Research Article

The Uniform Attractors for the Nonhomogeneous 2D Navier-Stokes Equations in Some Unbounded Domain

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We consider the attractors for the two-dimensional nonautonomous Navier-Stokes equations in some unbounded domain Ω with nonhomogeneous boundary conditions. We apply the so-called uniformly ω-limit compact approach to nonhomogeneous Navier-Stokes equation as well as a method to verify it. Assuming $f \in L^2_{\mathrm{loc}}((0,T); L^2(\Omega))$, which is translation compact and $\varphi \in C^1_b(\mathbb{R}_+; H^2(\mathbb{R}^1 \times \{\pm L\}))$ asymptotically almost periodic, we establish the existence of the uniform attractor in $H^1(\Omega)$.

1. Introduction

We study the long-time behavior of a uniform flow past and infinitely long cylindrical obstacle. We will assume that the flow is uniform in the direction x of the axis of the cylindrical obstacle and the flow approaches $U_\infty e_x$ farther away from the obstacle. In this respect, we can consider a two-dimensional flow and assume that the obstacles are a disk with radius r (more general obstacle can be treated in exactly the same way). A further simplification is to observe that since the flow is uniform at infinity, we may assume that the flow is in an infinitely long channel with width $2L$ $(L \gg r)$ and the obstacle is located at the center, while the flow at the boundary of the channel is almost the uniform flow at infinity. More precisely, we assume that the flow is governed by the following Navier-Stokes equations in $\Omega = \mathbb{R}^1 \times (-L, L) \setminus B_r(0) (L \gg r)$:

$$\frac{\partial u}{\partial t} - \nu \Delta u + (u \cdot \nabla)u + \nabla p = f,$$

$$\mathrm{div}\, u = 0,$$

$$u = \varphi \quad \text{on } \partial\Omega_1 = \{y = \pm L\},$$
$$u = 0 \quad \text{on } \partial\Omega_2 = \partial B_r,$$
$$u = U_\infty e_x \quad \text{if } x \longrightarrow \pm\infty,$$

$$(1.1)$$

where $f \in L^2_{\text{loc}}((0,T); L^2(\Omega))$ is translation compact and $\varphi - U_\infty e_x \in C^1_b(\mathbb{R}_+; H^2(\mathbb{R}^1 \times \{\pm L\}))$ is called asymptotically almost periodic, that is, for any $\varepsilon \geq 0$, there is a number $l = l(\varepsilon)$ such that for each interval $(\alpha, \alpha + l)$, $\alpha \in \mathbb{R}_+$, there exists a point $\tau \in (\alpha, \alpha + l)$ such that

$$\mu(\varphi(s + \tau), \varphi(s)) \leq \varepsilon, \quad \forall s \geq l \tag{1.2}$$

(the number τ is called the ε-period of the function φ) (see [1]), where μ is the metric in $C^1_b(\mathbb{R}_+; H^2(\mathbb{R}^1 \times \{\pm L\}))$.

The basic idea of our construction is motivated by the works of [2] where an attractor $\mathcal{A}$ in space $L^2(\Omega)$ to which all solutions approach as $t \to \infty$ was shown. In this paper, we verify the existence of uniform attractors in space H^1 by using a noncompactness measure method.

We assume that the following Poincaré inequality holds:

$$\text{there exists } \lambda_1 > 0 \text{ such that } \lambda_1 \int_\Omega \phi^2 \, dx \leq \int_\Omega |\nabla\phi|^2 dx, \quad \forall\phi \in H^1_0(\Omega). \tag{1.3}$$

Throughout this paper, we introduce the spaces

$v = \{u \in (\mathcal{D}(\Omega))^2;\ \nabla \cdot u = 0 \text{ in } \Omega,\ u \cdot n = 0 \text{ on } \partial\Omega\}$,

$H = $ closure of v in $L^2(\Omega)$,

$V = $ closure of v in $H^1_0(\Omega)$,

$|\cdot|$, the $L^2(\Omega)$ norm,

$\|\cdot\|$, the norm in V,

$(\cdot, \cdot)$ the inner product in H or the dual product between V and V',

$((\cdot, \cdot))$ the inner product in V.

Here V' is the dual of $V = V_1$. The constants $C(c)$ are considered in a generic sense, which is independent of the physical parameters in the equations and may be different from line to line and even in the same line.

2. Setting of the problem

The first simplification is to introduce the new variables

$$\tilde{u} = u - U_\infty e_x, \qquad \tilde{\varphi} = \varphi - U_\infty e_x. \tag{2.1}$$

Then $\tilde{u}$ satisfies the equations

$$\frac{\partial\tilde{u}}{\partial t} - v\Delta\tilde{u} + (\tilde{u} \cdot \nabla)\,\tilde{u} + U_\infty\,\partial_x\tilde{u} + \nabla p = f,$$
$$\text{div}\,\tilde{u} = 0,$$
$$\tilde{u} = \tilde{\varphi} \quad \text{on } \partial\Omega_1 = \{y = \pm L\},$$
$$\tilde{u} = -U_\infty\,e_x \quad \text{on } \partial\Omega_2 = \partial B_r,$$
$$\tilde{u} = 0 \quad \text{if } x \longrightarrow \pm\infty.$$

$$(2.2)$$

We observe that $\tilde{\varphi} \in C_b^1(\mathbb{R}_+; H^2(\mathbb{R}^1 \times \{\pm L\}))$ and is asymptotically almost periodic. Note that $\tilde{u}$ and $\tilde{\varphi}$ decay nicely near infinity. However, the boundary condition is not homogeneous, and thus we apply a modified Hopf's technique (see [2–5]) to homogenize the boundary condition. More specifically, we choose

$$\eta_i \in C^\infty([0,1]), \quad \operatorname{supp} \eta_i \subset \left[0, \frac{1}{2}\right],$$

$$\int_0^1 \eta_1(s)ds = 0, \quad \eta_1(0) = 1,$$

$$\eta_2(0) = 1, \qquad \eta_2'(0) = 0, \qquad \eta_2''(0) = 0, \tag{2.3}$$

$$\left|s\eta_2(s)\right| \leq \frac{\nu}{r'U_\infty}, \qquad \left|s\eta_2'(s)\right| \leq \frac{\nu}{r'U_\infty}, \qquad \left|s\eta_2''(s)\right| \leq \frac{\nu}{r'U_\infty},$$

and we define, for $\varepsilon < 1$,

$$\vartheta^1 = \begin{cases} -\tilde{\varphi}_1(x, L; t) \int_0^{L-y} \eta_1\left(\frac{s}{L\varepsilon}\right) ds, & \text{for } \frac{L}{2} < y < L, \\ \tilde{\varphi}_1(x, -L; t) \int_0^{L+y} \eta_1\left(\frac{s}{L\varepsilon}\right) ds, & \text{for } -L < y < -\frac{L}{2}, \\ 0, & \text{otherwise,} \end{cases} \tag{2.4}$$

$$\vartheta^2 = \begin{cases} -U_\infty \eta_2\left(\frac{\sqrt{x^2+y^2}}{r} - 1\right)y, & \text{for } r < \sqrt{x^2+y^2} < 2r, \\ 0, & \text{otherwise.} \end{cases}$$

we then define

$$\psi^i = \operatorname{curl} \vartheta^i = \left(\partial_y \vartheta^i, -\partial_x \vartheta^i\right), \quad i = 1, 2. \tag{2.5}$$

Observe that $\psi^1 = \tilde{\varphi}$ at $y = \pm L$ and $\psi^2 = -U_\infty e_x$ at ∂B_r. If we set

$$v = \tilde{u} - \psi, \quad \text{where } \psi = \psi^1 + \psi^2, \tag{2.6}$$

then v satisfies

$$\frac{\partial v}{\partial t} - \nu \Delta v + (v \cdot \nabla)v + (v \cdot \nabla)\psi + (\psi \cdot \nabla)v + U_\infty \partial_x v + \nabla p$$

$$= \frac{\partial \psi^1}{\partial t} + \nu \Delta \psi - (\psi \cdot \nabla)\psi - U_\infty \partial_x \psi + f$$

$$= G(\varepsilon, \nu, U_\infty, r, L, t) + f = F, \tag{2.7}$$

$$\operatorname{div} v = 0,$$

$$v = 0 \quad \text{on } \partial\Omega,$$

$$v(\tau) = v_\tau.$$

It is easy to check that for fixed ε, ν, U_∞, r, and L, the right-hand side of (2.7) $G \in C_b(\mathbb{R}_+; L^2(\Omega))$ (see [2]) is asymptotically almost periodic.

3. Abstract results

Let E be a Banach space, and let a two-parameter family of mappings $\{U(t,\tau)\} = \{U(t,\tau) \mid t \geq \tau, \ \tau \in \mathbb{R}\}$ act on E:

$$U(t,\tau) : E \longrightarrow E, \quad t \geq \tau, \ \tau \in \mathbb{R}. \tag{3.1}$$

Definition 3.1. A two-parameter family of mappings $\{U(t,\tau)\}$ is said to be a process in E if

$$U(t,s)U(s,\tau) = U(t,\tau), \quad \forall t \geq s \geq \tau, \ \tau \in \mathbb{R},$$
$$U(\tau,\tau) = \mathrm{Id}, \quad \tau \in \mathbb{R}. \tag{3.2}$$

A family of processes $\{U_\sigma(t,\tau)\}$, $\sigma \in \Sigma$, acting in E is said to be $(E \times \Sigma, E)$-*continous*, if for all fixed t and τ, $t \geq \tau$, $\tau \in \mathbb{R}$, the mapping $(u,\sigma) \mapsto U_\sigma(t,\tau)u$ is continuous from $E \times \Sigma$ into E.

A curve $u(s)$, $s \in \mathbb{R}$, is said to be *a complete trajectory* of the process $\{U(t,\tau)\}$ if

$$U(t,\tau)u(\tau) = u(t), \quad \forall t \geq \tau, \ \tau \in \mathbb{R}. \tag{3.3}$$

The *kernel* $\mathcal{K}$ of the process $\{U(t,\tau)\}$ consists of all bounded complete trajectories of the process $\{U(t,\tau)\}$:

$$\mathcal{K} = \{u(\cdot) \mid u(\cdot) \text{ satisfies (3.3) and } \|u(s)\|_E \leq M_u \text{ for } s \in \mathbb{R}\}. \tag{3.4}$$

The set

$$\mathcal{K}(s) = \{u(s) \mid u(\cdot) \in \mathcal{K}\} \subseteq E \tag{3.5}$$

is said to be the *kernel section* at time $t = s$, $s \in \mathbb{R}$.

We consider the two projectors Π_1 and Π_2 from $E \times \Sigma$ onto E and Σ, respectively,

$$\Pi_1(u,\sigma) = u, \quad \Pi_2(u,\sigma) = \sigma. \tag{3.6}$$

Now we recall the basic results in [1].

Theorem 3.2. *Let a family of processes* $\{U_\sigma(t,\tau)\}$, $\sigma \in \Sigma$, *acting in the space E be uniformly* (*w.r.t.* $\sigma \in \Sigma$) *asymptotically compact and* $(E \times \Sigma, E)$-*continuous. Also let Σ be a compact-metric space and let $\{T(t)\}$ be a continuous-invariant* $(T(t)\Sigma = \Sigma)$ *semigroup on Σ satisfying the translation identity*

$$U_\sigma(t+s,\tau+s) = U_{T(s)\sigma}(t,\tau), \quad \forall \, \sigma \in \Sigma, \ t \geq \tau, \ \tau \in \mathbb{R}, \ s \geq 0. \tag{3.7}$$

Then the semigroup $\{S(t)\}$ *corresponding to the family of processes* $\{U_\sigma(t,\tau)\}$, $\sigma \in \Sigma$, *and acting on* $E \times \Sigma$,

$$S(t)(u,\sigma) = (U_\sigma(t,0)u, T(t)\sigma), \quad t \geq 0, \ (u,\sigma) \in E \times \Sigma, \tag{3.8}$$

possesses the compact attractor $\mathcal{A}$ *which is strictly invariant with respect to* $\{S(t)\} : S(t)\mathcal{A} = \mathcal{A}$ *for all $t \geq 0$. Moreover,*

(i) $\Pi_1 \mathcal{A} = \mathcal{A}_1 = \mathcal{A}_\Sigma$ *is the uniform (w.r.t.* $\sigma \in \Sigma$*) attractor of the family of processes* $\{U_\sigma(t, \tau)\}$, $\sigma \in \Sigma$;

(ii) $\Pi_2 \mathcal{A} = \mathcal{A}_2 = \Sigma$;

(iii) *the global attractor satisfies*

$$\mathcal{A} = \bigcup_{\sigma \in \Sigma} \mathcal{K}_\sigma(0) \times \{\sigma\}; \tag{3.9}$$

(iv) *the uniform attractor satisfies*

$$\mathcal{A}_\Sigma = \mathcal{A}_1 = \bigcup_{\sigma \in \Sigma} \mathcal{K}_\sigma(0). \tag{3.10}$$

Here $\mathcal{K}_\sigma(0)$ is the section at $t = 0$ of the kernel $\mathcal{K}_\sigma$ of the process $\{U_\sigma(t, \tau)\}$ with symbol $\sigma \in \Sigma$.

For convenience, let $B_t = \bigcup_{\sigma \in \Sigma} \bigcup_{s \geq t} U_\sigma(s, t) B$, the closure $\overline{B}$ of the set B and $\mathbb{R}_\tau = \{t \in \mathbb{R} \mid t \geq \tau\}$. Define the uniform (w.r.t. $\sigma \in \Sigma$) ω-limit set $\omega_{\tau, \Sigma}(B)$ of B by $\omega_{\tau, \Sigma}(B) = \bigcap_{t \geq \tau} \overline{B}_t$ which can be characterized with, analogously to that for semigroups, the following:

$$y \in \omega_{\tau, \Sigma}(B) \Longleftrightarrow \text{there are sequences } \{x_n\} \subset B, \ \{\sigma_n\} \subset \Sigma, \ \{t_n\} \subset \mathbb{R}_\tau$$
$$\text{such that } t_n \longrightarrow +\infty, \ U_{\sigma_n}(t_n, \tau) x_n \longrightarrow y \ (n \longrightarrow \infty). \tag{3.11}$$

We will characterize the existence of the uniform attractor for a family of processes satisfying (3.7) in terms of the concept of measure of noncompactness that was put forward first by Kuratowski.

Let $B \in \mathcal{B}(E)$. The Kuratowski measure of noncompactness $\kappa(B)$ is defined by

$$\kappa(B) = \inf \{\delta > 0 \mid B \text{ admits a finite covering by sets of diameter } \leq \delta\}. \tag{3.12}$$

Definition 3.3. A family of processes $\{U_\sigma(t, \tau)\}$, $\sigma \in \Sigma$, is said to be uniformly (w.r.t. $\sigma \in \Sigma$) ω-limit compact if for any $\tau \in \mathbb{R}$ and $B \in \mathcal{B}(E)$ the set B_t is bounded for every t and $\lim_{t \to \infty} \kappa(B_t) = 0$.

We present now a method to verify the uniform (w.r.t. $\sigma \in \Sigma$) ω-limit compactness (see [6, 7]).

Definition 3.4. A family of processes $\{U_\sigma(t, \tau)\}$, $\sigma \in \Sigma$, is said to satisfy uniformly (w.r.t. $\sigma \in \Sigma$) condition (C) if for any fixed $\tau \in \mathbb{R}$, $B \in \mathcal{B}(E)$, and $\varepsilon > 0$, there exist $t_0 = t(\tau, B, \varepsilon) \geq \tau$ and a finite-dimensional subspace E_1 of E such that

(i) $P(\bigcup_{\sigma \in \Sigma} \bigcup_{t \geq t_0} U_\sigma(t, \tau) B)$ is bounded;

(ii) $\|(I - P)(\bigcup_{\sigma \in \Sigma} \bigcup_{t \geq t_0} U_\sigma(t, \tau) x)\| \leq \varepsilon, \ \forall x \in B$,

where $P : E \to E_1$ is a bounded projector.

Therefore we have the following results.

Theorem 3.5. *Let Σ be a compact metric space and let $\{T(t)\}$ be a continuous invariant semigroup $T(t)\Sigma = \Sigma$ on Σ satisfying the translation identity (3.7). A family of processes $\{U_\sigma(t,\tau)\}$, $\sigma \in \Sigma$, acting in E is $(E \times \Sigma, E)$ (weakly) continuous and possesses the compact uniform (w.r.t. $\sigma \in \Sigma$) attractor $\mathscr{A}_\Sigma$ satisfying*

$$\mathscr{A}_\Sigma = \omega_{0,\Sigma}(B_0) = \omega_{\tau,\Sigma}(B_0) = \bigcup_{\sigma \in \Sigma} \mathscr{K}_\sigma(0), \quad \forall \tau \in \mathbb{R}, \tag{3.13}$$

if it

  (i) *has a bounded uniformly (w.r.t. $\sigma \in \Sigma$) absorbing set B_0;*

  (ii) *satisfies uniformly (w.r.t. $\sigma \in \Sigma$) condition (C).*

Moreover, if E is a uniformly convex Banach space then the converse is true.

4. The attractor of the nonhomogeneous Navier-Stokes equations

We say that v is a weak solution of (2.7) if

$$v_\tau \in H, \quad v \in L^\infty(0,T;H) \cap L^2(0,T;V), \tag{4.1}$$

$$\frac{d}{dt}(v,w) + a(v,w) + b(v,\psi,w) + b(\psi,v,w) + b(v,v,w) + b(U_\infty e_x,v,w) = (F,w) \text{ in } V',$$

$$\text{for } t > 0, \ \forall \, w \in V,$$

$$v(\tau) = v_\tau \tag{4.2}$$

in the distributional sense, where $a(v,w) = (Av,w)_{V',V}$. The well-posedness of (4.2) can be derived using a standard Faedo-Galerkin approach. It can be viewed as a family of semiprocesses on H with the symbol space Σ defined as

$$\Sigma = \Sigma_1 \times \Sigma_2 = \overline{\{\psi(\cdot + \tau)\}_{\tau \geq 0}} \times \overline{\{F(\cdot + \tau)\}_{\tau \geq 0}}, \tag{4.3}$$

endowed with the product norm of the supremum norm on $C_b(\mathbb{R}_+; H^2(\Omega))$ (for ψ) and the supremum norm of $C_b(\mathbb{R}_+; L^2(\Omega))$ (for G) or the norm of $L^2_{\text{loc}}((0,T); L^2(\Omega))$ (for f). The symbol space Σ is a compact space by our assumptions on φ and f, and the explicit construction of ψ and F. For each $v_\tau \in H$ and $\sigma = (\sigma^1, \sigma^2) \in \Sigma$, $t \geq \tau$, $U_\sigma(t,\tau)v_\tau$ is the solution in V' to

$$\frac{dv}{dt} + \nu Av + B(v,v) + B(v,\sigma^1) + B(\sigma^1,v) + B(U_\infty e_x, v) = P\sigma^2, \quad t > \tau,$$

$$v(\tau) = v_\tau, \tag{4.4}$$

where $A : V \to V'$ is the Stokes operator defined by

$$(Av,w) = (\nabla v, \nabla w), \quad \forall v,w \in V. \tag{4.5}$$

$B(u, v)$ is a bilinear operator $H_0^1 \times H_0^1 \to V'$ defined by

$$(B(u, v), w) = b(u, v, w), \quad \forall u, v \in H_0^1, \ \forall w \in V, \tag{4.6}$$

and P is the Leray-Hopf projection from $L^2(\Omega)$ onto H.

We can also define on Σ the semigroup $\{T(s)\}_{s \geq 0}$ given by $T(s)\sigma(t) = (T(s)\sigma)(t) = \sigma(t + s)$, $\forall t \geq 0$, $\forall s \geq 0$, $\forall \sigma \in \Sigma$. Since the symbol space Σ is compact, the semigroup $\{T(s)\}_{s \geq 0}$ is continuous and compact and in particular asymptotically compact. It is then obvious that this family of semiprocesses satisfies the translation invariance property.

Now recall the following facts that can be found in [6].

Lemma 4.1. *Assume that $f(s) \in L_c^2(\mathbb{R}; E)$ is translation compact, then for any $\varepsilon > 0$, there exists $\eta > 0$ such that*

$$\sup_{t \in \mathbb{R}} \int_t^{t+\eta} \|f(s)\|_E^2 ds \leq \varepsilon. \tag{4.7}$$

Since A^{-1} is a continuous-compact operator in H, by the classical spectral theorem, there exists a sequence $\{\lambda_j\}_{j=1}^\infty$,

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_j \leq \cdots, \quad \lambda_j \longrightarrow +\infty, \text{ as } j \longrightarrow \infty. \tag{4.8}$$

Now we will write (4.4) in the operator form

$$\partial_t v = A_{\sigma(t)}(v), \quad v|_{t=\tau} = v_\tau, \tag{4.9}$$

where $\sigma(s) \in \Sigma$ is the symbol of (4.9). Thus, if $v_\tau \in H$, then problem (4.9) has a unique solution $v(t) \in C([0, T]; H) \cap L^2([0, T]; V)$. This implies that the process $\{U_\sigma(t, \tau)\}$ given by the formula $U_\sigma(t, \tau)v_\tau = v(t)$ is defined in H.

Let

$$\mathcal{K}_\sigma = \{v_\sigma(x, t) \text{ for } t \in \mathbb{R} \mid v_\sigma(x, t) \text{ is the solution of (4.9) satisfying } \|v_\sigma(\cdot, t)\|_H \leq M_\sigma \ \forall t \in \mathbb{R}\} \tag{4.10}$$

be the so-called kernel of the process $\{U_\sigma(t, \tau)\}$.

Proposition 4.2. *The process $\{U_\sigma(t, \tau)\} : V \to V$ associated with (4.9) possesses absorbing sets*

$$\mathcal{B} = \{v \in V \mid \|v\| \leq \rho\} \tag{4.11}$$

which absorb all bounded sets of V in the norm of V.

Proof. Multiplying (4.4) by Av, we have

$$\begin{aligned}
\left(\frac{dv}{dt}, Av\right) &+ (\nu Av, Av) + (B(v, v), Av) \\
&+ (B(v, \psi^1 + \psi^2), Av) + (B(\psi^1 + \psi^2, v), Av) + (B(U_\infty e_x, v), Av) \\
&= \left(\frac{\partial \psi^1}{\partial t}, Av\right) + \nu(\Delta(\psi^1 + \psi^2), Av) \\
&\quad - (B(\psi^1 + \psi^2, \psi^1 + \psi^2), Av) - (B(U_\infty e_x, \psi^1 + \psi^2), Av) + (f, Av) \\
&= (F, Av),
\end{aligned} \tag{4.12}$$

that is,

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|v\|^2 + \nu|Av|^2 &+ (B(v,v),Av) \\
&+ (B(v,\psi^1+\psi^2),Av) + (B(\psi^1+\psi^2,v),Av) + (B(U_\infty e_x,v),Av) \\
&= (F,Av),
\end{aligned}
\tag{4.13}
$$

thus,

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|v\|^2 + \nu|Av|^2 &\leq \big|(B(v,v),Av)\big| + \big|(B(v,\,\psi^1+\psi^2),Av)\big| \\
&+ \big|(B(\psi^1+\psi^2,v),Av)\big| + \big|B(U_\infty e_x,v),Av)\big| + \big|(F,Av)\big|.
\end{aligned}
\tag{4.14}
$$

We have to estimate each term in the right-hand side of (4.14).

First, we recall some inequalities [8]

$$
|B(u,v)| \leqslant C \begin{cases} |u|^{1/2}\|u\|^{1/2}\|v\|^{1/2}|Av|^{1/2}, & \forall u \in V,\ v \in D(A), \\ |u|^{1/2}|Au|^{1/2}\|v\|, & \forall u \in D(A),\ v \in V. \end{cases}
\tag{4.15}
$$

By using Young's inequality, we have

$$
\big|(B(v,v),Av)\big| \leqslant C|v|^{1/2}\|v\||Av|^{3/2} \leqslant \frac{\nu}{10}|Av|^2 + \frac{C}{\nu^3}|v|^2\|v\|^4.
\tag{4.16}
$$

By (2.3)–(2.5) and Cauchy's inequality, we have

$$
\begin{aligned}
\big|(B(v,\psi^1),Av)\big| &\leq \int_\Omega |v||\nabla\psi^1||Av| \leq \left(\left\|\frac{v}{L-y}\right\|_{L^2(1-\varepsilon<y/L<1)} + \left\|\frac{v}{L+y}\right\|_{L^2(-1<y/L<-1+\varepsilon)}\right) \\
&\quad \cdot \left(\left\|(L-y)\nabla\psi^1\right\|_{L^\infty(1-\varepsilon<y/L<1)} + \left\|(L+y)\nabla\psi^1\right\|_{L^\infty(-1<y/L<-1+\varepsilon)}\right)|Av| \\
&\left(\text{since the support of } \psi^1 \text{ is in } \left\{1-\varepsilon<\frac{y}{L}<1\right\}\cup\left\{-1<\frac{y}{L}<-1+\varepsilon\right\}\right) \\
&\leq k|\nabla v|\big((L\varepsilon)^2\big(\|\varphi_{1x}\|_{L^\infty(\mathbb{R}^1\times\{\pm L\})} + \|\varphi_{1xx}\|_{L^\infty(\mathbb{R}^1\times\{\pm L\})}\big) + L\varepsilon\|\varphi_1 - U_\infty\|_{L^\infty(\mathbb{R}^1\times\{\pm L\})}\big)|Av|.
\end{aligned}
\tag{4.17}
$$

Thanks to Hardy's inequality and (2.3)–(2.5), we have

$$
\begin{aligned}
\big|(B(v,\psi^2),Av)\big| &\leq \int_\Omega |v||\nabla\psi^2||Av| \\
&\leq r\left\|\frac{v}{\sqrt{x^2+y^2}-r}\right\|_{L^2(B_{2r}\setminus B_r)}\left\|\left(\frac{\sqrt{x^2+y^2}}{r}-1\right)\nabla\psi^2\right\|_{L^\infty(B_{2r}\setminus B_r)}|Av| \\
&\leq 4r|\nabla v||U_\infty|\big(\big|s\eta_1(s)\big|_{L^\infty} + 4\big|s\big(\eta_1'(s)+\eta_1''(s)\big)\big|_{L^\infty}\big)|Av|.
\end{aligned}
\tag{4.18}
$$

Using (2.3)–(2.5), provided we choose ε small enough (see [2]), one can obtain

$$\left|\left(B(v,\psi^1),Av\right)\right| \le \frac{\nu}{10}|Av|^2,$$

$$\left|\left(B(v,\psi^2),Av\right)\right| \le \frac{\nu}{10}|Av|^2. \tag{4.19}$$

Since $F \in L^2(\Omega)$, we have

$$|(F,Av)| \le |F|_{L^2(\Omega)}|Av| \le \frac{\nu}{10}|Av|^2 + \frac{5|F|^2_{L^2(\Omega)}}{2\nu}. \tag{4.20}$$

Putting (4.16)–(4.20) together, there exists a constant $M > 0$ such that

$$\frac{d}{dt}\|v\|^2 + \nu\lambda_1\|v\|^2 \le C\left(\frac{1}{\nu}|F|^2 + c\|v\|^2 + \frac{c}{\nu^3}|v|^2\|v\|^4 + M\right). \tag{4.21}$$

Similarly to [9, III 2.2], applying the uniform Gronwall's lemma, we can obtain $\|v\|^2 \le \rho^2$. $\quad\square$

The main results in this section are as follows.

Theorem 4.3. *If $f \in L^2_{\mathrm{loc}}((0,T);L^2(\Omega))$ is translation compact, then the processes $\{U_\sigma(t,\tau)\}$ corresponding to problem (4.9) possess the compact uniform (w.r.t. $\tau \in \mathbb{R}$) attractor $\mathfrak{A}_0$ in V which coincides with the uniform (w.r.t. $\sigma \in \Sigma$) attractor $\mathfrak{A}_\Sigma$ of the family of processes $\{U_\sigma(t,\tau)\}$, $\sigma \in \Sigma$,*

$$\mathfrak{A}_0 = \mathfrak{A}_\Sigma = \omega_{0,\Sigma}(\mathcal{B}) = \bigcup_{\sigma \in \Sigma}\mathcal{K}_\sigma(0), \tag{4.22}$$

where $\mathcal{B}$ is a uniformly (w.r.t. $\sigma \in \Sigma$) absorbing set in V and $\mathcal{K}_\sigma$ is the kernel of the process $\{U_\sigma(t,\tau)\}$. Furthermore, the kernel $\mathcal{K}_\sigma$ is nonempty for all $\sigma \in \Sigma$.

Proof. Using Proposition 4.2, the family of processes $\{U_\sigma(t,\tau)\}$, $\sigma \in \Sigma$, corresponding to (4.9) possesses the uniformly (w.r.t. $\sigma \in \Sigma$) absorbing set in V.

Now we prove the existence of the compact uniform (w.r.t. $\sigma \in \Sigma$) attractor in V by applying the method established in Section 3, that is, we prove that the family of processes $\{U_\sigma(t,\tau)\}$, $\sigma \in \Sigma$, corresponding to (4.9) satisfies uniformly (w.r.t. $\sigma \in \Sigma$) condition (C).

As in the previous section, for fixed N, let H_1 be the subspace spanned by $w_1,\ldots,w_N$, and H_2 the orthogonal complement of H_1 in H. We write

$$v = v_1 + v_2, \quad v_1 \in H_1, \quad v_2 \in H_2 \quad \text{for any } v \in H, \tag{4.23}$$

where $v(t) = v_1(t) + v_2(t)$ is a solution of (4.4).

Multiplying (4.4) by $Av_2(t)$, similarly to the proof of Proposition 4.2, we have

$$\begin{aligned}
&\left(\frac{dv}{dt},Av_2\right) + (\nu Av,Av_2) + (B(v,v),Av_2) \\
&\quad + (B(v,\psi^1+\psi^2),Av_2) + (B(\psi^1+\psi^2,v),Av_2) + (B(U_\infty e_x,v),Av_2) \\
&= (F,Av_2),
\end{aligned} \tag{4.24}$$

thus,

$$\frac{1}{2}\frac{d}{dt}\|v_2\|^2 + \nu|Av_2|^2 \leq |(B(v,v), Av_2)| + |(B(v, \psi^1 + \psi^2), Av_2)| \tag{4.25}$$
$$+ |(B(\psi^1 + \psi^2, v), Av_2)| + |B(U_\infty e_x, v), Av_2)| + |(F, Av_2)|.$$

To estimate $(B(v,v), Av_2)$, we recall some inequalities [10]

$$|w|_{L^\infty(\Omega)^2} \leq c\|w\|\left(1 + \log\frac{|Aw|}{\lambda_1\|w\|^2}\right)^{1/2} \tag{4.26}$$

from which we deduce that

$$|B(u,v)| \leq \begin{cases} |u|_{L^\infty(\Omega)}|\nabla v|, \\ |u|\|\nabla v|_{L^\infty(\Omega)}, \end{cases} \tag{4.27}$$

and using (4.26)

$$|B(u,v)| \leq c \begin{cases} \|u\|\|v\|\left(1 + \log\frac{|Au|^2}{\lambda_1\|u\|^2}\right)^{1/2}, \\ |u||Av|\left(1 + \log\frac{|A^{3/2}v|^2}{\lambda_1\|Av\|^2}\right)^{1/2}. \end{cases} \tag{4.28}$$

Expanding and using Young's inequality, together with the first one of (4.28) and the second one of (4.15), we have

$$|(B(v,v), Av_2)| \leq |(B(v_1, v_1 + v_2), Av_2)| + |(B(v_2, v_1 + v_2), Av_2)|$$
$$\leq CD^{1/2}\|v_1\||Av_2|(\|v_1\| + \|v_2\|) + C|v_2|^{1/2}|Av_2|^{3/2}(\|v_1\| + \|v_2\|) \tag{4.29}$$
$$\leq \frac{\nu}{10}|Av_2|^2 + \frac{C}{\nu}\rho^4 D + \frac{C}{\nu^3}\rho^6, \quad t \geq t_0 + 1,$$

where we use

$$|Av_1|^2 \leq \lambda_m\|v_1\|^2, \tag{4.30}$$

and set

$$D = \left(1 + \log\frac{\lambda_{m+1}}{\lambda_1}\right). \tag{4.31}$$

Next, similarly to (4.19), we have

$$|(B(v, \psi^1), Av_2)| \leq \frac{\nu}{10}|Av_2|^2,$$
$$|(B(v, \psi^2), Av_2)| \leq \frac{\nu}{10}|Av_2|^2. \tag{4.32}$$

Since $G \in C_b(\mathbb{R}_+; L^2(\Omega))$ and $f \in L^2_{\text{loc}}((0,T); L^2(\Omega))$, one can obtain

$$\left|(F, Av_2)\right| \leq \left|(G, Av_2)\right| + \left|(f, Av_2)\right|$$
$$\leq \frac{\nu}{10}|Av_2|^2 + \frac{5}{2\nu}|f|^2 + c. \tag{4.33}$$

Combining with (4.29)–(4.33), there exists a constant $M = M(\rho, \nu, D) > 0$ such that

$$\frac{d}{dt}\|v_2\|^2 + \nu\lambda_{m+1}\|v_2\|^2 \leqslant C\left(\frac{1}{\nu}|f|^2 + \frac{M}{\nu}\right). \tag{4.34}$$

By the uniform Gronwall's lemma, it follows from (4.34) that

$$\|v_2\|^2 \leqslant \|v_2(t_0+1)\|^2 e^{-\nu\lambda_{m+1}(t-(t_0+1))} + \frac{CM}{\nu^2\lambda_{m+1}} + \frac{C}{\nu}\int_{t_0+1}^t e^{-\nu\lambda_{m+1}(t-s)}|f|^2 ds. \tag{4.35}$$

Since $f \in L^2(\Omega)$ is also translation compact, using [1, Lemmas 4.1 and II-1.3] by Chepyzhov and Vishik, for any ε, we have

$$\frac{CM}{\nu^2\lambda_{m+1}} \leqslant \frac{\varepsilon}{3},$$
$$\frac{C}{\nu}\int_{t_0+1}^t e^{-\nu\lambda_{m+1}(t-s)}|f|^2 ds \leqslant \frac{\varepsilon}{3}. \tag{4.36}$$

Using (4.8) and letting $t_1 = t_0 + 1 + (2/\nu\lambda_{m+1})\ln(3\rho^2/\varepsilon)$, then $t \geq t_1$ implies

$$\rho^2 e^{-\nu\lambda_{m+1}(t-(t_0+1))} \leqslant \frac{\varepsilon}{3}. \tag{4.37}$$

Therefore, we deduce from (4.35) that

$$\|v_2(t)\|^2 \leqslant \varepsilon, \quad \forall t \geqslant t_1, \tag{4.38}$$

which indicates that $\{U_f(t,\tau)\}$, $f \in \Sigma$ satisfying uniformly (w.r.t. $f \in \Sigma$) condition (C) in V. Applying Theorem 3.5, the proof is complete. $\square$

Acknowledgment

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