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Research Article

Multipoint Singular Boundary-Value Problem for Systems of Nonlinear Differential Equations

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A singular Cauchy-Nicoletti problem for a system of nonlinear ordinary differential equations is considered. With the aid of combination of Ważewski's topological method and Schauder's principle, the theorem concerning the existence of a solution of this problem (having the graph in a prescribed domain) is proved.

1. Introduction

In the present paper the following Cauchy-Nicoletti problem

$$y_i' = f_i(x, y), \quad i = 1, \dots, n, \tag{1.1}$$

$$\begin{gathered} y_p\left(x_p^+\right) = A_p, \qquad y_q\left(x_q^\pm\right) = A_q, \qquad y_r\left(x_r^-\right) = A_r, \\ p = 1, \dots, k; \qquad q = k+1, \dots, s; \qquad r = s+1, \dots, n \end{gathered} \tag{1.2}$$

is considered, where $y = (y_1, \dots, y_n)$, $x \in I = [a, b]$ and $a = x_1 = \cdots = x_k < x_{k+1} \le \cdots \le x_s < x_{s+1} = \cdots = x_n = b$; A_i, $i = 1, \dots, n$ are real constants. Denote $I_i = I \setminus \{x_i\}$, $i = 1, \dots, n$ and $J = \bigcap_{i=1}^n I_i$. We will suppose $f_i \in C(\Theta_i, \mathbb{R})$, $i = 1, \dots, n$ where the domain $\Theta_i \subset I_i \times \mathbb{R}^n$ (satisfying a relation $\Theta_i \cap \{x = x^*\} \ne \emptyset$ for every $x^* \in I_i$) is more precisely specified in Section 2. The continuity of the function f_i is not required at the point x_i, $i = 1, \dots, n$. Solution of the problem (1.1), (1.2) is defined in the following sense.

Definition 1.1. A vector-function $y(x) = (y_1(x), \dots, y_n(x)) \in C(I, \mathbb{R}^n)$ with $y_i \in C^1(I_i, \mathbb{R})$, $i = 1, \dots, n$, is said to be a solution of the problem (1.1), (1.2) if it satisfies the system (1.1) on J and if, moreover, conditions (1.2) hold.

Although singular boundary value problems were widely considered by using various methods (see, e.g., [1–11]), the method used here is based on a different approach. Namely, it uses simultaneously the topological method of Ważewski and Schauder's principle. Note that the method of Ważewski (see, e.g., [12–14]) was used for the investigation of various asymptotic and singular problems, for example, in [3–6, 11, 12, 15]. For successful generalization of this to multipoint boundary-value problems, the basic obstacle must be overcome: the applying of topological method assumes that every intersection of so-called regular polyfacial set and the plane $x = x^* =$ const, $x^* \in (a, b)$ is an open set in the space of dependent variables. Nature of the problems considered, as followed from problem (1.1), (1.2) does not permit straightforward generalization of this approach since the cross-section by the plane $x = x_i$, $i = 1, \dots, n$ is not an open set in the space y. The above mentioned obstacle is overcome in the present paper by connecting the topological method and the fixed point theorem.

Let us explain the main idea of this approach. Each equation of the system (1.1) is considered separately (as a scalar equation) under the supposition that nondiagonal variables are changed by functions taken from a prescribed set M of vector functions. For every scalar equation (together with the corresponding Cauchy initial condition which is subtracted from (1.2)) it is, with the aid of Ważewski's method and qualitative properties of solutions of differential equations, showed that there exists its solution having the same properties which were supposed for the corresponding coordinate of vector functions from M. In this way an operator $\mathfrak{T}$ is defined. For verification of conditions of Schauder's principle (namely, the continuity of operator $\mathfrak{T}$), Ważewski's method is used again. Stationary point of operator $\mathfrak{T}$ defines a solution of the problem (1.1), (1.2). The paper is organized as follows. In Section 2 the main result is formulated. Illustrative examples are contained in Section 3. Auxiliary results are stated in Section 4. In Section 5 we prove results concerning scalar singular problems and the last section contains the proof of the main result.

2. Existence of Solutions of the Problem (1.1), (1.2)

Let $\alpha_i, \beta_i \in C^1(I, \mathbb{R})$, $i = 1, 2, \dots, n$ be functions satisfying $\alpha_i(x_i) = \beta_i(x_i) = A_i$ and $\alpha_i(x) < \beta_i(x)$ on I_i. Define

$$\begin{gathered} \Omega = \{(x, y_1, \dots, y_n) : x \in I,\ \alpha_i(x) \le y_i \le \beta_i(x),\ i = 1, \dots, n\}, \\ \Omega_i = \{(x, y_1, \dots, y_n) : x \in I_i,\ (x, y_1, \dots, y_n) \in \Omega\}. \end{gathered} \tag{2.1}$$

Let us suppose that there exists a domain Θ_i, $i = 1, 2 \dots, n$ such that $\Omega_i \subset \Theta_i \subset I_i \times \mathbb{R}^n$; cross section $S_i(x) = \{(x, y) \in \Theta_i\}$ is an open set for every fixed $x \in I_i$ and $f_i \in C(\Theta_i, \mathbb{R})$. These assumptions are supposed in the sequel. Define, moreover,

$$\begin{gathered} \Gamma_i = \{(x, y_{i_1}, \dots, y_{i_{n-1}}) : x \in I_i,\ \{i_1, \dots, i_{n-1}\} = \{1, \dots, n\} \setminus \{i\}, \\ \alpha_s(x) \le y_s \le \beta_s(x),\ s = i_1, \dots, i_{n-1}\}, \\ F_i(x, y) \equiv f_i(x, y) - y_i', \quad i = 1, \dots, n. \end{gathered} \tag{2.2}$$

Result of the paper is given in the following theorem.

Theorem 2.1. *Assume that*

$$\sum_{j=1}^{n} M_{ij}(x)\left|y_j - z_j\right| \le f_i(x,y) - f_i(x,z) \le \sum_{j=1}^{n} N_{ij}(x)\left|y_j - z_j\right| \tag{2.3}$$

for every $(x, y_1, \ldots, y_n)$, $(x, z_1, \ldots, z_n) \in \Omega_i$ *with* $y_i > z_i$ *where* $M_{ij}(x)$, $N_{ij}(x)$, $i, j = 1, \ldots, n$, *are continuous on* I_i *functions, such that for a constant* $\xi > 0$

$$|M_{ii}(x)| > \xi \sum_{j=1, j\neq i}^{n} \left|M_{ij}(x)\right|, \qquad |N_{ii}(x)| > \xi \sum_{j=1, j\neq i}^{n} \left|N_{ij}(x)\right|. \tag{2.4}$$

Let, moreover,

$$\begin{gathered} F_i(x,y)\big|_{y_i=\alpha_i(x)} \cdot F_i(x,y)\big|_{y_i=\beta_i(x)} < 0, \\ \operatorname{sign} M_{ii}(x) = \operatorname{sign} N_{ii}(x) = \operatorname{sign} F_i(x,y)\big|_{y_i=\beta_i(x)}, \end{gathered} \tag{2.5}$$

if $(x, y_1, \ldots, y_{i-1}, y_{i+1}, \ldots, y_n) \in \Gamma_i$, $i = 1, \ldots, n$. *Then there exists at least one solution* $y(x) = (y_1(x), \ldots, y_n(x))$ *of the problem* (1.1), (1.2) *such that on* I_i, $i = 1, \ldots, n$:

$$\alpha_i(x) < y_i(x) < \beta_i(x). \tag{2.6}$$

Remark 2.2. In the formulation of the problem (1.1), (1.2) the inequalities $x_k < x_{k+1}$ and $x_s < x_{s+1}$ were supposed. Analyzing the method of proof of Theorem 2.1 we conclude that the result remains valid in the cases when $x_k \le x_{k+1}$ and $x_s \le x_{s+1}$ too. This means, for example, that a singular Cauchy problem

$$y_i' = f_i(x,y), \quad y_i(x_1^+) = A_i, \quad i = 1, \ldots, n \tag{2.7}$$

is a partial case of given result as well as a two-point boundary-value problem:

$$\begin{gathered} y_i' = f_i(x,y), \quad i = 1, \ldots, n \\ y_p(x_1^+) = A_p, \quad p = 1, \ldots, k, \qquad y_r(x_n^-) = A_r, \quad r = k+1, \ldots, n. \end{gathered} \tag{2.8}$$

Remark 2.3. In [9] a technique based on Kneser's theorem is introduced to extend the topological method of Ważewski for Carathéodory systems. It has, for example, been used to study the asymptotic behavior of the solutions of a perturbed linear system:

$$\dot{x} = [A(t) + B(t)]x + g(t,x), \tag{2.9}$$

where the $n \times n$ matrices A (diagonal) and B are locally integrable, $g \in \mathrm{Car}_{\mathrm{loc}}((t_0, \infty) \times \mathbb{C}^n)$, and the solutions are unique with respect to their initial values. The existence of solutions $x_p = (x_{p1}, x_{p2}, \dots, x_{pn})$, $p = 1, 2, \dots, n$ such that for any $i \neq p$

$$\lim_{t \to \infty} \frac{x_{pi}(t)}{x_{pp}(t)} = 0 \tag{2.10}$$

is studied. This is accomplished with the above-mentioned extension of the topological retract method for Carathédory systems which can be applied due to the construction of a suitable regular polyfacial sets. This technique makes it possible to extend the results initially proved (by Ważewski's method) for ordinary differential equations with continuous right-hand sides to Carathédory systems. Similar method has, for example, been used in a recent paper [6] where the technique developed in [9] is utilized. Along these lines, we can analyse our result in terms of its possible extension to systems (1.1) with Carathéodory right-hand sides. Since the Lipschitz-type condition (2.3) is necessary in the proof of Theorem 2.1 for verifying the continuity of the operator $\mathfrak{T}$, it cannot be omitted. Therefore, our result seems to be extendable for Carathéodory systems (1.1) if the uniqueness of the solutions is ensured with respect to their initial values (save at singular points).

3. Examples

Let us consider two illustrative nonlinear systems. The first one has a linear part which determines the existence of the solution of the problem considered. The second one is a perturbation of a system for which we know analytic solution of singular problem.

Example 3.1. Let us consider a singular problem:

$$\begin{gathered}
y_1' = 2\frac{y_1}{x} - \frac{x(y_3^2 + 1)}{10}, \\
y_2' = \frac{y_2}{(x - 1/2)^2} - \frac{y_1 + y_3 + 1}{20(x - 1/2)}, \\
y_3' = -\frac{3y_3}{1 - x} + \frac{(1 - x)(y_2^2 + 1)}{10}, \\
y_1(0^+) = 0, \qquad y_2\left(\frac{1}{2^\pm}\right) = 0, \qquad y_3(1^-) = 0.
\end{gathered} \tag{3.1}$$

For this problem all conditions of Theorem 2.1 are valid for

$$\begin{gathered}
\alpha_1(x) = 0, \qquad \beta_1(x) = 3x, \\
\alpha_2(x) = \begin{cases} 3\left(x - \dfrac{1}{2}\right), & \text{if } x \in \left[0, \dfrac{1}{2}\right], \\ 0, & \text{if } x \in \left(\dfrac{1}{2}, 1\right], \end{cases}
\end{gathered}$$

$$\beta_2(x) = \begin{cases} 0, & \text{if } x \in \left[0, \dfrac{1}{2}\right], \\ 3\left(x - \dfrac{1}{2}\right), & \text{if } x \in \left(\dfrac{1}{2}, 1\right], \end{cases}$$
$$\alpha_3(x) = 0, \qquad \beta_3(x) = 3(1-x)^2 \tag{3.2}$$

and, for example, for $\xi = 1$, $M_{11}(x) = N_{11}(x) = 2/x$, $M_{12}(x) = N_{12}(x) = M_{31}(x) = N_{31}(x) = 0$, $M_{13}(x) = -N_{13}(x) = -3x(1-x)^2/5$, $M_{21}(x) = M_{23}(x) = -N_{21}(x) = -N_{23}(x) = -1/(20|x - 1/2|)$, $M_{22}(x) = N_{22}(x) = 1/(x-1/2)^2$, $M_{32}(x) = -N_{32}(x) = -|x - 1/2|$, and $M_{33}(x) = -N_{33}(x) = -3/(1-x)$. Consequently, there is at least one solution to this problem $y(x) = (y_1(x), y_2(x), y_3(x))$ such that

$$0 < y_1(x) < 3x \quad \text{for } x \in (0,1],$$
$$\min\left\{3\left(x - \frac{1}{2}\right); 0\right\} < y_2(x) < \max\left\{0; 3\left(x - \frac{1}{2}\right)\right\} \quad \text{for } x \in \left[0, \frac{1}{2}\right) \cup \left(\frac{1}{2}, 1\right], \tag{3.3}$$
$$0 < y_3(x) < 3(1-x)^2 \quad \text{for } x \in [0,1).$$

Example 3.2. Let us consider a singular problem:

$$y_1' = \frac{2}{x^3} y_1^2 + \varepsilon x^2 y_2^2,$$
$$y_2' = \frac{3}{(x-1/2)^4} y_2^2 + \varepsilon \left(x - \frac{1}{2}\right)^3 y_3^2,$$
$$y_3' = \frac{-2}{(1-x)^3} y_3^2 + \varepsilon (1-x)^2 y_1^2, \tag{3.4}$$
$$y_1(0^+) = 0, \qquad y_2\left(\frac{1}{2^{\pm}}\right) = 0, \qquad y_3(1^-) = 0,$$

where ε is a real constant, $|\varepsilon| < 0.01$. For this problem all conditions of Theorem 2.1 are valid, for example, for

$$\alpha_1(x) = x^2 - 0.1x^3, \qquad \beta_1(x) = x^2 + 0.1x^3,$$
$$\alpha_2(x) = \left(x - \frac{1}{2}\right)^3 - 0.1\left(x - \frac{1}{2}\right)^4, \qquad \beta_2(x) = \left(x - \frac{1}{2}\right)^3 + 0.1\left(x - \frac{1}{2}\right)^4, \tag{3.5}$$
$$\alpha_3(x) = (1-x)^2 - 0.1(1-x)^3, \qquad \beta_3(x) = (1-x)^2 + 0.1(1-x)^3$$

and for $\xi = 1$, $M_{11}(x) = 2/x$, $M_{12}(x) = M_{31}(x) = -N_{12}(x) = -N_{31}(x) = -M_{23}(x) = N_{23}(x) = -0.1$, $N_{11}(x) = 6/x$, $M_{13}(x) = N_{13}(x) = M_{21}(x) = N_{21}(x) = M_{32}(x) = N_{32}(x) = 0$, $M_{22}(x) = -12/(1/2 - x)$ if $x < 1/2$, $N_{22}(x) = -5/(1/2 - x)$ if $x < 1/2$, $M_{22}(x) = 3/(x - 1/2)$ if $x > 1/2$, $N_{22}(x) = 12/(x - 1/2)$ if $x > 1/2$, $M_{33}(x) = -5/(1-x)$, and $N_{33}(x) = -3/(1-x)$.

Consequently, there is at least one solution to this problem $y(x) = (y_1(x), y_2(x), y_3(x))$ such that

$$x^2 - 0.1x^3 < y_1(x) < x^2 + 0.1x^3 \quad \text{for } x \in (0,1],$$
$$\left(x-\frac{1}{2}\right)^3 - 0.1\left(x-\frac{1}{2}\right)^4 < y_2(x) < \left(x-\frac{1}{2}\right)^3 + 0.1\left(x-\frac{1}{2}\right)^4 \quad \text{for } x \in [0,1] \setminus \left\{\frac{1}{2}\right\}, \tag{3.6}$$
$$(1-x)^2 - 0.1(1-x)^3 < y_3(x) < (1-x)^2 + 0.1(1-x)^3 \quad \text{for } x \in [0,1).$$

If $\varepsilon = 0$, then the considered system turns into system

$$y_1' = \frac{2}{x^3}y_1^2, \qquad y_2' = \frac{3}{(x-1/2)^4}y_2^2, \qquad y_3' = \frac{-2}{(1-x)^3}y_3^2, \tag{3.7}$$

having solution

$$y_1 = x^2, \qquad y_2 = \left(x-\frac{1}{2}\right)^3, \qquad y_3 = (1-x)^2, \tag{3.8}$$

which satisfies (3.4).

4. Preliminaries

In the sequel we will apply topological method of Ważewski (see, e.g., [12–14]). Therefore we give a short summary of it. Let us consider the system of ordinary differential equations

$$y' = g(x,y) \tag{4.1}$$

with $y \in \mathbb{R}^n$. Below, it will be assumed that the right-hand sides of the system (4.1) are continuous functions defined on an open (x,y)-set $\Omega^* \subset \mathbb{R} \times \mathbb{R}^n$.

Definition 4.1 (see [12]). An open subset Ω^0 of the set Ω^* is called an (n,p)-subset of Ω^* with respect to the system (4.1) if the following conditions are satisfied.

(1) There exist continuously differentiable functions $n_i : \Omega^* \to \mathbb{R}$, $i = 1,\dots,\ell$ and $p_j : \Omega^* \to \mathbb{R}$, $j = 1,\dots,m$; $\ell + m > 0$ such that

$$\Omega^0 = \{(x,y) \in \Omega^* : n_i(x,y) < 0,\ p_j(x,y) < 0\ \forall i,j\}. \tag{4.2}$$

(2) $\dot{n}_\alpha(x,y) < 0$ holds for the derivatives of the functions $n_\alpha(x,y)$, $\alpha = 1,\dots,\ell$ along trajectories of the system (4.1) on the set

$$N_\alpha = \{(x,y) \in \Omega^*,\ n_\alpha(x,y) = 0,\ n_i(x,y) \le 0,\ p_j(x,y) \le 0\ \forall i \ne \alpha \text{ and } j\}. \tag{4.3}$$

(3) $\dot{p}_\beta(x,y) > 0$ holds for the derivatives of the functions $p_\beta(x,y)$, $\beta = 1,\dots,m$ along trajectories of the system (4.1) on the set

$$P_\beta = \{(x,y) \in \Omega^*,\ p_\beta(x,y) = 0,\ n_i(x,y) \le 0,\ p_j(x,y) \le 0\ \forall i \text{ and } j \ne \beta\}. \tag{4.4}$$

As usual, if $\omega \subset \mathbb{R}\times\mathbb{R}^n$, then $\operatorname{int}\omega, \partial\omega$ and $\overline{\omega}$ denote the interior, the boundary, and the closure of ω, respectively.

Definition 4.2. The point $(x_0,y_0) \in \Omega^*\cap\partial\Omega^0$ is called an *egress point* (or *ingress point*) of Ω^0 with respect to the system (4.1) if, for every fixed solution of the problem $y(x_0) = y_0$, there exists an $\varepsilon > 0$ such that $(x,y(x)) \in \Omega^0$ for $x_0 - \varepsilon \le x < x_0$ ($x_0 < x \le x_0 + \varepsilon$). An egress point (ingress point) (x_0,y_0) of Ω^0 is called a *strict egress point* (*strict ingress point*) of Ω^0 if $(x,y(x)) \notin \overline{\Omega}^{\,0}$ on interval $x_0 < x \le x_0 + \varepsilon_1$ ($x_0 - \varepsilon_1 \le x < x_0$) for an $\varepsilon_1 > 0$.

The set of all points of egress (strict egress) is denoted by Ω_e^0 (Ω_{se}^0). It is proved in [12, page 281], that when a set Ω^0 is an (n,p)-subset of Ω^* then $\Omega_e^0 \equiv \Omega_{se}^0$.

Theorem 4.3 (see [12, page 282]). *Let Ω^0 be some (n,p)-subset of Ω^* with respect to the system* (4.1). *Let S be a nonempty compact subset of $\Omega^0 \cup \Omega_e^0$ such that the set $S \cap \Omega_e^0$ is not a retract of S but is a retract of Ω_e^0. Then there is at least one point $(x_0,y_0) \in S \cap \Omega^0$ such that the graph of a solution $y(x)$ of the Cauchy problem $y(x_0) = y_0$ for* (4.1) *lies in Ω^0 on its right-hand maximal interval of existence.*

5. Partial Singular Problems

At this part we are interested in existence of solutions of some auxiliary singular problems for *one scalar equation*. We consider two cases below with respect to the location of singular point (at the left end or at the right end of the interval considered).

5.1. Singular Point Coincides with the Left End of Interval

Consider the initial problem

$$y' = B(x,y), \tag{5.1}$$

$$y(u^+) = K, \quad (K \in \mathbb{R}) \tag{5.2}$$

on an interval $(u,v]$ with $u < v$. By a solution of problem (5.1), (5.2) on interval $(u,v]$ we mean the function $y \in C([u,v],\mathbb{R}) \cap C^1((u,v],\mathbb{R})$ which satisfies (5.1) on $(u,v]$ and the condition (5.2). Let functions $\lambda(x), \mu(x)$ be continuously differentiable on $(u,v]$, $\lambda(u^+) = \mu(u^+) = K$ and $\lambda(x) < \mu(x)$ on $(u,v]$. Denote

$$\Theta^+ = \{(x,y) : x \in (u,v],\ \lambda(x) < y < \mu(x)\}. \tag{5.3}$$

Let us suppose that there exists a domain $\widetilde{\Theta} \subset (u,v] \times \mathbb{R}$, such that $\Theta^+ \subset \widetilde{\Theta}$ and the cross section $S^+(x) = \{(x,y) \in \widetilde{\Theta}\}$ is an open set for every $x \in (u,v]$. Define an auxiliary function

$$H(x,y) \equiv B(x,y) - y'. \tag{5.4}$$

Lemma 5.1. *Suppose that $B \in C(\widetilde{\Theta},\mathbb{R})$ satisfies the local Lipschitz condition with respect to the variable y in $\widetilde{\Theta}$ and, moreover,*

$$H(x,\lambda(x)) < 0 < H(x,\mu(x)) \quad \text{if } x \in (u,v]. \tag{5.5}$$

Then each point (v,y^*) where $y^* \in [\lambda(v),\mu(v)]$ defines a solution $y = y^*(x)$ of (5.1) *on $(u,v]$ such that* (5.2) *holds, $y^*(v) = y^*$, and*

$$\lambda(x) < y^*(x) < \mu(x), \quad x \in (u,v]. \tag{5.6}$$

Proof. Let us evaluate the derivative of the function $w(x,y) \equiv (y-\lambda(x))(y-\mu(x))$ along the trajectories of (5.1) if $(x,y) \in \mathcal{N}$, where

$$\mathcal{N} = \{(x,y) : x \in (u,v],\ w(x,y) = 0\}. \tag{5.7}$$

We get

$$\begin{aligned} \frac{dw(x,y)}{dx} &= (y' - \lambda'(x)) \cdot (y - \mu(x)) + (y - \lambda(x)) \cdot (y' - \mu'(x)) \\ &= [B(x,y) - \lambda'(x)](y - \mu(x)) + (y - \lambda(x))[B(x,y) - \mu'(x)]. \end{aligned} \tag{5.8}$$

Since $(x,y) \in \mathcal{N}$, then either $y = \mu(x)$ or $y = \lambda(x)$. In the first case we have

$$\left.\frac{dw(x,y)}{dx}\right|_{y=\mu(x)} = (\mu(x) - \lambda(x)) \cdot H(x,\mu(x)) \tag{5.9}$$

and in the second one

$$\left.\frac{dw(x,y)}{dx}\right|_{y=\lambda(x)} = -H(x,\lambda(x)) \cdot (\mu(x) - \lambda(x)). \tag{5.10}$$

Thus, in view of condition (5.5),

$$\left.\frac{dw(x,y)}{dx}\right|_{(x,y)\in\mathcal{N}} > 0, \tag{5.11}$$

and, consequently, all points of the set $\mathcal{N} = \widetilde{\Theta} \cap \partial\Theta^+$ are for $x \in (u,v)$ the points of strict egress of Θ^+ with respect to (5.1).

Let us consider behaviour of a solution $y = y^*(x)$ of the problem $y^*(v) = y^* \in [\lambda(v), \mu(v)]$ for decreasing values of $x \in (u, v]$. Let us suppose that this solution leaves the domain Θ^+ passing through a boundary point $(x^0, y^*(x^0)) \in \mathcal{N}$ where $x^0 \in (u, v)$ and $(x, y(x)) \in \Theta^+$ for $x \in (x^0, v]$. In this is case this point a point of ingress (for increasing x) with respect to (5.1) and this contradicts the fact that each point of the set $\mathcal{N}$ is for $x \in (u, v)$ a point of strict egress. Only one possibility remains valid: solution $y^*(x)$ is simultaneously a solution of the problem (5.1), (5.2). The lemma is proved. □

Lemma 5.2. *Let all assumptions of Lemma 5.1 hold except condition* (5.5) *which is replaced by the condition:*

$$H(x, \mu(x)) < 0 < H(x, \lambda(x)) \quad \text{if } x \in (u, v]. \tag{5.12}$$

Then there is at least one solution $y = y^*(x)$ of the problem (5.1), (5.2) *on $(u, v]$ such that inequalities* (5.6) *hold.*

Proof. Let us define the set $\mathcal{N}$ and the function $w(x, y)$ in the same way as in the proof of Lemma 5.1. Then the derivative of $w(x, y)$ along the trajectories of (5.1) satisfies, in view of condition (5.12), the inequality

$$\left.\frac{dw(x, y)}{dx}\right|_{(x,y)\in\mathcal{N}} < 0. \tag{5.13}$$

This means that all points of the set $\mathcal{N}$ are for $x \in (u, v)$ the points of strict ingress of Θ^+ with respect to (5.1).

Let us change the orientation of the x-axis *into reverse*. Then all points of the set $\mathcal{N}$ are for $x \in (u, v)$ the points of strict egress of Θ^+ with respect to (5.1).

Is it easy to see that the two-point set $\{\lambda(v - \varrho), \mu(v - \varrho)\}$, where ϱ is a small positive number, is a retract of the set $\mathcal{N}$ in view of existence of the retraction

$$r(x, y) = \left(v - \varrho, \mu(v - \varrho) + [\lambda(v - \varrho) - \mu(v - \varrho)] \frac{y - \mu(x)}{\lambda(x) - \mu(x)}\right), \tag{5.14}$$

where $(x, y) \in \mathcal{N}$. Clearly, the nonempty compact set $S = [\lambda(v - \varrho), \mu(v - \varrho)]$ is not a retract of its boundary $\partial S = \{\lambda(v - \varrho), \mu(v - \varrho)\}$ (see, e.g., [16]). All assumptions of topological principle of Ważewski are valid, and, by Theorem 4.3 (in its formulation we put $\Omega^0 \equiv \operatorname{int}\Theta^+$, $p_1(x, y) \equiv w(x, y)$, $j = 1$, $n_1 \equiv x - v + \varrho$ and $\ell = 1$), there exists at least one solution $y = y^*(x)$ of the problem (5.1), (5.2) with graph belonging to the domain Θ^+ on $(u, v - \varrho]$. By the same arguments, as in the proof of Lemma 5.1, this solution can be continued on the interval $(u, v]$. The lemma is proved. □

5.2. *Singular Point Coincides with the Right End of Interval*

Let us consider the initial problem (5.1), (5.15) where

$$y(v^-) = K, \quad (K \in \mathbb{R}) \tag{5.15}$$

on an interval $[u, v)$, with $u < v$. By a solution of (5.1), (5.15) on interval $[u, v)$ we mean the function $y \in C([u, v], \mathbb{R}) \cap C^1([u, v), \mathbb{R})$ which satisfies (5.1) on interval $[u, v)$ and condition (5.15). Let $\lambda(x), \mu(x)$ be continuously differentiable functions on $[u, v), \lambda(v^-) = \mu(v^-) = K$ and $\lambda(x) < \mu(x)$ on $[u, v)$. Denote

$$\Theta^- = \{(x, y) : x \in [u, v),\ \lambda(x) < y < \mu(x)\}. \tag{5.16}$$

Let us suppose that there exists a domain $\overline{\Theta} \subset [u, v) \times \mathbb{R}$, such that $\Theta^- \subset \overline{\Theta}$ and the cross section $S^-(x) = \{(x, y) \in \overline{\Theta}\}$ is an open set for every $x \in [u, v)$. The proofs of following Lemmas 5.3 and 5.4 can be made by the similar manner as the proofs of Lemmas 5.1 and 5.2. Hence, they are omitted.

Lemma 5.3. *Suppose that $B \in C(\overline{\Theta}, \mathbb{R})$ satisfies the local Lipschitz condition with respect to the variable y in $\overline{\Theta}$ and, moreover,*

$$H(x, \lambda(x)) < 0 < H(x, \mu(x)) \quad \text{if } x \in [u, v). \tag{5.17}$$

Then there is at least one solution $y = y^{**}(x)$ of the problem (5.1), (5.15) *on $(u, v]$ such that*

$$\lambda(x) < y^{**}(x) < \mu(x). \tag{5.18}$$

Lemma 5.4. *Let all assumptions of Lemma 5.3 hold except condition* (5.17) *which is replaced by the condition:*

$$H(x, \mu(x)) < 0 < H(x, \lambda(x)) \quad \text{if } x \in [u, v). \tag{5.19}$$

Then each point (u, y^{**}) where $y^{**} \in [\lambda(u), \mu(u)]$ defines a solution $y = y^{**}(x)$ of (5.1) *on $[u, v)$, $y^{**}(u) = y^{**}$ and the inequalities* (5.18) *hold.*

6. Proof of Theorem 2.1

6.1. Construction of Operator

Let us consider the system

$$y_i' = f_i(x, \varphi_1(x), \ldots, \varphi_{i-1}(x), y_i, \varphi_{i+1}(x), \ldots, \varphi_n(x)), \quad i = 1, 2, \ldots, n \tag{6.1}$$

with $(\varphi_1(x), \ldots, \varphi_n(x)) \in M$, where

$$\begin{aligned} M = \{ & (\varphi_1(x), \varphi_2(x), \ldots, \varphi_n(x)),\ x \in I,\ \varphi_i \in C(I, \mathbb{R}), \\ & \alpha_i(x) \le \varphi_i(x) \le \beta_i(x),\ i = 1, 2, \ldots, n\} \end{aligned} \tag{6.2}$$

This system, strictly speaking, consists of separated scalar equations. Therefore in the following we will consider equations of system (6.1) separately.

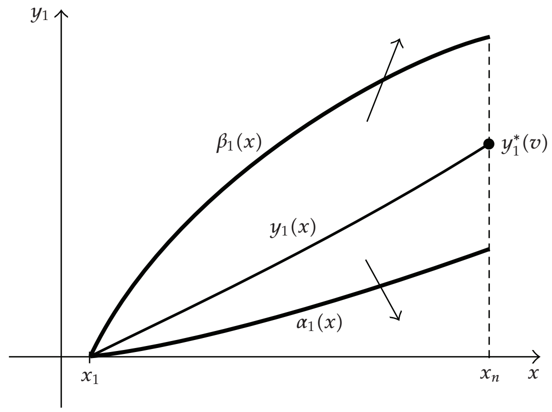

Figure 1: Using Lemma 5.1.

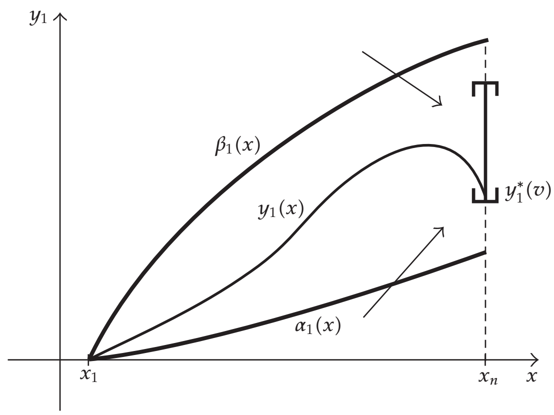

Figure 2: Using Lemma 5.2.

(a) Let us consider the first equation of system (6.1) (which corresponds to the value $i = 1$) together with corresponding initial value which is subtracted from (1.2):

$$\begin{gathered} y_1'(x) = f_1(x, y_1, \varphi_2(x), \dots, \varphi_n(x)), \\ y_1(x_1^+) = A_1. \end{gathered} \tag{6.3}$$

Let us put $B(x, y) \equiv f_1(x, y, \varphi_2(x), \dots, \varphi_n(x))$, $\lambda(x) \equiv \alpha_1(x)$, $\mu(x) \equiv \beta_1(x)$, $u = x_1$, $v = x_n$ and $K = A_1$. In view of condition (2.5) we see that either condition (5.5) or condition (5.12) holds for $H(x, y) \equiv F_1(x, y)$. From Lemmas 5.1 and 5.2 (it is easy to see that their assumptions are valid) the existence of a solution of the problem (6.3) satisfying inequalities (5.6) follows. An illustration to the cases where Lemma 5.1 or Lemma 5.2 is used is given in Figures 1 and 2.

In the sequel we will consider a solution $y_1(x) = y_1^*(x)$ of problem (6.3) chosen in a unique way. We define this solution (in the case when Lemma 5.1 was used) by means of the additional condition

$$y_1(x_n) = y_1^*(v) = y_1^* = \frac{1}{2}\left(\alpha_1(x_n) + \beta_1(x_n)\right). \tag{6.4}$$

If Lemma 5.2 was used, then denote the set of all solutions of problem (6.3) with the indicated properties as a set Y and put $y_1(x_n) = y_1^*(v) = \min\{y(v) : y \in Y\}$. Obviously this minimum exists and $y_1^*(v) > \lambda(v)$.

Define the first coordinate T_1 of operator $\mathcal{T}$ by relation

$$T_1(\varphi_2, \ldots, \varphi_n) = y_1^*. \tag{6.5}$$

From inequalities (5.6) it follows that $(y_1^*, \varphi_2, \ldots, \varphi_n) \in M$. The same reasoning can be repeated for $i = 2, \ldots, k$.

(b) Now consider the last equation of system (6.1) (which corresponds to the value $i = n$) together with the corresponding initial value which is subtracted from (1.2):

$$\begin{gathered} y_n' = f_n(x, \varphi_1(x), \ldots, \varphi_{n-1}(x), y_n), \\ y_n(x_n^-) = A_n. \end{gathered} \tag{6.6}$$

Let us put $B(x,y) \equiv f_n(x, \varphi_1(x), \ldots, \varphi_{n-1}(x), y)$, $\lambda(x) \equiv \alpha_n(x)$, $\mu(x) \equiv \beta_n(x)$, $u = x_1$, $v = x_n$ and $K = A_n$. In view of condition (2.5) we see that either condition (5.17) or condition (5.19) holds for $H(x,y) \equiv F_n(x,y)$. From Lemmas 5.3 and 5.4 the existence of a solution of the problem (6.6) satisfying inequalities (5.18) follows. Similarly as in the part (a) above, we chose the solution $y_n(x) = y_n^{**}(x)$ of problem (6.6), which is uniquely defined.

Define the last coordinate T_n of operator $\mathcal{T}$ by relation

$$T_n(\varphi_1, \ldots, \varphi_{n-1}) = y_n^*. \tag{6.7}$$

From inequalities (5.18) it follows that $(\varphi_1, \ldots, \varphi_{n-1}, y_n^*) \in M$. The same reasoning can be repeated for $i = s+1, \ldots, n-1$.

(c) Let us consider the equation of system (6.1) which corresponds to the value $i = s$ together with corresponding initial value which follows from (1.2):

$$\begin{gathered} y_s' = f_s(x, \varphi_1(x), \ldots, \varphi_{s-1}(x), y_s, \varphi_{s+1}(x), \ldots, \varphi_n(x)), \\ y_s(x^{\pm}) = A_s. \end{gathered} \tag{6.8}$$

Let us put $B(x,y) \equiv f_s(x, \varphi_1(x), \ldots, \varphi_{s-1}(x), y, \varphi_{s+1}(x), \ldots, \varphi_n(x)$, $\lambda(x) \equiv \alpha_s(x)$, $\mu(x) \equiv \beta_s(x)$ and $K = A_s$. Consider, at first, the problem (6.8) on interval $[x_1, x_s)$. For this, let us put $u = x_1$, $v = x_s$. In view of condition (2.5) we see that either condition (5.17) or condition (5.19) holds for $H(x,y) = F_s(x,y)$ and with the aid of Lemmas 5.3 and 5.4 (as in the part (b)) we can define the unique solution $y_s(x) = y^{\Delta\Delta}(x)$ of (6.8) on interval $[x_1, x_s)$. Now consider the problem (6.8) on interval $(x_s, x_n]$. Put $u = x_s$, $v = x_n$. In view of condition (2.5) we see that either condition (5.5) or condition (5.12) holds for $H(x,y) \equiv F_s(x,y)$ and with the aid of Lemmas 5.1 and 5.2 (as in part (a)) we define the unique solution $y_s(x) = y^{\Delta}(x)$ of (6.8) on interval $(x_s, x_n]$.

At the end we define, by a unique manner, the solution $y_s^*(x)$ of the problem (6.8) as

$$y_s^*(x) = \begin{cases} y^{\Delta\Delta}(x), & x \in [x_1, x_s), \\ y^{\Delta}(x), & x \in (x_s, x_n]. \end{cases} \tag{6.9}$$

Define the sth coordinate T_s of operator $\mathfrak{T}$ by relation

$$T_s(\varphi_1, \ldots, \varphi_{s-1}, \varphi_{s+1}, \ldots, \varphi_n) = y_s^*. \tag{6.10}$$

It is easy to see that $(\varphi_1, \ldots, \varphi_{s-1}, y_s^*, \varphi_{s+1}, \ldots, \varphi_n) \in M$. The same reasoning can be repeated for $i = k+1, \ldots, s-1$.

(d) Now we are able to define operator $\mathfrak{T}$. For every $\varphi = (\varphi_1, \ldots, \varphi_n) \in M$ define $\mathfrak{T}\varphi = y^*$ with

$$\mathfrak{T} = (T_1, T_2, \ldots, T_n), \tag{6.11}$$

where $y^* = (y_1^*, \ldots, y_n^*) \in M$. Note that y^* is defined in the unique way. Obviously, $\mathfrak{T}(M) \subset M$.

6.2. Verification of Schauder's Assumptions

Let us consider the Banach space Ψ of functions $\varphi(x) = (\varphi_1(x), \varphi_2(x), \ldots, \varphi_n(x))$, continuous on I, with the norm

$$\|\varphi\| = \max_{i=1,2,\ldots,n} \left\{ \max_I |\varphi_i(x)| \right\}. \tag{6.12}$$

Clearly $M \subset \Psi$ and, as follows from the properties of the functions $\alpha_i(x)$, $\beta_i(x)$, $i = 1, 2, \ldots, n$, M is a closed, bounded and convex set. It remains to prove that $\mathfrak{T}$ is a continuous mapping such that $\mathfrak{T}(M)$ is a relatively compact subset of Ψ. Then all the assumptions of Schauder's fixed-point theorem will be satisfied (e.g., [15, page 29]). With respect to the relative compactness of $\mathfrak{T}(M)$ it is sufficient to prove in accordance with Arzelà-Ascoli theorem that $\mathfrak{T}(M)$ is uniformly bounded and equicontinuous on I.

(α) The *uniform boundedness* follows from the inequality

$$\|\varphi\| < L, \tag{6.13}$$

where $L = \max_I\{|\alpha_i(x)|, |\beta_i(x)|, i = 1, 2, \ldots, n\}$, which holds for every $\varphi \in M$.

(β) Let us prove the *equicontinuity* of each function $\varphi \in \mathcal{T}(M)$. On I_1 the first coordinate φ_1 of φ satisfies (as it follows from the construction of $\mathcal{T}$) an equation of the type

$$\varphi_1' = f_1\big(x, \varphi_1, \nu_2(x), \dots, \nu_n(x)\big) \tag{6.14}$$

with $(\varphi_1, \nu_2, \dots, \nu_n) \in M$. Since $f_1 \in C(\Theta_1, \mathbb{R})$, (6.14) yields

$$\left|\varphi_1'(x)\right| < K_\delta, \quad x \in [x_1 + \delta, x_n], \quad x_1 + \delta < x_n, \quad 0 < \delta = \text{const}, \tag{6.15}$$

where the constant K_δ exists and depends on δ. Let us put $\delta_1 = \min(\delta/2, \varepsilon^*/K_{\delta/2})$ where ε^* is an arbitrary positive number and δ is so small that

$$\max_{[x_1, x_1+\delta]} \left|\beta_1(x) - A_1\right| < \frac{\varepsilon^*}{2}, \qquad \max_{[x_1, x_1+\delta]} |\alpha_1(x) - A_1| < \frac{\varepsilon^*}{2}. \tag{6.16}$$

Let us suppose that $|z_1 - z_2| < \delta_1$, $z_1, z_2 \in [x_1, x_n]$. Then either $z_1, z_2 \in [x_1, x_1 + \delta]$ or $z_1, z_2 \in [x_1 + \delta/2, x_n]$. In the first case

$$\left|\varphi_1(z_1) - \varphi_1(z_2)\right| \le \left|\varphi_1(z_1) - A_1\right| + \left|\varphi_1(z_2) - A_1\right| < \frac{\varepsilon^*}{2} + \frac{\varepsilon^*}{2} = \varepsilon^* \tag{6.17}$$

and in the second one (by Lagrange's mean value theorem)

$$\left|\varphi_1(z_1) - \varphi_1(z_2)\right| \le K_{\delta/2}|z_1 - z_2| < \varepsilon^*. \tag{6.18}$$

So, for each positive ε^* there is a $\delta_1 > 0$ such that $|\varphi_1(z_1) - \varphi_1(z_2)| < \varepsilon^*$ for $|z_1 - z_2| < \delta_1$ and each function of the type of $\varphi_1(x)$ is equicontinuous. By analogy we can show that the functions of the type $\varphi_j(x)$, $j = 2, \dots, n$ are equicontinuous too. Finally, for $|z_1 - z_2| < \delta_1$, we get $\|\varphi(z_1) - \varphi(z_2)\| < \varepsilon^*$ and the equicontinuity of the set $\mathcal{T}(M)$ is proved.

(γ) *Continuity* of operator $\mathcal{T}$. Let us suppose that $y^0, \tilde{y} \in M$ and

$$Y^0 = \mathcal{T} y^0, \qquad \tilde{Y} = \mathcal{T} \tilde{y}. \tag{6.19}$$

In the sequel we prove that the operator $\mathcal{T}$ is continuous. We prove that

$$\left\|Y^0 - \tilde{Y}\right\| < \varepsilon \quad \text{if } \left\|y^0 - \tilde{y}\right\| < \delta, \tag{6.20}$$

where $\delta \le \varepsilon\xi$ and ξ was defined in formulation of Theorem 2.1. The following construction will show that operator $\mathcal{T}$ is continuous. All expressions in the following will be well defined (supposing, if necessary, ε sufficiently small).

Consider the *identity* (see the definition of $\mathcal{T}$)

$$Y_i^{0'}(x) \equiv f_i\Big(x, \eta_1^0(x), \eta_2^0(x), \ldots, \eta_n^0(x)\Big), \tag{6.21}$$

where $i = 1, 2, \ldots, n$, $\eta_i^0(x) \equiv Y_i^0(x)$, $\eta_j^0(x) \equiv y_j^0(x)$, $j \neq i$, $(x, \eta_1^0, \eta_2^0, \ldots, \eta_n^0) \in \Omega_i$ and the *equation*

$$\widetilde{Y}_i' = f_i(x, \widetilde{\eta}_1(x), \widetilde{\eta}_2(x), \ldots, \widetilde{\eta}_n(x)), \tag{6.22}$$

where $i = 1, 2, \ldots, n$, $\widetilde{\eta}_i = \widetilde{Y}_i$, $\widetilde{\eta}_j = \widetilde{\eta}_j(x) \equiv \widetilde{y}_j(x)$, $j \neq i$, $(x, \widetilde{\eta}_1(x), \widetilde{\eta}_2(x), \ldots, \widetilde{\eta}_n(x)) \in \Omega_i$. Note that in view of definition of $\mathcal{T}$ a *solution* of (6.22) *is given* by $\widetilde{Y}_i \equiv \widetilde{Y}_i(x)$. Define, for $i = 1, 2, \ldots, n$, the functions

$$W_i\Big(x, \widetilde{Y}_i\Big) = \Big(\widetilde{Y}_i - Y_i^0(x) - \varepsilon\Big)\Big(\widetilde{Y}_i - Y_i^0(x) + \varepsilon\Big) \tag{6.23}$$

and the sets

$$\mathcal{D}_i = \Big\{\Big(x, \widetilde{Y}_i\Big) : \Big(x, \widetilde{Y}_i\Big) \in \Omega_i, W_i\Big(x, \widetilde{Y}_i\Big) = 0\Big\}. \tag{6.24}$$

(γ_1) Let us evaluate the derivative of $W_1(x, \widetilde{Y}_1)$ along the trajectories of (6.22) for $i = 1$ if $(x, \widetilde{Y}_1) \in \mathcal{D}_1$. We get,

$$\frac{dW_1\Big(x, \widetilde{Y}_1\Big)}{dx} = \Big[\widetilde{Y}_1' - Y_1^{0'}(x)\Big]\Big(\widetilde{Y}_1 - Y_1^0(x) + \varepsilon\Big) + \Big(\widetilde{Y}_1 - Y_1^0(x) - \varepsilon\Big)\Big[\widetilde{Y}_1' - Y_1^{0'}(x)\Big]. \tag{6.25}$$

Since $(x, \widetilde{Y}_1) \in \mathcal{D}_1$, then either $\widetilde{Y}_1 = Y_1^0(x) + \varepsilon$ or $\widetilde{Y}_1 = Y_1^0(x) - \varepsilon$. So,

$$\begin{aligned}
&\left.\frac{dW_1(x, \widetilde{Y}_1)}{dx}\right|_{\widetilde{Y}_1 = Y_1^0(x) \pm \varepsilon} \\
&\quad = \pm 2\varepsilon\Big[f_1\Big(x, Y_1^0(x) \pm \varepsilon, \widetilde{y}_2(x), \ldots, \widetilde{y}_n(x)\Big) - f_1\Big(x, Y_1^0(x), y_2^0(x), \ldots, y_n^0(x)\Big)\Big].
\end{aligned} \tag{6.26}$$

According to (2.3) and (6.20):

$$\begin{aligned}
&\varepsilon\left(M_{11}(x)-\xi\sum_{j=2}^{n}\left|M_{1j}(x)\right|\right)\leq\varepsilon M_{11}(x)+\sum_{j=2}^{n}M_{1j}(x)\left|\tilde{y}_j(x)-y_j^0(x)\right|\\
&\quad\leq f_1\left(x,Y_1^0(x)+\varepsilon,\tilde{y}_2(x),\ldots,\tilde{y}_n(x)\right)-f_1\left(x,Y_1^0(x),y_2^0(x),\ldots,y_n^0(x)\right)\\
&\quad\leq\varepsilon N_{11}(x)+\sum_{j=2}^{n}N_{1j}(x)\left|\tilde{y}_j(x)-y_j^0(x)\right|\leq\varepsilon\left(N_{11}(x)+\xi\sum_{j=2}^{n}\left|N_{1j}(x)\right|\right),\\
&\varepsilon\left(-N_{11}(x)-\xi\sum_{j=2}^{n}\left|N_{1j}(x)\right|\right)\leq-\varepsilon N_{11}(x)-\sum_{j=2}^{n}N_{1j}(x)\left|\tilde{y}_j(x)-y_j^0(x)\right|\\
&\quad\leq f_1\left(x,Y_1^0(x)-\varepsilon,\tilde{y}_2(x),\ldots,\tilde{y}_n(x)\right)-f_1\left(x,Y_1^0(x),y_2^0(x),\ldots,y_n^0(x)\right)\\
&\quad\leq-\varepsilon M_{11}(x)-\sum_{j=2}^{n}M_{1j}(x)\left|\tilde{y}_j(x)-y_j^0(x)\right|\leq\varepsilon\left(-M_{11}(x)+\xi\sum_{j=2}^{n}\left|M_{1j}(x)\right|\right).
\end{aligned}\tag{6.27}$$

Therefore (in view of (2.4), (6), and (6.20))

$$\left.\frac{dW_1(x,\tilde{Y}_1)}{dx}\right|_{(x,\tilde{Y}_1)\in p_1}>0\quad\text{if } N_{11}(x)>0 \text{ on } I_1,\tag{6.28}$$

$$\left.\frac{dW_1(x,\tilde{Y}_1)}{dx}\right|_{(x,\tilde{Y}_1)\in p_1}<0\quad\text{if } N_{11}(x)<0 \text{ on } I_1.\tag{6.29}$$

If inequality (6.28) and suppositions of Lemma 5.1 (in the situation, described in Section 6.1, (a)) hold simultaneously, then points of the set ∂Q_1, where

$$Q_1=\{(x,Y_1):x\in(x_1,x_n],w(x,Y_1)<0,W_1(x,Y_1)<0\}\tag{6.30}$$

with w defined in the proof of Lemma 5.1, are (for all $x\in(x_1,x_n)$) the points of strict egress for Q_1 with respect to (6.22) with $i=1$ (this equation is at the same time an equation of the type (6.1) for $i=1$). Since $Y_1^0(x_1^+)=\tilde{Y}_1(x_1^+)$ and (in view of construction of operator $\mathcal{T}$) $Y_1^0(x_n^-)=\tilde{Y}_1(x_n^-)$, then $|Y_1^0(x)-\tilde{Y}_1(x)|<\varepsilon$ (see Figure 3).

Indeed, if this inequality does not hold then there is a $x^*\in I_1$ such that $|Y_1^0(x^*)-\tilde{Y}_1(x^*)|=\varepsilon$ and by (6.28): $|Y_1^0(x)-\tilde{Y}_1(x)|>\varepsilon$ on $(x^*,x_n]$. This is impossible.

If inequality (6.29) and suppositions of Lemma 5.2 (in the situation, described in Section 6.1, (a)) hold simultaneously, then all points of the set ∂Q_1 are, for all $x\in(x_1,x_n)$, the points of strict ingress for Q_1 with respect to (6.22) with $i=1$ (see Figure 4).

In view of construction $(x,Y_1^0(x))\in\Omega_1$ and $(x,\tilde{Y}_1(x))\in\Omega_1$. If inequality $|Y_1^0(x)-\tilde{Y}_1(x)|<\varepsilon$ does not hold, then there is a $x^*\in I_1$ such that $|Y_1^0(x^*)-\tilde{Y}_1(x^*)|=\varepsilon$ and $|Y_1^0(x)-\tilde{Y}_1(x)|<\varepsilon$ on (x_1,x^*). This is impossible, since point $(x^*,\tilde{Y}_1(x^*)$ is the point of strict ingress.

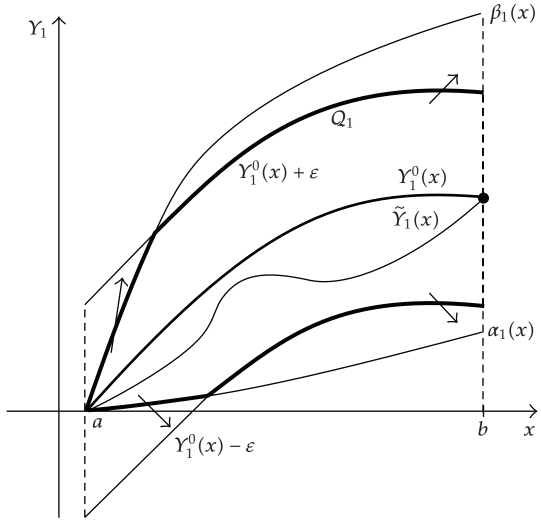

Figure 3: Continuity of $\mathfrak{T}$ (the first case).

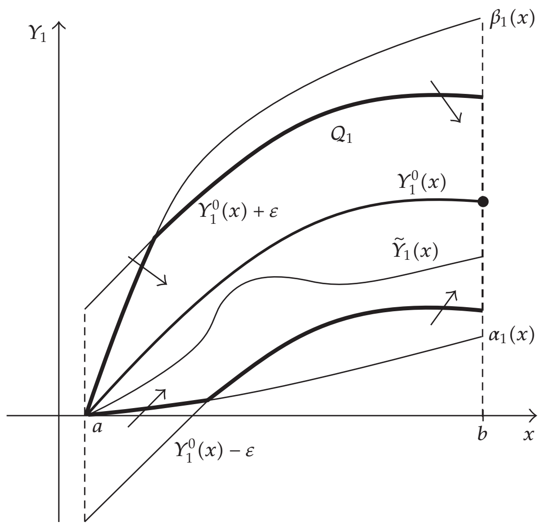

Figure 4: Continuity of $\mathfrak{T}$ (the second case).

In both considered cases, $|Y_1^0(x) - \tilde{Y}_1(x)| < \varepsilon$ on I_1 and, consequently, on I too. We conclude that

$$\left|\tilde{Y}_1(x) - Y_1^0(x)\right| < \varepsilon \quad \text{on } I \text{ if } \left\|y^0 - \tilde{y}\right\| < \delta. \tag{6.31}$$

If inequality (6.29) and suppositions of Lemma 5.1 (in the situation, described in part 6.1, (a)) hold simultaneously, then for small ε : $\mathcal{N}_1 \cap \mathcal{P}_1 \neq \emptyset$, where $\mathcal{N}_1 \equiv \mathcal{N}$ ($\mathcal{N}$ was defined in the proof of Lemma 5.1), and there exist a point $(x^\Delta, y^\Delta) \in \mathcal{N}_1 \cap \mathcal{P}_1$ which is at the same time a point of strict egress and a point of strict ingress for Q_1. This is excluded by condition (6).

For the same reason is the case when (6.28) and Lemma 5.2 hold simultaneously impossible. Analogously we can investigate (6.22) if $i = 2, \dots, k$.

(γ_2) Let us evaluate the derivative of $W_n(x, \widetilde{Y}_n)$ along the trajectories of (6.22) for $i = n$ if $(x, \widetilde{Y}_n) \in \mathcal{P}_n$. The similar computations as above lead to inequalities

$$\left.\frac{dW_n(x,\widetilde{Y}_n)}{dx}\right|_{(x,\widetilde{Y}_n)\in\mathcal{P}_n} > 0 \quad \text{if } N_{nn}(x) > 0 \text{ on } I_n, \tag{6.32}$$

$$\left.\frac{dW_n(x,\widetilde{Y}_n)}{dx}\right|_{(x,\widetilde{Y}_n)\in\mathcal{P}_n} < 0 \quad \text{if } N_{nn}(x) < 0 \text{ on } I_n. \tag{6.33}$$

If inequality (6.32) and suppositions of Lemma 5.3 (in the situation described in Section 6.1, (b)) hold simultaneously, then all points of the set ∂Q_n, where

$$Q_n = \{(x, Y_n) : x \in (x_1, x_n), w(x, Y_n) < 0, W_n(x, Y_n) < 0\} \tag{6.34}$$

with w defined as in the proof of Lemma 5.1, are (for $x \in (x_1, x_n)$) the points of strict egress for Q_n with respect to (6.22) with $i = n$ (since this equation is at the same time an equation of the type (6.1) for $i = n$).

If inequality (6.33) and suppositions of Lemma 5.4 (in the situation described in Section 6.1, (b)) hold simultaneously, then all points of the set ∂Q_n for $x \in (x_1, x_n)$ are points of strict ingress.

In both of these cases we conclude similarly, as in part (γ_1), that $|Y_n^0(x) - \widetilde{Y}_n^0(x)| < \varepsilon$ on I if $\|y^0 - \widetilde{y}\| < \delta$. The cases when inequality (6.32) and suppositions of Lemma 5.4 hold simultaneously or when inequality (6.33) and suppositions of Lemma 5.3 hold simultaneously are impossible according to (6).

Analogously we can proceed if $i = s + 1, \dots, n - 1$.

(γ_3) Let us evaluate the derivative of W_q along the trajectories of (6.22) for $q = k + 1, \dots, s$ if $(x, \widetilde{Y}_q) \in \mathcal{P}_q$. It is easy to see (by analogy with γ_1)) that the following four cases (6.35)–(6.38) are possible:

$$\frac{dW_q\left(x,\widetilde{Y}_q\right)}{dx} > 0 \text{ if } N_{qq}(x) > 0, \text{ on } I_q, \tag{6.35}$$

$$\begin{aligned} &\frac{dW_q\left(x,\widetilde{Y}_q\right)}{dx} > 0 \quad \text{if } N_{qq}(x) > 0 \text{ on } \left[x_1, x_q\right), \\ &\frac{dW_q\left(x,\widetilde{Y}_q\right)}{dx} < 0 \quad \text{if } N_{qq}(x) < 0 \text{ on } \left(x_q, x_n\right], \end{aligned} \tag{6.36}$$

$$\frac{dW_q\left(x,\widetilde{Y}_q\right)}{dx} < 0 \quad \text{if } N_{qq}(x) < 0 \text{ on } I_q, \tag{6.37}$$

$$\begin{aligned} &\frac{dW_q\left(x,\widetilde{Y}_q\right)}{dx} < 0 \quad \text{if } N_{qq}(x) < 0 \text{ on } [x_1, x_q), \\ &\frac{dW_q\left(x,\widetilde{Y}_q\right)}{dx} > 0 \quad \text{if } N_{qq}(x) > 0 \text{ on } (x_q, x_n]. \end{aligned} \tag{6.38}$$

Each of the admissible cases (i.e., if suppositions of Lemmas 5.1 , 5.3 and inequality (6.35) hold; or if suppositions of Lemmas 5.2 , 5.3 and inequalities (6.36) hold; or if suppositions of Lemmas 5.2 , 5.4 and inequality (6.37) hold; or if suppositions of Lemmas 5.1 , 5.4 and inequalities (6.38) hold) can be considered as above (see parts (γ_1) and (γ_2)) and, therefore, for $q = k+1,\ldots,s : |Y_q^0(x) - \widetilde{Y}_q(x)| < \varepsilon$ on I if $\|y^0 - \widetilde{y}\| < \delta$. The remaining cases are impossible in view of (6). Connecting all parts (γ_1)–(γ_3) we conclude that (6.20) holds and, consequently, operator $\mathfrak{T}$ is continuous. All conditions of Schauder's principle are valid. Therefore, the operator $\mathfrak{T}$ has a fixed point, that is, the problem (1.1), (1.2) has a solution with indicated properties which follow from the form of the set M. Strong inequalities in (2.6) are a consequence of the fact that boundaries of considered sets are transversal with respect to integral curves. The proof is complete.

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