,x,y)$ replaced by $k_u(t,x,y,\xi(t,y))$ and

$$\lambda > L_1 + \max_{\overline{\Omega}}\left\{\frac{L_2\,|\,\nabla\phi(x)\,|\, + \mathbf{a}\nabla^2\phi(x)}{\phi(x)}\right\}. \tag{2.25}$$

If there is a point $(t,x) \in (0,T]\times\overline{\Omega}$ such that $w(t,x) < 0$, then V will attain its negative minimum at some point (t_1,x_1) with

$$V(t_1,x_1) < 0, \qquad V_t(t_1,x_1) \leq 0, \qquad \nabla V(t_1,x_1) = 0. \tag{2.26}$$

Hence, using the hypotheses on f, we obtain a contradiction since we have

$$0 \geq V_t \geq -\left(\lambda - L_1 - \frac{L_2|\nabla\phi|}{\phi} - \frac{\mathbf{a}\nabla^2\phi}{\phi}\right)V > 0 \quad \text{at } (t_1,x_1) \text{ if } x_1 \in \Omega. \tag{2.27}$$

We obtain also a contradiction if $x_1 \in \Gamma$ since we have

$$\int_\Omega |\,k_u(t_1,x_1,y,\xi(t_1,y))\,|\,\phi(y)\,dy < 1. \tag{2.28}$$

We thus conclude that $V \geq 0$, and therefore, $w(t,x) \geq 0$ on $\overline{Q}_T$. □

A similar result can be obtained for parabolic systems with changing-sign kernels. Note that in [9, Example 2.1], the kernel K_{ij} appearing in the boundary condition is assumed to be positive.

Remark 2.8. From the above discussion, the result of Theorem 2.7 holds true if we just assume k and f to be locally (one side) Lipschitz continuous, respectively, on u and ∇u, that is, $k(\cdot,u) \in C([0,T]\times\Gamma\times\overline{\Omega})$ for any fixed u and there exists $L, L_1, L_2 > 0$ such that

$$\left.\begin{aligned} &|k(t,x,y,u)-k(t,x,y,v)| \le L(\rho)|u-v|;\\ &f(t,x,u,P)-f(t,x,v,P) \le L_1(\rho)(u-v), \quad \text{if } u \ge v;\\ &|f(t,x,u,P)-f(t,x,u,Q)| \le L_2(\rho)|P-Q| \end{aligned}\right\} \text{ when } |u|,|v| \le \rho. \tag{2.29}$$

The uniqueness of the solution of problem (SP) is a direct consequence of Theorem 2.7. Using the upper and lower solutions, some existence theorems of the solutions for problem (SP) will be obtained by monotonicity methods (see [2]). We can also discuss the quadric convergence of iterative series constructed using upper and lower solutions (see [10]). Here we do not give more details about that.

3. A fully nonlinear equation

Let us consider a general nonlinear parabolic equation with nonlinear and nonlocal boundary conditions

$$\begin{gathered} u_t = f(t,x,u,\nabla u,\nabla^2 u) \quad \text{in } Q_T,\\ \beta\partial_\nu u + u = \int_\Omega k(t,x,y;u)\,dy \quad \text{on } (0,T]\times\Gamma,\\ u(0,x) = u_0(x) \quad \text{in } \Omega, \end{gathered} \tag{Pf}$$

where $f \in C(\overline{Q}_T \times \mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^{n^2},\mathbb{R})$, $\nabla u = (u_{x_1},\dots,u_{x_n})$, and $\nabla^2 u = (u_{x_1x_1},u_{x_1x_2},\dots,u_{x_nx_n})$.

In order to establish the comparison principle, we give a definition of elliptic function. We say that $f \in C(\overline{Q}_T\times\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^{n^2},\mathbb{R})$ is *elliptic* at point (t_0,x_0) if for any u, P, R, S with $R=(R_{ij})_{n\times n}$, $S=(S_{ij})_{n\times n}$, verifying $\Lambda^{\mathrm{T}}(R-S)\Lambda \ge 0$ for any vector $\Lambda\in\mathbb{R}^n$, we have $f(t_0,x_0,u,P,R) \ge f(t_0,x_0,u,P,S)$. If f is elliptic for every $(t,x)\in Q_T$, then f is said to be *elliptic* in Q_T. In the remainder of this paper, we assume f to be elliptic in Q_T.

A function $u(t,x) \in C^{1,2}(Q_T)\cap C^{0,1}(\overline{Q}_T)$ is said to be an upper solution (resp., a lower solution) of problem (Pf) on $\overline{Q}_T$ if u satisfies the following system:

$$\begin{gathered} u_t \ge (\le) f(t,x,u,\nabla u,\nabla^2 u) \quad \text{in } Q_T,\\ \beta\partial_\nu u + u \ge (\le) \int_\Omega k(t,x,y;u)\,dy \quad \text{on } (0,T]\times\Gamma,\\ u(0,x) \ge (\le) u_0(x) \quad \text{in } \Omega. \end{gathered} \tag{3.1}$$

Assuming β to be positive, k to be continuous, and there exists a nonnegative $C([0,T]\times\Gamma\times\overline{\Omega})$-function L_2 verifying

$$k(t,x,y,u)-k(t,x,y,v)\geq L_2(t,x,y)(u-v) \quad \text{if } u\geq v, \tag{3.2}$$

we get the following theorem.

Theorem 3.1. *Let u and v be, respectively, an upper and lower solutions of problem (Pf). Suppose $u(0,x)>v(0,x)$ and one of the first two inequalities in (3.1) to be strict. Then $u(t,x)>v(t,x)$ on $\overline{Q}_T$.*

Proof. Let us consider the function $U(t,x)=u(t,x)-v(t,x)$. If the conclusion was not true, then the initial condition implies that $U(t,x)>0$ for some $t>0$ and there exists $(t_1,x_1)\in\overline{Q}_T$ such that $U(t_1,x_1)=0$. We can assume that (t_1,x_1) is the first nonnegative maximum point, that is,

$$U(t,x)>0, \quad \forall t<t_1,\ x\in\overline{\Omega}. \tag{3.3}$$

We have that $(t_1,x_1)\notin Q_T$. In fact, if $(t_1,x_1)\in Q_T$, then we have

$$U_t\leq 0, \quad \nabla U=0, \quad \Lambda^{\mathrm{T}}\left(U_{x_ix_j}\right)_{n\times n}\Lambda\geq 0 \quad \text{at } (t_1,x_1). \tag{3.4}$$

Using the ellipticity of f, we obtain that

$$U_t(t_1,x_1)>f\left(t_1,x_1,u,\nabla u,\nabla^2 u\right)-f\left(t_1,x_1,v,\nabla v,\nabla^2 v\right)\geq 0, \tag{3.5}$$

which is in contradiction with (3.4). Hence, $U(t,x)>0$ in Q_{t_1}. We have also $(t_1,x_1)\notin(0,T]\times\Gamma$. Otherwise,

$$0\geq\beta\partial_\nu U+U\geq\int_\Omega L_2U\,dy>0, \quad \text{at } (t_1,x_1), \tag{3.6}$$

which leads to a contradiction again.

Finally, we conclude that $U(t,x)>0$, that is, $u(t,x)>v(t,x)$ on $\overline{Q}_T$. □

Let us now assume β to be positive, f satisfying locally one-side Lipschitz conditions, that is, for $|u|\leq\rho$ and $|v|\leq\rho$, there exists a constant $L_1(\rho)$ such that

$$f(t,x,u,P,R)-f(t,x,v,P,R)\leq L_1(u-v), \quad \text{if } u\geq v. \tag{3.7}$$

We also assume k to be continuous and there exist two nonnegative $C([0,T]\times\Gamma\times\overline{\Omega})$-functions, L_2 and $\overline{L}_2$, such that

$$L_2(t,x,y)(u-v)\leq k((t,x,y);u)-k((t,x,y);v)\leq\overline{L}_2(t,x,y)(u-v), \quad \text{if } u\geq v. \tag{3.8}$$

Then, for $\varepsilon > 0$, it is obvious that

$$\left(\varepsilon e^{\delta t}\right)_t = \delta\varepsilon e^{\delta t} > f\left(t,x,u+\varepsilon e^{\delta t},\nabla\left(u+\varepsilon e^{\delta t}\right),\nabla^2\left(u+\varepsilon e^{\delta t}\right)\right) - f\left(t,x,u,\nabla u,\nabla^2 u\right) \tag{3.9}$$

whenever $\delta > L_1$.

Let $\widetilde{u} = u + \varepsilon e^{\delta t}$ with $\delta > L_1$ and suppose $\overline{L}_2\,|\Omega| < 1$, then

$$\begin{gathered}
\widetilde{u}_t = u_t + \delta\varepsilon e^{\delta t} > f\left(t,x,\widetilde{u},\nabla\widetilde{u},\nabla^2\widetilde{u}\right), \quad \text{in } Q_T, \\
\beta\partial_\nu\widetilde{u} + \widetilde{u} \geq \varepsilon e^{\delta t} + \int_\Omega k(t,x,y;u)dy > \int_\Omega k(t,x,y;\widetilde{u})dy, \quad \text{on } (0,T]\times\Gamma, \\
\widetilde{u}(0,x) = u(0,x) + \varepsilon, \quad \text{in } \Omega.
\end{gathered} \tag{3.10}$$

This means that $\widetilde{u}$ is a (strict) upper solution as well as u. Letting $\varepsilon \to 0^+$ and using Theorem 3.1, we obtain the following corollary.

Corollary 3.2. *Under the above assumptions, if u and v are, respectively, the upper and the lower solutions of problem (Pf) and if $\overline{L}_2|\Omega| < 1$, then $u(t,x) \geq v(t,x)$ on $\overline{Q}_T$.*

The uniqueness of the solution for problem (Pf) can be easily obtained and an extension to a fully nonlinear system can be derived.

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