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Research Article

Global Behavior of the Components for the Second Order *m*-Point Boundary Value Problems

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We consider the nonlinear eigenvalue problems $u'' + rf(u) = 0$, $0 < t < 1$, $u(0) = 0$, $u(1) = \sum_{i=1}^{m-2}\alpha_i u(\eta_i)$, where $m \geq 3$, $\eta_i \in (0,1)$, and $\alpha_i > 0$ for $i = 1,\ldots,m-2$, with $\sum_{i=1}^{m-2}\alpha_i < 1$; $r \in \mathbb{R}$; $f \in C^1(\mathbb{R},\mathbb{R})$. There exist two constants $s_2 < 0 < s_1$ such that $f(s_1) = f(s_2) = f(0) = 0$ and $f_0 := \lim_{u\to 0}(f(u)/u) \in (0,\infty)$, $f_\infty := \lim_{|u|\to\infty}(f(u)/u) \in (0,\infty)$. Using the global bifurcation techniques, we study the global behavior of the components of nodal solutions of the above problems.

1. Introduction

In [1], Ma and Thompson were concerned with determining values of real parameter r, for which there exist nodal solutions of the boundary value problems:

$$\begin{gathered} u'' + ra(t)f(u) = 0, \quad 0 < t < 1, \\ u(0) = u(1) = 0, \end{gathered} \tag{1.1}$$

where a and f satisfy the following assumptions:

(H1) $f \in C(\mathbb{R},\mathbb{R})$ with $sf(s) > 0$ for $s \neq 0$;

(H2) there exist f_0, $f_\infty \in (0,\infty)$ such that

$$f_0 = \lim_{|s|\to 0}\frac{f(s)}{s}, \qquad f_\infty = \lim_{|s|\to\infty}\frac{f(s)}{s}; \tag{1.2}$$

(H3) $a : [0,1] \to [0,\infty)$ is continuous and $a(t) \not\equiv 0$ on any subinterval of $[0,1]$.

Using Rabinowitz global bifurcation theorem, Ma and Thompson established the following theorem.

Theorem 1.1. *Let (H1), (H2), and (H3) hold. Assume that for some $k \in \mathbb{N}$, either*

$$\frac{\lambda_k}{f_\infty} < r < \frac{\lambda_k}{f_0} \tag{1.3}$$

or

$$\frac{\lambda_k}{f_0} < r < \frac{\lambda_k}{f_\infty}. \tag{1.4}$$

Then (1.1) *have two solutions u_k^+ and u_k^- such that u_k^+ has exactly $k-1$ zeros in $(0,1)$ and is positive near* 0, *and u_k^- has exactly $k-1$ zeros in $(0,1)$ and is negative near* 0. *In* (1.3) *and* (1.4), *λ_k is the kth eigenvalue of*

$$\varphi'' + \lambda a(t)\varphi = 0, \quad 0 < t < 1, \qquad \varphi(0) = \varphi(1) = 0. \tag{1.5}$$

Recently, Ma [2] extended this result and studied the global behavior of the components of nodal solutions of (1.1) under the following conditions:

(H1′) $f \in C(\mathbb{R}, \mathbb{R})$ and there exist two constants $s_2 < 0 < s_1$, such that $f(s_1) = f(s_2) = f(0) = 0$ and $sf(s) > 0$ for $s \in \mathbb{R} \setminus \{0, s_1, s_2\}$;

(H4) f satisfies Lipschitz condition in $[s_2, s_1]$.

Using Rabinowitz global bifurcation theorem, Ma established the following theorem.

Theorem 1.2. *Let* (*H*1′)*, (H2), (H3), and (H4) hold. Assume that for some $k \in \mathbb{N}$,*

$$\frac{\lambda_k}{f_\infty} < \frac{\lambda_k}{f_0}. \tag{1.6}$$

Then

(i) *if $r \in (\lambda_k/f_\infty, \lambda_k/f_0]$, then* (1.1) *have at least two solutions $u_{k,\infty}^{\pm}$, such that $u_{k,\infty}^+$ has exactly $k-1$ zeros in $(0,1)$ and is positive near* 0, *and $u_{k,\infty}^-$ has exactly $k-1$ zeros in $(0,1)$ and is negative near* 0,

(ii) *if $r \in (\lambda_k/f_0, \infty)$, then* (1.1) *have at least four solutions $u_{k,\infty}^{\pm}$ and $u_{k,0}^{\pm}$, such that $u_{k,\infty}^+$ (resp., $u_{k,0}^+$) has exactly $k-1$ zeros in $(0,1)$ and is positive near* 0; *$u_{k,\infty}^-$ (resp., $u_{k,0}^-$) has exactly $k-1$ zeros in $(0,1)$ and is negative near* 0.

Remark 1.3. Let (H1′), (H2), (H3), and (H4) hold. Assume that for some $k \in \mathbb{N}$, $\lambda_k/f_0 < \lambda_k/f_\infty$. Similar results to Theorem 1.2 have also been obtained.

Making a comparison between the above two theorems, we see that as f has two zeros $s_1, s_2 : s_2 < 0 < s_1$, the bifurcation structure of the nodal solutions of (1.1) becomes more complicated: two new nodal solutions are obtained when $r > \max\{\lambda_k/f_0, \lambda_k/f_\infty\}$.

In [3], Ma and O'Regan established some existence results (which are similar to Theorem 1.1) of the nodal solutions of the m-point boundary value problems

$$\begin{gathered} u'' + f(u) = 0, \quad 0 < t < 1, \\ u(0) = 0, \qquad u(1) = \sum_{i=1}^{m-2} \alpha_i u(\eta_i) \end{gathered} \tag{1.7}$$

under the following condition:

(H1″) $f \in C^1(\mathbb{R}, \mathbb{R})$ with $sf(s) > 0$ for $s \neq 0$.

Remark 1.4. For other results about the existence of nodal solution of multipoint boundary value problems, we can see [4–7].

Of course an interesting question is, as for m-point boundary value problems, when f possesses zeros in $\mathbb{R} \setminus \{0\}$, whether we can obtain some new results which are similar to Theorem 1.2.

We consider the eigenvalue problems

$$u'' + rf(u) = 0, \quad 0 < t < 1, \tag{1.8}$$

$$u(0) = 0, \quad u(1) = \sum_{i=1}^{m-2} \alpha_i u(\eta_i), \tag{1.9}$$

where $m \geq 3$, $\eta_i \in (0,1)$, and $\alpha_i > 0$ for $i = 1, \dots, m-2$. Also using the global bifurcation techniques, we study the global behavior of the components of nodal solutions of (1.8), (1.9) and give a positive answer to the above question. However, when m-point boundary value condition (1.9) is concerned, the discussion is more difficult since the problem is nonsymmetric and the corresponding operator is disconjugate.

In the following paper, we assume that

(H0) $\alpha_i > 0$ for $i = 1, \dots, m-2$, with $0 < \sum_{i=1}^{m-2} \alpha_i < 1$;

($\widetilde{\text{H1}}$) $f \in C^1(\mathbb{R}, \mathbb{R})$ and there exist two constants $s_2 < 0 < s_1$, such that $f(s_1) = f(s_2) = f(0) = 0$;

(H2) there exist $f_0, f_\infty \in (0, \infty)$ such that

$$f_0 = \lim_{|s| \to 0} \frac{f(s)}{s}, \qquad f_\infty = \lim_{|s| \to \infty} \frac{f(s)}{s}. \tag{1.10}$$

The rest of the paper is organized as follows. Section 2 contains preliminary definitions and some eigenvalue results of corresponding linear problems of (1.8), (1.9). In Section 3, we give two Rabinowize-type global bifurcation theorems. Finally, in Section 4, we consider two bifurcation problems related to (1.8), (1.9), and use the global bifurcation theorems from Section 3 to analyze the global behavior of the components of nodal solutions of (1.8), (1.9).

2. Preliminary definitions and eigenvalues of corresponding linear problems

Let $Y = C[0,1]$ with the norm

$$\|u\|_\infty = \max_{t \in [0,1]} |u(t)|. \tag{2.1}$$

Let

$$X = \left\{ u \in C^1[0,1] \mid u(0) = 0,\ u(1) = \sum_{i=1}^{m-2} \alpha_i u(\eta_i) \right\},$$
$$E = \left\{ u \in C^2[0,1] \mid u(0) = 0,\ u(1) = \sum_{i=1}^{m-2} \alpha_i u(\eta_i) \right\} \tag{2.2}$$

with the norm

$$\|u\|_X = \max\{\|u\|_\infty, \|u'\|_\infty\}, \qquad \|u\|_E = \max\{\|u\|_\infty, \|u'\|_\infty, \|u''\|_\infty\}, \tag{2.3}$$

respectively. Define $L : E \to Y$ by setting

$$Lu := -u'', \quad u \in E. \tag{2.4}$$

Then L has a bounded inverse $L^{-1} : Y \to E$ and the restriction of L^{-1} to X, that is, $L^{-1} : X \to X$ is a compact and continuous operator, see [3, 4, 8].

Let $\mathbb{E} = \mathbb{R} \times E$ under the product topology. As in [9], we add the points $\{(\lambda, \infty) \mid \lambda \in \mathbb{R}\}$ to our space $\mathbb{E}$. For any C^1 function u, if $u(x_0) = 0$, then x_0 is a simple zero of u if $u'(x_0) \neq 0$. For any integer $k \geq 1$ and any $\nu \in \{\pm\}$, define sets $S_k^\nu, T_k^\nu \subset C^2[0,1]$ consisting of functions $u \in C^2[0,1]$ satisfying the following conditions:

S_k^ν:

(i) $u(0) = 0$, $\nu u'(0) > 0$;

(ii) u has only simple zeros in $[0,1]$ and has exactly $k-1$ zeros in $(0,1)$;

T_k^ν:

(i) $u(0) = 0$, $\nu u'(0) > 0$, and $u'(1) \neq 0$;

(ii) u' has only simple zeros in $(0,1)$ and has exactly k zeros in $(0,1)$;

(iii) u has a zero strictly between each two consecutive zeros of u'.

Remark 2.1. Obviously, if $u \in T_k^\nu$, then $u \in S_k^\nu$ or $u \in S_{k+1}^\nu$. The sets T_k^ν are open in E and disjoint.

Remark 2.2. The nodal properties of solutions of nonlinear Sturm-Liouville problems with separated boundary conditions are usually described in terms of sets similar to S_k^ν, see [1, 2, 5, 9–11]. However, Rynne [4] stated that T_k^ν are more appropriate than S_k^ν when the multipoint boundary condition (1.9) is considered.

Next, we consider the eigenvalues of the linear problem

$$Lu = \lambda u, \quad u \in E. \tag{2.5}$$

We call the set of eigenvalues of (2.5) the spectrum of L, and denote it by $\sigma(L)$. The following lemmas can be found in [3, 4, 12].

Lemma 2.3. *Let (H0) hold. The spectrum $\sigma(L)$ consists of a strictly increasing positive sequence of eigenvalues λ_k, $k = 1, 2, \ldots$, with corresponding eigenfunctions $\varphi_k(x) = \sin(\sqrt{\lambda_k}x)$. In addition,*

(i) $\lim_{k\to\infty}\lambda_k = \infty$;

(ii) $\varphi_k \in T_k^+$, *for each* $k \geq 1$, *and* φ_1 *is strictly positive on* $(0,1)$.

We can regard the inverse operator $L^{-1} : Y \to E$ as an operator $L^{-1} : Y \to Y$. In this setting, each λ_k, $k = 1, 2, \ldots$, is a characteristic value of L^{-1}, with algebraic multiplicity defined to be $\dim \bigcup_{j=1}^{\infty} N((I - \lambda_k L^{-1})^j)$, where N denotes null-space and I is the identity on Y.

Lemma 2.4. *Let (H0) hold. For each $k \geq 1$, the algebraic multiplicity of the characteristic value λ_k, $k = 1, 2, \ldots$, of $L^{-1} : Y \to Y$ is equal to 1.*

3. Global bifurcation

Let $g \in C^1(\mathbb{R}, \mathbb{R})$ and satisfy

$$g(0) = g'(0) = 0. \tag{3.1}$$

Consider the following bifurcation problem:

$$Lu = \mu u + g(u), \quad (\mu, u) \in \mathbb{R} \times X. \tag{3.2}$$

Obviously, $u \equiv 0$ is a trivial solution of (3.2) for any $\mu \in \mathbb{R}$. About nontrivial solutions of (3.2), we have the following.

Lemma 3.1 (see [4, Proposition 4.1]). *Let (H0) hold. If $(\mu, u) \in \mathbb{E}$ is a nontrivial solution of* (3.2), *then $u \in T_k^\nu$ for some k, ν.*

Remark 3.2. From Lemmas 2.3 and 3.1, we can see that T_k^ν are more effectual than the set S_k^ν when the multipoint boundary condition (1.9) is considered. In fact, eigenfunctions $\varphi_k(x) = \sin(\sqrt{\lambda_k}x)$, $k = 1, 2, \ldots$, of (2.5) do not necessarily belong to S_k^+. In [3, 4], there were some special examples to show this problem.

Also, in [4], Rynne obtained the following Rabinowitz-type global bifurcation result for (3.2).

Lemma 3.3 (see [4, Theorem 4.2]). *Let (H0) hold. For each $k \geq 1$ and ν, there exists a continuum $\mathcal{C}_k^\nu \subset \mathbb{E}$ of solution of* (3.2) *with the following properties:*

- (1°) $(\lambda_k, 0) \in \mathcal{C}_k^\nu$;
- (2°) $\mathcal{C}_k^\nu \setminus \{(\lambda_k, 0)\} \subset \mathbb{R} \times T_k^\nu$;
- (3°) $\mathcal{C}_k^\nu$ *is unbounded in* $\mathbb{E}$.

Now, we consider another bifurcation problem

$$Lu = \mu u + h(u), \quad (\mu, u) \in \mathbb{R} \times X, \tag{3.3}$$

where we suppose that $h \in C^1(\mathbb{R}, \mathbb{R})$ and satisfy

$$\lim_{|x| \to \infty} \frac{h(x)}{x} = 0. \tag{3.4}$$

Take $\Lambda \subset \mathbb{R}$ as an interval such that $\Lambda \cap \{\lambda_j \mid j \in \mathbb{N}\} = \{\lambda_k\}$ and $\mathcal{M}$ as a neighborhood of (λ_k, ∞) whose projection on $\mathbb{R}$ lies in Λ and whose projection on E is bounded away from 0.

Lemma 3.4. *Let (H0) and* (3.4) *hold. For each $k \geq 1$ and ν, there exists a continuum $\mathfrak{D}_k^\nu \subset \mathbb{E}$ of solution of* (3.3) *which meets (λ_k, ∞) and either*

- (1°) $\mathfrak{D}_k^\nu \setminus \mathcal{M}$ *is bounded in* $\mathbb{E}$ *in which case* $\mathfrak{D}_k^\nu \setminus \mathcal{M}$ *meets* $\{(\lambda, 0) \mid \lambda \in \mathbb{R}\}$ *or*
- (2°) $\mathfrak{D}_k^\nu \setminus \mathcal{M}$ *is unbounded in* $\mathbb{E}$.

Moreover, if (2^o) *occurs and* $\mathfrak{D}_k^\nu \setminus \mathcal{M}$ *has a bounded projection on* $\mathbb{R}$*, then* $\mathfrak{D}_k^\nu \setminus \mathcal{M}$ *meets* $(\widehat{\mu}, \infty)$*, where* $\widehat{\mu} \in \{\lambda_j \mid j \in \mathbb{N}\}$ *with* $\widehat{\mu} \neq \lambda_k$.

In every case, there exists a neighborhood $\mathcal{O} \subset \mathcal{M}$ *of* (λ_k, ∞) *such that* $(\mu, u) \in \mathfrak{D}_k^\nu \cap \mathcal{O}$ *and* $(\mu, u) \neq (\lambda_k, \infty)$ *implies* $(\mu, u) \in \mathbb{R} \times T_k^\nu$.

Remark 3.5. A continuum $\mathfrak{D}_k^\nu \subset \mathbb{E}$ of solution of (3.3) meets (λ_k, ∞) which means that there exists a sequence $\{(\lambda_n, u_n)\} \subset \mathfrak{D}_k^\nu$ such that $\|u_n\|_E \to \infty$ and $\lambda_n \to \lambda_k$.

Proof. Obviously, (3.3) is equivalent to the problem

$$u = \mu L^{-1} u + L^{-1} h(u), \quad (\mu, u) \in \mathbb{R} \times \mathrm{X}. \tag{3.5}$$

Note that $L^{-1} : X \to X$ is a compact and continuous linear operator. In addition, the mapping $u \to L^{-1} h(u)$ is continuous and compact, and satisfies $L^{-1} h(u) = o(\|u\|_X)$ at $u = \infty$; moreover, $\|u\|_X^2 L^{-1} h(u / \|u\|_X^2)$ is compact (similar proofs can be found in [9]). Hence, the problem (3.3) is of the form considered in [9], and satisfies the general hypotheses imposed in that paper. Then by [9, Theorem 1.6 and Corollary 1.8] together with Lemmas 2.3 and 2.4 in Section 2, there exists a continuum $\mathfrak{D}_k^\nu \subset \mathbb{R} \times \mathrm{X}$ of solutions of (3.3) which meets (λ_k, ∞) and either

(1^o) $\mathfrak{D}_k^\nu \setminus \mathcal{M}$ is bounded in $\mathbb{R} \times X$ in which case $\mathfrak{D}_k^\nu \setminus \mathcal{M}$ meets $\{(\lambda, 0) \mid \lambda \in \mathbb{R}\}$ or

(2^o) $\mathfrak{D}_k^\nu \setminus \mathcal{M}$ is unbounded in $\mathbb{R} \times \mathrm{X}$.

Moreover, if *(2^o)* occurs and $\mathfrak{D}_k^\nu \setminus \mathcal{M}$ has a bounded projection on $\mathbb{R}$, then $\mathfrak{D}_k^\nu \setminus \mathcal{M}$ meets $(\widehat{\mu}, \infty)$ where $\widehat{\mu} \in \{\lambda_j \mid j \in \mathbb{N}\}$ with $\widehat{\mu} \neq \lambda_k$.

In every case, there exists a neighborhood $\mathcal{O} \subset \mathcal{M}$ of (λ_k, ∞) such that $(\mu, u) \in \mathfrak{D}_k^\nu \cap \mathcal{O}$ and $(\mu, u) \neq (\lambda_k, \infty)$ implies $(\mu, u) \in \mathbb{R} \times T_k^\nu$.

On the other hand, by (3.5) and the continuity of the operator $L^{-1} : Y \to E$, the set $\mathfrak{D}_k^\nu$ lies in $\mathbb{E}$ and the injection $\mathfrak{D}_k^\nu \to \mathbb{E}$ is continuous. Thus, $\mathfrak{D}_k^\nu$ is also a continuum in $\mathbb{E}$ and the above properties hold in $\mathbb{E}$. □

Now, we assume that

$$h(0) = 0. \tag{3.6}$$

Lemma 3.6. *Let (H0) and* (3.6) *hold. If* $(\mu, u) \in \mathbb{E}$ *is a nontrivial solution of* (3.3)*, then* $u \in T_k^\nu$ *for some* k, ν.

Proof. The proof of Lemma 3.6 is similar to the proof of Lemma 3.1 ([4, Proposition 4.1]); we omit it. □

Remark 3.7. If (3.6) holds, Lemma 3.6 guarantees that $\mathfrak{D}_k^\nu$ in Lemma 3.4 is a component of solutions of (3.3) in T_k^ν which meets (λ_k, ∞). Otherwise, if there exist $(\eta_1, y_1) \in \mathfrak{D}_k^\nu \cap T_k^\nu$ and $(\eta_2, y_2) \in \mathfrak{D}_k^\nu \cap T_h^\nu$ for some $k \neq h \in \mathbb{N}$, then by the connectivity of $\mathfrak{D}_k^\nu$, there exists $(\eta_*, y_*) \in \mathfrak{D}_k^\nu$ such that y'_* has a multiple zero point in $(0, 1)$. However, this contradicts Lemma 3.6. Hence, if (3.6) holds and $\mathfrak{D}_k^\nu$ in Lemma 3.4 is unbounded in $\mathbb{R} \times E$, then $\mathfrak{D}_k^\nu$ has unbounded projection on $\mathbb{R}$.

4. Statement of main results

We return to the problem (1.8), (1.9). Let $(\widetilde{H1})$, (H2) hold and let ζ, $\xi \in C^1(\mathbb{R},\mathbb{R})$ be such that

$$f(u) = f_0 u + \zeta(u), \qquad f(u) = f_\infty u + \xi(u). \tag{4.1}$$

Clearly

$$\begin{gathered}\zeta(0) = 0, \qquad \xi(0) = 0,\\ \lim_{|u|\to 0}\frac{\zeta(u)}{u} = \zeta'(0) = 0, \qquad \lim_{|u|\to\infty}\frac{\xi(u)}{u} = 0.\end{gathered} \tag{4.2}$$

Let us consider

$$Lu - rf_0 u = r\zeta(u) \tag{4.3}$$

as a bifurcation problem from the trivial solution $u \equiv 0$, and

$$Lu - rf_\infty u = r\xi(u) \tag{4.4}$$

as a bifurcation problem from infinity. We note that (4.3) and (4.4) are the same, and each of them is equivalent to (1.8), (1.9).

The results of Lemma 3.3 for (4.3) can be stated as follows: for each integer $k \geq 1$, $\nu \in \{+,-\}$, there exists a continuum $\mathcal{C}_{k,0}^{\nu}$ of solutions of (4.3) joining $(\lambda_k/f_0, 0)$ to infinity, and $\mathcal{C}_{k,0}^{\nu} \setminus \{(\lambda_k/f_0, 0)\} \subset \mathbb{R}\times T_k^{\nu}$.

The results of Lemma 3.4 for (4.4) can be stated as follows: for each integer $k \geq 1$, $\nu \in \{+,-\}$, there exists a continuum $\mathfrak{D}_{k,\infty}^{\nu}$ of solutions of (4.4) meeting $(\lambda_k/f_\infty, \infty)$.

Theorem 4.1. *Let (H0), $(\widetilde{H1})$, and (H2) hold. Then*

(i) *for* $(r,u) \in \mathcal{C}_{k,0}^{+} \cup \mathcal{C}_{k,0}^{-}$,

$$s_2 < u(t) < s_1, \quad t \in [0,1]; \tag{4.5}$$

(ii) *for* $(r,u) \in \mathfrak{D}_{k,\infty}^{+} \cup \mathfrak{D}_{k,\infty}^{-}$,

$$\max_{t\in[0,1]} u(t) > s_1, \qquad \text{or} \qquad \min_{t\in[0,1]} u(t) < s_2. \tag{4.6}$$

Proof of Theorem 4.1. Suppose on the contrary that there exists $(r,u) \in \mathcal{C}_{k,0}^{+} \cup \mathcal{C}_{k,0}^{-} \cup \mathfrak{D}_{k,\infty}^{+} \cup \mathfrak{D}_{k,\infty}^{-}$ such that either

$$\max\{u(t) \mid t \in [0,1]\} = s_1 \tag{4.7}$$

or

$$\min\{u(t) \mid t \in [0,1]\} = s_2. \tag{4.8}$$

Since $u \in T_k^{\nu}$, by Remark 2.1, $u \in S_k^{\nu}$ or $u \in S_{k+1}^{\nu}$. We assume $u \in S_k^{\nu}$. When $u \in S_{k+1}^{\nu}$, we can prove all the following results with small modifications. Let

$$0 = \tau_0 < \tau_1 < \cdots < \tau_{k-1} < 1 \tag{4.9}$$

denote the zeros of u. We divide the proof into two cases.

Case 1 ($\max\{u(t) \mid t \in [0, 1]\} = s_1$). In this case, there exists $j \in \{0, \dots, k-2\}$ such that

$$\max\left\{u(t) \mid t \in \left[\tau_j, \tau_{j+1}\right]\right\} = s_1 \qquad \text{or} \qquad \max\left\{u(t) \mid t \in \left[\tau_{k-1}, 1\right]\right\} = s_1,$$
$$0 \le u(t) \le s_1, \quad t \in \left[\tau_j, \tau_{j+1}\right], \qquad \text{or} \qquad t \in \left[\tau_{k-1}, 1\right]. \tag{4.10}$$

Since $u(1) = \sum_{i=1}^{m-2} \alpha_i u(\eta_i)$ and (H0), we claim $u(1) < s_1$.

Let $t_0 \in (\tau_j, \tau_{j+1})$ or $t_0 \in (\tau_{k-1}, 1)$ such that $u(t_0) = s_1$, then $u'(t_0) = 0$. Note that

$$f\left(u(t_0)\right) = f(s_1) = 0. \tag{4.11}$$

By the uniqueness of solutions of (1.8) subject to initial conditions, we see that $u(t) \equiv s_1$ on $[0, 1]$. This contradicts (1.9) and (H0).

Therefore,

$$\max\left\{u(t) \mid t \in [0, 1]\right\} \ne s_1. \tag{4.12}$$

Case 2 ($\min\{u(t) \mid t \in [0, 1]\} = s_2$). In this case, the proof is similar to Case 1, we omit it. □

Consequently, we obtain the results (i) and (ii).

Theorem 4.2. *Let (H0), ($\widetilde{H1}$), and (H2) hold. Assume that for some $k \in \mathbb{N}$,*

$$\frac{\lambda_k}{f_\infty} < \frac{\lambda_k}{f_0} \quad \left(\textit{resp.}, \ \frac{\lambda_k}{f_0} < \frac{\lambda_k}{f_\infty}\right). \tag{4.13}$$

Then

(i) *if $r \in (\lambda_k/f_\infty, \lambda_k/f_0]$ (resp., $r \in (\lambda_k/f_0, \lambda_k/f_\infty]$), then* (1.8), (1.9) *have at least two solutions $u_{k,\infty}^{\pm}$ (resp., $u_{k,0}^{\pm}$), such that $u_{k,\infty}^{+} \in T_k^{+}$ and $u_{k,\infty}^{-} \in T_k^{-}$ (resp., $u_{k,0}^{+} \in T_k^{+}$ and $u_{k,0}^{-} \in T_k^{-}$),*

(ii) *if $r \in (\lambda_k/f_0, \infty)$ (resp., $r \in (\lambda_k/f_\infty, \infty)$), then* (1.8), (1.9) *have at least four solutions $u_{k,\infty}^{\pm}$ and $u_{k,0}^{\pm}$, such that $u_{k,\infty}^{+}$, $u_{k,0}^{+} \in T_k^{+}$, and $u_{k,\infty}^{-}$, $u_{k,0}^{-} \in T_k^{-}$.*

Remark 4.3. Making a comparison between results in [3] and the above theorem, we see that as f has two zeros $s_1, s_2 : s_2 < 0 < s_1$, the bifurcation structure of the nodal solutions of (1.8), (1.9) becomes more complicated: the component of the solutions of (1.8), (1.9) from the trivial solution at $(\lambda_k/f_0, 0)$ and the component of the solutions of (1.8), (1.9) from infinity at $(\lambda_k/f_\infty, \infty)$ are disjoint; two new nodal solutions are born when $r > \max\{\lambda_k/f_0, \lambda_k/f_\infty\}$.

Proof of Theorem 4.2. Since (1.8), (1.9) have a unique solution $u \equiv 0$, we get

$$\left(\mathcal{C}_{k,0}^{+} \cup \mathcal{C}_{k,0}^{-} \cup \mathfrak{D}_{k,\infty}^{+} \cup \mathfrak{D}_{k,\infty}^{-}\right) \subset \left\{(\mu, z) \in \mathbb{E} \mid \mu \ge 0\right\}. \tag{4.14}$$

Take $\Lambda \subset \mathbb{R}$ as an interval such that $\Lambda \cap \{\lambda_j/f_\infty \mid j \in \mathbb{N}\} = \{\lambda_k/f_\infty\}$ and $\mathcal{M}$ as a neighborhood of $(\lambda_k/f_\infty, \infty)$ whose projection on $\mathbb{R}$ lies in Λ and whose projection on E is bounded away from 0. Then by Lemma 3.4, Remark 3.7, and Lemma 3.6 we have that each $\nu \in \{+, -\}$,

$\mathfrak{D}^{\nu}_{k,\infty} \setminus \mathcal{M}$ satisfies one of the following:

(1^o) $\mathfrak{D}^{\nu}_{k,\infty} \setminus \mathcal{M}$ is bounded in $\mathbb{E}$ in which case $\mathfrak{D}^{\nu}_{k,\infty} \setminus \mathcal{M}$ meets $\{(\lambda, 0) \mid \lambda \in \mathbb{R}\}$;

(2^o) $\mathfrak{D}^{\nu}_{k,\infty} \setminus \mathcal{M}$ is unbounded in $\mathbb{E}$ in which case $\mathrm{Proj}_{\mathbb{R}}(\mathfrak{D}^{+}_{k,\infty} \setminus \mathcal{M})$ is unbounded. □

Obviously, Theorem 4.1(ii) implies that (1^o) does not occur. So $\mathfrak{D}^{+}_{k,\infty} \setminus \mathcal{M}$ is unbounded in $\mathbb{E}$. Thus

$$\mathrm{Proj}_{\mathbb{R}}\left(\mathfrak{D}^{+}_{k,\infty}\right) \supset \left(\frac{\lambda_k}{f_\infty}, +\infty\right),$$
$$\mathrm{Proj}_{\mathbb{R}}\left(\mathfrak{D}^{-}_{k,\infty}\right) \supset \left(\frac{\lambda_k}{f_\infty}, +\infty\right). \tag{4.15}$$

By Theorem 4.1, for any $(r, u) \in (\mathcal{C}^{+}_{k,0} \cup \mathcal{C}^{-}_{k,0})$,

$$\|u\|_\infty < \max\left\{s_1, |s_2|\right\} := s^*. \tag{4.16}$$

Equations (4.16), (1.8), and (1.9) imply that

$$\|u\|_E < \max\left\{r \max_{|s|\le s^*}|f(s)|, s^*\right\}, \tag{4.17}$$

which means that the sets $\{(\mu, z) \in \mathcal{C}^{+}_{k,0} \mid \mu \in [0, d]\}$ and $\{(\mu, z) \in \mathcal{C}^{-}_{k,0} \mid \mu \in [0, d]\}$ are bounded for any fixed $d \in (0, \infty)$. This, together with the fact that $\mathcal{C}^{+}_{k,0}$ (resp., $\mathcal{C}^{-}_{k,0}$) joins $(\lambda_k/f_0, 0)$ to infinity, yields that

$$\mathrm{Proj}_{\mathbb{R}}\left(\mathcal{C}^{+}_{k,0}\right) \supset \left(\frac{\lambda_k}{f_0}, +\infty\right),$$
$$\mathrm{Proj}_{\mathbb{R}}\left(\mathcal{C}^{-}_{k,0}\right) \supset \left(\frac{\lambda_k}{f_0}, +\infty\right). \tag{4.18}$$

Combining (4.15) with (4.18), we conclude the desired results.

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