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Research Article

Variational Method to the Impulsive Equation with Neumann Boundary Conditions

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We study the existence and multiplicity of classical solutions for second-order impulsive Sturm-Liouville equation with Neumann boundary conditions. By using the variational method and critical point theory, we give some new criteria to guarantee that the impulsive problem has at least one solution, two solutions, and infinitely many solutions under some different conditions, respectively. Some examples are also given in this paper to illustrate the main results.

1. Introduction

In this paper, we consider the boundary value problem of second-order Sturm-Liouville equation with impulsive effects

$$\begin{gathered} -\left(p(t)u'(t)\right)' + r(t)u'(t) + q(t)u(t) = g(t,u(t)), \quad t \neq t_k, \text{ a.e. } t \in [0,1], \\ -\Delta\left(p(t_k)u'(t_k)\right) = I_k(u(t_k)), \quad k = 1,2,\ldots,p-1, \\ u'(0^+) = u'\left(1^-\right) = 0, \end{gathered} \tag{1.1}$$

where $0 = t_0 < t_1 < t_2 < \cdots < t_{p-1} < t_p = 1, p \in C^1([0,1]), r, q \in C([0,1])$ with p and q positive functions, $g : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a continuous function, $I_k : \mathbb{R} \to \mathbb{R}, 1 \le k \le p-1$ are continuous, $-\Delta(p(t_k)u'(t_k)) = -p(t_k)(u'(t_k^+) - u'(t_k^-)), u'(t_k^+)$ and $u'(t_k^-)$ denote the right and the left limits, respectively, of $u'(t)$ at $t = t_k$, $u'(0^+)$ is the right limit of $u'(0)$, and $u'(1^-)$ is the left limit of $u'(1)$.

In the recent years, a great deal of work has been done in the study of the existence of solutions for impulsive boundary value problems (IBVPs), by which a number of chemotherapy, population dynamics, optimal control, ecology, industrial robotics, and physics phenomena are described. For the general aspects of impulsive differential equations, we refer the reader to the classical monograph [1]. For some general and recent works on the theory of impulsive differential equations, we refer the reader to [2–9]. Some classical tools or techniques have been used to study such problems in the literature. These classical techniques include the coincidence degree theory of Mawhin [10], the method of upper and lower solutions with monotone iterative technique [11], and some fixed point theorems in cones [12–14].

On the other hand, in the last two years, some researchers have used variational methods to study the existence of solutions for impulsive boundary value problems. Variational method has become a new powerful tool to study impulsive differential equations, we refer the reader to [15–20]. More precisely, in [15], the authors studied the following equation with impulsive effects:

$$\begin{gathered}
-\left(\rho(t)\phi_p\left(u'(t)\right)\right)' + s(t)\phi_p(u(t)) = f(t,u(t)), \quad t \neq t_j, \ \text{a.e. } t \in [a,b], \\
-\Delta\left(\rho(t_j)\phi_p\left(u'(t_j)\right)\right) = I_j\left(u(t_j)\right), \quad j = 1,2,\ldots,l, \\
\alpha u'(a) - \beta u(a) = A, \qquad \gamma u'(b) + \sigma u(b) = B,
\end{gathered} \tag{1.2}$$

where $f : [a,b] \times [0,+\infty) \to [0,+\infty)$ is continuous, $I_j : [0,+\infty) \to [0,+\infty)$, $j = 1,2,\ldots,l$, are continuous, and $\alpha,\beta,\gamma,\sigma > 0$. They essentially proved that IBVP (1.2) has at least two positive solutions via variational method. Recently, in [16], using variational method and critical point theory, Nieto and O'Regan studied the existence of solutions of the following equation:

$$\begin{gathered}
-u''(t) + \lambda u(t) = f(t,u(t)), \quad t \neq t_j, \ \text{a.e. } t \in [0,T], \\
\Delta\left(u'(t_j)\right) = I_j\left(u(t_j)\right), \quad j = 1,2,\ldots,l, \\
u(0) = u(T) = 0,
\end{gathered} \tag{1.3}$$

where $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ is continuous, and $I_j : \mathbb{R} \to \mathbb{R}, j = 1,2,\ldots,l$ are continuous. They obtained that IBVP (1.3) has at least one solution. Shortly, in [17], authors extended the results of IBVP (1.3).

In [19],Zhou and Li studied the existence of solutions of the following equation:

$$\begin{gathered}
-u''(t) + g(t)u(t) = f(t,u(t)), \quad t \neq t_j, \ \text{a.e. } t \in [0,T], \\
\Delta\left(u'(t_j)\right) = I_j\left(u(t_j)\right), \quad j = 1,2,\ldots,p, \\
u(0) = u(T) = 0,
\end{gathered} \tag{1.4}$$

where $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ is continuous, and $I_j : \mathbb{R} \to \mathbb{R}$, $j = 1,2,\ldots,p$, are continuous. They proved that IBVP (1.4) has at least one solution and infinitely many solutions by using variational method and critical point theorem.

Motivated by the above facts, in this paper, our aim is to study the variational structure of IBVP (1.1) in an appropriate space of functions and obtain the existence and multiplicity of solutions for IBVP (1.1) by using variational method. To the best of our knowledge, there is no paper concerned impulsive differential equation with Neumann boundary conditions via variational method. In addition, this paper is a generalization of [21], in which impulse effects are not involved.

In this paper, we will need the following conditions.

(H1) There is constants $\beta > 2, M > 0$ such that for every $t \in [0,1]$ and $u \in \mathbb{R}$ with $|u| \geq M$,

$$0 < \beta G(t,u) \leq ug(t,u), \qquad 0 < \beta \int_0^u I_k(s)ds \leq uI_k(u), \tag{1.5}$$

where $G(t,u) = \int_0^u g(t,s)ds$.

(H2) $\lim_{u \to 0}(g(t,u))/u = 0$ uniformly for $t \in [0,1]$, and $\lim_{u \to 0}(I_k(u))/u = 0$.

(H3) There exist numbers $h_1, h_2 > 0$ and $p_1 > 1$ such that

$$g(t,u) \leq h_1 + h_2|u|^{p_1} \quad \text{for } u \in \mathbb{R},\ t \in [0,1]. \tag{1.6}$$

(H4) There exist numbers $a_k, b_k > 0$ and $\gamma_k \in [0,1)$ such that

$$I_k(u) \leq a_k + b_k|u|^{\gamma_k} \quad \text{for } u \in \mathbb{R}. \tag{1.7}$$

(H5) There exist numbers $r_1, r_2 > 0$ and $\mu \in [0,1)$ such that

$$g(t,u) \leq r_1 + r_2|u|^{\mu} \quad \text{for } u \in \mathbb{R},\ t \in [0,1]. \tag{1.8}$$

(H6) There exist numbers $a'_k, b'_k > 0$ and $\gamma'_k \in (1,+\infty)$ such that

$$I_k(u) \leq a'_k + b'_k|u|^{\gamma'_k} \quad \text{for } u \in \mathbb{R}. \tag{1.9}$$

This paper is organized as follows. In Section 2, we present some preliminaries. In Section 3, we discuss the existence and multiplicity of classical solutions to IBVP (1.1). Some examples are presented in this section to illustrate our main results in the last section.

2. Preliminaries

Take $L(t) = \int_0^t (r(s)/p(s))ds$. Then $e^{-L(t)} \in C^1([0,1])$. We transform IBVP (1.1) into the following equivalent form:

$$\begin{gathered}
-\left(e^{-L(t)}p(t)u'(t)\right)' + e^{-L(t)}q(t)u(t) = e^{-L(t)}g(t,u(t)), \quad t \neq t_k,\ \text{a.e. } t \in [0,1], \\
-\Delta\left(e^{-L(t_k)}p(t_k)u'(t_k)\right) = e^{-L(t_k)}I_k(u(t_k)), \quad k = 1,2,\dots,p-1, \\
u'(0^+) = u'(1^-) = 0.
\end{gathered} \tag{2.1}$$

Obviously, the solutions of IBVP (2.1) are solutions of IBVP (1.1). So it suffices to consider IBVP (2.1).

In this section, the following theorem will be needed in our argument. Suppose that E is a Banach space (in particular a Hilbert space) and $\varphi \in C^1(E, \mathbb{R})$. We say that φ satisfies the Palais-Smale condition if any sequence $\{u_j\} \subset E$ for which $\varphi(u_j)$ is bounded and $\varphi'(u_j) \to 0$ as $j \to +\infty$ possesses a convergent subsequence in X. Let B_r be the open ball in X with the radius r and centered at 0 and ∂B_r denote its boundary.

Theorem 2.1 ([22, Theorem 38.A]). *For the functional* $F : M \subseteq X \to [-\infty, +\infty]$ *with* $M \neq \emptyset, \min_{u \in M} F(u) = \alpha$ *has a solution for which the following hold:*

(i) X *is a real reflexive Banach space;*

(ii) M *is bounded and weakly sequentially closed;*

(iii) F *is weakly sequentially lower semicontinuous on* M*; that is, by definition, for each sequence* $\{u_n\}$ *in* M *such that* $u_n \rightharpoonup u$ *as* $n \to \infty$*, one has* $F(u) \leq \liminf_{n\to\infty} F(u_n)$ *holds.*

Theorem 2.2 ([16, Theorem 2.2]). *Let* E *be a real Banach space and let* $\varphi \in C^1(E, \mathbb{R})$ *satisfy the Palais-Smale condition. Assume there exist* $u_0, u_1 \in E$ *and a bounded open neighborhood* Ω *of* u_0 *such that* $u_1 \in E \setminus \overline{\Omega}$ *and*

$$\max\{\varphi(u_0), \varphi(u_1)\} < \inf_{x \in \partial\Omega} \varphi(u). \tag{2.2}$$

Let

$$\begin{gathered} \Gamma = \{h \mid h : [0,1] \longrightarrow E \text{ is continuous and } h(0) = u_0,\ h(1) = u_1\}, \\ c = \inf_{h \in \Gamma} \max_{s \in [0,1]} \varphi(h(s)). \end{gathered} \tag{2.3}$$

Then c *is a critical value of* φ*; that is, there exists* $u^* \in E$ *such that* $\varphi'(u^*) = \Theta$ *and* $\varphi(u^*) = c$*, where* $c > \max\{\varphi(u_0), \varphi(u_1)\}$*.*

Theorem 2.3 ([23]). *Let* E *be a real Banach space, and let* $\varphi \in C^1(E, \mathbb{R})$ *be even satisfying the Palais-Smale condition and* $\varphi(0) = 0$*. If* $E = V \oplus Y$*, where* V *is finite dimensional, and* φ *satisfies that*

(A1) *there exist constants* $\rho, \alpha > 0$ *such that* $\varphi|_{\partial B_r \cap Y} \geq \alpha$*,*

(A2) *for each finite dimensional subspace* $W \subset E$*, there is* $R = R(W)$ *such that* $\varphi(u) \leq 0$ *for all* $u \in W$ *with* $\|u\| \geq R$*.*

Then φ *possesses an unbounded sequence of critical values.*

Let us recall some basic knowledge. Denote by X the Sobolev space $W^{1,2}([0,1])$, and consider the inner product

$$(u, v) = \int_0^1 u'(t)v'(t)dt + \int_0^1 u(t)v(t)dt \tag{2.4}$$

which induces the usual norm

$$\|u\| = \left[\int_0^1 \left|u'(t)\right|^2 dt + \int_0^1 |u(t)|^2 dt\right]^{1/2}. \tag{2.5}$$

We also consider the inner product

$$(u,v)_X = \int_0^1 e^{-L(t)} p(t) u'(t) v'(t) dt + \int_0^1 e^{-L(t)} q(t) u(t) v(t) dt, \tag{2.6}$$

and the norm

$$\|u\|_X = \left(\int_0^1 e^{-L(t)} p(t) \left|u'(t)\right|^2 dt + \int_0^1 e^{-L(t)} q(t) |u(t)|^2 dt\right)^{1/2}, \tag{2.7}$$

then the norm $\|\cdot\|_X$ is equivalent to the usual norm $\|\cdot\|$ in $W^{1,2}([0,1])$. Hence, X is reflexive. We define the norm in $C([0,1]), L^2([0,1])$ as $\|u\|_\infty = \max_{t\in[0,1]}|u(t)|$ and $\|u\|_2 = [\int_0^1 |u|^2 dt]^{1/2}$, respectively.

For $u \in W^{2,2}([0,1])$, we have that u, u' are absolutely continuous, and $u'' \in L^2([0,1])$, hence $-\Delta(e^{-L(t_k)} p(t_k) u'(t_k)) = -e^{-L(t_k)} p(t_k)(u'(t_k^+) - u'(t_k^-)) = 0$, for any $t_k \in [0,1]$. If $u \in X$, then u is absolutely continuous and $u' \in L^2(0,1)$. In this case, the one-side derivatives $u'(0^+), u'(1^-), u'(t_k^+), u'(t_k^-)$, $k = 1,2,\ldots,p-1$ may not exist. As a consequence, we need to introduce a different concept of solution. We say that $u \in C([0,1])$ is *a classical solution* of IBVP (2.1) if it satisfies the equation in IBVP (2.1) a.e. on $[0,1]$, the limits $u'(t_k^+), u'(t_k^-)$, $k = 1,2,\ldots,p-1$ exist and impulsive conditions in IBVP (2.1) hold, $u'(0^+), u'(1^-)$ exist and $u'(0^+) = u'(1^-) = 0$. Moreover, for every $k = 0,1,\ldots,p-1, u_k = u|_{(t_k,t_{k+1})}$ satisfy $u_k \in W^{2,2}(t_k, t_{k+1})$.

For each $u \in X$, consider the functional φ defined on X by

$$\varphi(u) = \frac{1}{2}\|u\|_X^2 - \sum_{k=1}^{p-1} e^{-L(t_k)} \int_0^{u(t_k)} I_k(s) ds - \int_0^1 e^{-L(t)} G(t,u) dt. \tag{2.8}$$

It is clear that φ is differentiable at any $u \in$ X and

$$\begin{aligned}\varphi'(u)(v) = &\int_0^1 \left[e^{-L(t)} p(t) u'(t) v'(t) + e^{-L(t)} q(t) u(t) v(t)\right] dt \\ &- \sum_{k=1}^{p-1} e^{-L(t_k)} I_k(u(t_k)) v(t_k) - \int_0^1 e^{-L(t)} g(t,u(t)) v(t) dt\end{aligned} \tag{2.9}$$

for any $v \in$ X. Obviously, φ' is continuous.

Lemma 2.4. *If $u \in X$ is a critical point of the functional φ, then u is a classical solution of IBVP* (2.1).

Proof. Let $u \in X$ be a critical point of the functional φ. It shows that

$$\int_0^1 \left[e^{-L(t)}p(t)u'(t)v'(t)+e^{-L(t)}q(t)u(t)v(t)\right]dt \\ -\sum_{k=1}^{p-1} e^{-L(t_k)}I_k(u(t_k))v(t_k) - \int_0^1 e^{-L(t)}g(t,u(t))v(t)dt=0 \tag{2.10}$$

holds for any $v \in X$. Choose any $j \in \{0,1,2,\ldots,p-1\}$ and $v \in X$ such that $v(t) = 0$ if $t \in [t_k, t_{k+1}]$ for $k \neq j$. Equation (2.10) implies

$$\int_{t_j}^{t_{j+1}} \left[e^{-L(t)}p(t)u'(t)v'(t) + e^{-L(t)}q(t)u(t)v(t) - e^{-L(t)}g(t,u(t))v(t)\right]dt = 0. \tag{2.11}$$

This means, for any $w \in W_0^{1,2}(t_j, t_{j+1})$,

$$\int_{t_j}^{t_{j+1}} \left[e^{-L(t)}p(t)u_j'(t)w'(t) + e^{-L(t)}q(t)u_j(t)w(t) - e^{-L(t)}g(t,u_j(t))w(t)\right]dt = 0, \tag{2.12}$$

where $u_j = u|_{(t_j,t_{j+1})}$. Thus u_j is a weak solution of the following equation:

$$-\left(e^{-L(t)}p(t)u'(t)\right)' + e^{-L(t)}q(t)u(t) = e^{-L(t)}g(t,u(t)) \quad t \in (t_j, t_{j+1}), \tag{2.13}$$

and therefore $u_j \in W_0^{1,2}(t_j, t_{j+1}) \subset C([t_j, t_{j+1}])$. Let $h(t) := e^{-L(t)}(g(t,u) - qu)$, then (2.13) becomes the following form:

$$-\left(e^{-L(t)}p(t)u'(t)\right)' = h(t) \text{ on } (t_j, t_{j+1}), \quad j = 0,1,2,\ldots,p-1. \tag{2.14}$$

Then the solution of (2.14) can be written as

$$u_j(t) = C_1 + C_2\int_{t_j}^{t} e^{(L(s)-\ln p(s))}ds - \int_{t_j}^{t}\left(e^{(L(s)-\ln p(s))}\int_{t_j}^{s}\frac{h(r)}{p(r)}e^{\ln p(r)}dr\right)ds \quad t \in (t_j, t_{j+1}), \tag{2.15}$$

where C_1 and C_2 are two constants. Then $u_j' \in C(t_j, t_{j+1})$ and $u_j'' \in C(t_j, t_{j+1})$. Therefore, u_j is a classical solution of (2.13) and u satisfies the equation in IBVP (2.1) a.e. on $[0,1]$. By the previous equation, we can easily get that the limits $u'(t_j^+), u'(t_j^-)$, $j = 1, 2, \ldots, p-1, u'(t_0^+)$ and $u'(t_p^-)$ exist. By integrating (2.10), one has

$$\begin{aligned}
&\int_0^1 \left[e^{-L(t)}p(t)u'(t)v'(t) + e^{-L(t)}q(t)u(t)v(t)\right]dt \\
&\quad - \sum_{k=1}^{p-1} e^{-L(t_k)} I_k(u(t_k))v(t_k) - \int_0^1 e^{-L(t)}g(t,u(t))v(t)dt \\
&= -\sum_{k=1}^{p-1} \Delta\left(e^{-L(t_k)}p(t_k)u'(t_k)\right)v(t_k) + e^{-L(1)}p(1)u'(1^-)v(1) \\
&\quad - e^{-L(0)}p(0)u'(0^+)v(0) - \sum_{k=1}^{p-1} e^{-L(t_k)} I_k(u(t_k))v(t_k) \\
&\quad + \int_0^1 \left[-\left(e^{-L(t)}p(t)u'(t)\right)' + e^{-L(t)}q(t)u(t) - e^{-L(t)}g(t,u(t))\right]v(t)dt \\
&= -\sum_{k=1}^{p-1} \left[\Delta\left(e^{-L(t_k)}p(t_k)u'(t_k)\right) + e^{-L(t_k)} I_k(u(t_k))\right]v(t_k) \\
&\quad + e^{-L(1)}p(1)u'(1^-)v(1) - e^{-L(0)}p(0)u'(0^+)v(0) \\
&\quad + \int_0^1 \left[-\left(e^{-L(t)}p(t)u'(t)\right)' + e^{-L(t)}q(t)u(t) - e^{-L(t)}g(t,u(t))\right]v(t)dt = 0,
\end{aligned} \tag{2.16}$$

and combining with (2.13) we get

$$\begin{aligned}
&-\sum_{k=1}^{p-1} \left[\Delta\left(e^{-L(t_k)}p(t_k)u'(t_k)\right) + e^{-L(t_k)} I_k(u(t_k))\right]v(t_k) \\
&\quad + e^{-L(1)}p(1)u'(1^-)v(1) - e^{-L(0)}p(0)u'(0^+)v(0) = 0.
\end{aligned} \tag{2.17}$$

Next we will show that u satisfies the impulsive conditions in IBVP (2.1). If not, without loss of generality, we assume that there exists $i \in \{1, 2, \ldots, p-1\}$ such that

$$e^{-L(t_i)} I_i(u(t_i)) + \Delta\left(e^{-L(t_i)}p(t_i)u'(t_i)\right) \neq 0. \tag{2.18}$$

Let

$$v(t) = \prod_{k=0,\, k \neq i}^{p} (t - t_k). \tag{2.19}$$

Obviously, $v \in X$. Substituting them into (2.17), we get

$$\left(\Delta e^{-L(t_i)} p(t_i) u'(t_i) + e^{-L(t_i)} I_i(u(t_i))\right) v(t_i) = 0 \tag{2.20}$$

which contradicts (2.18). So u satisfies the impulsive conditions in IBVP (2.1). Thus, (2.17) becomes the following form:

$$e^{-L(1)} p(1) u'(1^-) v(1) - e^{-L(0)} p(0) u'(0^+) v(0) = 0, \tag{2.21}$$

for all $v \in X$. Since $v(0), v(1)$ are arbitrary, (2.21) shows that $e^{-L(1)} p(1) u'(1^-) = e^{-L(0)} p(0) u'(0^+) = 0$, and it implies $u'(1^-) = u'(0^+) = 0$. Therefore, u is a classical solution of IBVP (2.1). □

Lemma 2.5. *Let $u \in X$. Then $\|u\|_\infty \leq M_1 \|u\|_X$, where*

$$M_1 = 2^{1/2} \max\left\{ \frac{1}{\left(\min_{t\in[0,1]} e^{-L(t)} p(t)\right)^{1/2}}, \frac{1}{\left(\min_{t\in[0,1]} e^{-L(t)} q(t)\right)^{1/2}} \right\}. \tag{2.22}$$

Proof. By using the same methods of [15, Lemma 2.6], we easily obtain the above result, and we omit it here. □

3. Main Results

In this section, we will show our main results and prove them.

Theorem 3.1. *Assume that (H1) and (H2) hold. Moreover, $g(t,u)$ and the impulsive functions $I_k(u)$ are odd about u, then IBVP* (1.1) *has infinitely many classical solutions.*

Proof. Obviously, φ is an even functional and $\varphi(0) = 0$. We divide our proof into three parts in order to show Theorem 3.1.

Firstly, We will show that φ satisfies the Palais-Smale condition. Let $\{\varphi(u_n)\}$ be a bounded sequence such that $\lim_{n\to+\infty} \varphi'(u_n) = 0$. Then there exists constants $C_3 > 0$ such that

$$\left|\varphi(u_n)\right| \leq C_3, \qquad \left\|\varphi'(u_n)\right\|_X \leq C_3. \tag{3.1}$$

By (2.8), (2.9), (3.1), and (H1), we have

$$\begin{aligned}
\left(\frac{\beta}{2}-1\right)\|u_n\|_X^2 = \frac{\beta}{2}\|u_n\|_X^2 - \|u_n\|_X^2 \\
&= \beta\varphi(u_n) - \varphi'(u_n)u_n + \beta\sum_{k=1}^{p-1} e^{-L(t_k)}\int_0^{u_n(t_k)} I_k(s)ds + \beta\int_0^1 e^{-L(t)}G(t,u_n)dt \\
&\quad - \sum_{k=1}^{p-1} e^{-L(t_k)} I_k(u_n(t_k))u_n(t_k) - \int_0^1 e^{-L(t)} g(t,u_n)u_n dt \\
&= \sum_{k=1}^{p-1} e^{-L(t_k)}\left(\beta\int_0^{u_n(t_k)} I_k(s)ds - I_k(u_n(t_k))u_n(t_k)\right) \\
&\quad + \int_0^1 e^{-L(t)}(\beta G(t,u_n) - g(t,u_n)u_n)dt + \beta\varphi(u_n) - \varphi'(u_n)u_n \\
&\le \beta C_3 + M_1^2 C_3\|u_n\|_X \\
&\quad + \int_0^1 e^{-L(t)}dt \max_{t\in[0,1],\, u_n(t)\in[-M,M]} \left|\beta G(t,u_n) - g(t,u_n)u_n\right| \\
&\quad + \sum_{k=1}^{p-1} e^{-L(t_k)} \max_{u_n(t_k)\in[-M,M]} \left|\beta\int_0^{u_n(t_k)} I_k(s)ds - I_k(u_n(t_k))u_n(t_k)\right|.
\end{aligned} \tag{3.2}$$

It follows that $\{u_n\}$ is bounded in X. From the reflexivity of X, we may extract a weakly convergent subsequence that, for simplicity, we call $\{u_n\}, u_n \rightharpoonup u$ in X. In the following we will verify that $\{u_n\}$ strongly converges to u in X. By (2.9) we have

$$\begin{aligned}
&(\varphi'(u_n) - \varphi'(u))(u_n - u) \\
&\quad = \|u_n - u\|_X^2 \\
&\qquad - \sum_{k=1}^{p-1} e^{-L(t_k)}(I_k(u_n(t_k)) - I_k(u(t_n)))(u_n(t_k) - u(t_k)) \\
&\qquad - \int_0^1 e^{-L(t)}(g(t,u_n(t)) - g(t,u(t)))(u_n(t) - u(t))dt.
\end{aligned} \tag{3.3}$$

By $u_n \rightharpoonup u$ in X, we see that $\{u_n\}$ uniformly converges to u in $C([0,1])$. So

$$\begin{gathered}
\int_0^1 e^{-L(t)}(g(t,u_n(t)) - g(t,u(t)))(u_n(t) - u(t))dt \longrightarrow 0, \\
\sum_{k=1}^{p-1} e^{-L(t_k)}(I_k(u_n(t_k)) - I_k(u(t_k)))(u_n(t_k) - u(t_k)) \longrightarrow 0, \\
(\varphi'(u_{\mathrm{n}}) - \varphi'(u))(u_n - u) \longrightarrow 0 \quad \text{as } n \longrightarrow +\infty.
\end{gathered} \tag{3.4}$$

By (3.3), (3.4), we obtain $\|u_n - u\|_X \to 0$ as $n \to +\infty$. That is, $\{u_n\}$ strongly converges to u in X, which means the that P. S. condition holds for φ.

Secondly, we verify the condition (A1) in Theorem 2.3. Let $V = \mathbb{R}, Y = \{u \in X \mid \int_0^1 u(t)dt = 0\}$, then $X = V \oplus Y$, where $\dim V = 1 < +\infty$. In view of (H2), take $\varepsilon = \min\{1/8M_1^2\int_0^1 e^{-L(t)}dt, 1/8M_1^2\sum_{k=1}^{p-1} e^{-L(t_k)}\} > 0$, there exists an $\delta > 0$ such that for every u with $|u| < \delta$,

$$G(t,u) \leq \varepsilon|u|^2, \qquad \int_0^u I_k(s)ds \leq \varepsilon|u|^2. \tag{3.5}$$

Hence, for any $u \in Y$ with $\|u\|_X \leq \delta/M_1$, by (2.8) and (3.5) , we have

$$\begin{aligned}
\varphi(u) &= \frac{1}{2}\|u\|_X^2 - \sum_{k=1}^{p-1} e^{-L(t_k)}\int_0^{u(t_k)} I_k(s)ds - \int_0^1 e^{-L(t)}G(t,u)dt \\
&\geq \frac{1}{2}\|u\|_X^2 - \sum_{k=1}^{p-1} e^{-L(t_k)}\varepsilon|u_k(t_k)|^2 - \int_0^1 e^{-L(t)}\varepsilon|u(t)|^2 dt \\
&\geq \frac{1}{2}\|u\|_X^2 - \varepsilon M_1^2\sum_{k=1}^{p-1} e^{-L(t_k)}\|u\|_X^2 - \varepsilon M_1^2\int_0^1 e^{-L(t)}dt\|u\|_X^2 \\
&\geq \frac{1}{2}\|u\|_X^2 - \frac{1}{8}\|u\|_X^2 - \frac{1}{8}\|u\|_X^2 \\
&= \frac{1}{4}\|u\|_X^2.
\end{aligned} \tag{3.6}$$

Take $\alpha = \delta^2/4M_1^2, \rho = \delta/M_1$, then $\varphi(u) \geq \alpha, \forall u \in Y \cap \partial B_\rho$.

Finally, we verify condition (A2) in Theorem 2.3. According to (H1), for any $u \geq M > 0$ and $t \in [0,1]$ we have that

$$\left(\frac{G(t,u)}{u^\beta}\right)_u' = \frac{u^\beta g(t,u) - \beta u^{\beta-1}G(t,u)}{u^{2\beta}} = \frac{ug(t,u) - \beta G(t,u)}{u^{\beta+1}} \geq 0. \tag{3.7}$$

Hence

$$\frac{G(t,u)}{u^\beta} \geq \frac{G(t,M)}{M^\beta} \geq M^{-\beta}\min_{t\in[0,1]} G(t,M) = C' > 0 \tag{3.8}$$

for all $t \in [0,1]$ and $u \geq M > 0$. This implies that $G(t,u) \geq C'u^\beta$ for all $t \in [0,1]$ and $u \geq M > 0$. Similarly, we can prove that there is a constant $C'' > 0$ such that $G(t,u) \geq C''|u|^\beta$ for all $t \in [0,1]$ and $u \leq -M$. Since $G(t,u) - C_4|u|^\beta$ is continuous on $[0,1] \times [-M,M]$, there exists $C_5 > 0$ such that $G(t,u) - C_4|u|^\beta > -C_5$ on $[0,T] \times [-M,M]$. Thus, we have

$$G(t,u) \geq C_4|u|^\beta - C_5 \quad \forall (t,u) \in [0,1] \times \mathbb{R}, \tag{3.9}$$

where $C_4 = \min\{C', C''\}$.

Similarly, there exist constants $C_6, C_7 > 0$ such that

$$\int_0^u I_k(s)ds \geq C_6|u|^\beta - C_7 \quad \forall (t,u) \in [0,1] \times \mathbb{R}. \tag{3.10}$$

For every $\xi \in \mathbb{R} \setminus \{0\}$ and $u \in W \setminus \{0\}$, by (2.8), (3.9), and (3.10), we have that the following inequality:

$$\begin{aligned}\varphi(\xi u) &\leq \frac{1}{2}\|\xi u\|_X^2 - \sum_{k=1}^{p-1} e^{-L(t_k)}\left(C_6|\xi u(t_k)|^\beta - C_7\right) - \int_0^1 e^{-L(t)}\left(C_4|\xi u|^\beta - C_5\right)dt \\ &\leq \frac{\xi^2}{2}\|u\|_X^2 - C_6|\xi|^\beta \sum_{k=1}^{p-1} e^{-L(t_k)}|u(t_k)|^\beta + C_7\sum_{k=1}^{p-1} e^{-L(t_k)} - C_4|\xi|^\beta \int_0^1 e^{-L(t)}|u(t)|^\beta dt + C_5\int_0^1 e^{-L(t)}dt\end{aligned} \tag{3.11}$$

holds. Take $w \in W$ such that $\|w\|_X = 1$, since $\beta > 2$, (3.11) implies that there exists $\xi' > 0$ such that $\|\xi w\|_X > \rho$ and $\varphi(\xi w) < 0$ for $\xi \geq \xi' > 0$. Since W is a finite dimensional subspace, there exists $R(W) > 0$ such that $\varphi(u) \leq 0$ on $W \setminus B_{R(W)}$. By Theorem 2.3, φ possesses infinite many critical points; that is, IBVP (1.1) has infinite many classical solutions. □

Theorem 3.2. *Assume that (H1) and the first equality in (H2) hold. Moreover, $g(t,u)$ is odd about u and the impulsive functions $I_k(u)$ are odd and nonincreasing. Then IBVP* (1.1) *has infinitely many classical solutions.*

Proof. We only verify (A1) in Theorem 2.3. Since $I_k(u)$ are odd and nonincreasing continuous functions, then for any $u \in \mathbb{R}$, $\int_0^u I_k(s)ds < 0$. So we have $\sum_{k=1}^{p-1} e^{-L(t_k)} \int_0^{u_n(t_k)} I_k(s)ds < 0$. Take $\varepsilon = 1/8M_1^2\int_0^1 e^{-L(t)}dt > 0$, $\alpha = 3\delta^2/8M_1^2$, $\rho = \delta/M_1$, like in (3.6) we can obtain the result. □

Theorem 3.3. *Suppose that the first inequalities in (H1), (H3), and (H4) hold. Furthermore, one assumes that $g(t,u)$ and the impulsive functions $I_k(u)$ are odd about u and we have the following.*

(H7) *There exists $A_0 > 0$ such that*

$$\frac{A_0}{2} > M_1 \sum_{k=1}^{p-1} e^{-L(t_k)}\left(b_k M_1^{\gamma_k} A_0^{\gamma_k} + a_k\right) + M_1\left(h_2 M_1^{p_1} A_0^{p_1} + h_1\right)\int_0^1 e^{-L(t)}dt. \tag{3.12}$$

Then IBVP (1.1) *has infinitely many classical solutions.*

Proof. Obviously, φ is an even functional and $\varphi(0) = 0$. Firstly, we will show that φ satisfies the Palais-Smale condition. As in the proof of Theorem 3.1, by (2.8), (2.9), (3.1), the first inequalities in (H1) and (H4), we have

$$\begin{aligned}
\left(\frac{\beta}{2}-1\right)\|u_n\|_X^2 &= \frac{\beta}{2}\|u_n\|_X^2 - \|u_n\|_X^2 \\
&= \beta\varphi(u_n) - \varphi'(u_n)u_n + \beta\sum_{k=1}^{p-1} e^{-L(t_k)}\int_0^{u_n(t_k)} I_k(s)ds + \beta\int_0^1 e^{-L(t)}G(t,u_n)dt \\
&\quad - \sum_{k=1}^{p-1} e^{-L(t_k)} I_k(u_n(t_k))u_n(t_k) - \int_0^1 e^{-L(t)}g(t,u_n)u_n dt \\
&= \beta\varphi(u_n) - \varphi'(u_n)u_n + \int_0^1 e^{-L(t)}\left(\beta G(t,u_n) - g(t,u_n)u_n\right)dt \\
&\quad + \beta\sum_{k=1}^{p-1} e^{-L(t_k)}\int_0^{u_n(t_k)} I_k(s)ds - \sum_{k=1}^{p-1} e^{-L(t_k)} I_k(u_n(t_k))u_n(t_k) \\
&\le \beta C_3 + M_1^2 C_3\|u_n\|_X + \int_0^1 e^{-L(t)}dt \max_{t\in[0,1],\, u_n(t)\in[-M,M]}\left|\beta G(t,u_n) - g(t,u_n)u_n\right| \\
&\quad + (\beta+1)\sum_{k=1}^{p-1} e^{-L(t_k)}\left(a_k M_1\|u_n\|_X + b_k M_1^{\gamma_k+1}\|u_n\|_X^{\gamma_k+1}\right).
\end{aligned} \tag{3.13}$$

It follows that $\{u_n\}$ is bounded in X. In the following, the proof of P. S. condition is the same as that in Theorem 3.1, and we omit it here.

Secondly, as in Theorem 3.1, we can obtain that condition (A2) in Theorem 2.1 is satisfied.

Take the same direct sum decomposition $X = V \oplus Y$ as in Theorem 3.1. For any $u \in Y$, by (2.8), (H3), and (H4), we obtain

$$\begin{aligned}
\varphi(u) &= \frac{1}{2}\|u\|_X^2 - \sum_{k=1}^{p-1} e^{-L(t_k)}\int_0^{u(t_k)} I_k(s)ds - \int_0^1 e^{-L(t)}G(t,u)dt \\
&\ge \frac{1}{2}\|u\|_X^2 - \sum_{k=1}^{p-1} e^{-L(t_k)}\left(a_k M_1\|u\|_X + b_k M_1^{\gamma_k+1}\|u\|_X^{\gamma_k+1}\right) \\
&\quad - \int_0^1 e^{-L(t)}dt\left(h_1 M_1\|u\|_X + h_2 M_1^{p_1+1}\|u\|_X^{p_1+1}\right) \\
&= \frac{1}{2}\|u\|_X^2 - \sum_{k=1}^{p-1} e^{-L(t_k)} b_k M_1^{\gamma_k+1}\|u\|_X^{\gamma_k+1} - h_2 M_1^{p_1+1}\int_0^1 e^{-L(t)}dt\|u\|_X^{p_1+1} \\
&\quad - M_1\|u\|_X\left(\sum_{k=1}^{p-1} e^{-L(t_k)} a_k + h_1\int_0^1 e^{-L(t)}dt\right).
\end{aligned} \tag{3.14}$$

In view of (H7), set $\|u\|_X = \rho := A_0 > 0$, then we have

$$\begin{aligned}\varphi(u) \geq \alpha = \frac{1}{2}A_0^2 - \sum_{k=1}^{p-1} e^{-L(t_k)} b_k M_1^{\gamma_k+1} A_0^{\gamma_k+1} - h_2 M_1^{p_1+1} \int_0^1 e^{-L(t)} dt A_0^{p_1+1} \\ - M_1 A_0 \left(\sum_{k=1}^{p-1} e^{-L(t_k)} a_k + h_1 \int_0^1 e^{-L(t)} dt \right) > 0.\end{aligned} \tag{3.15}$$

Therefore, $\varphi(u) \geq \alpha, \forall u \in Y \cap \partial B_\rho$. By Theorem 2.3, φ possesses infinite many critical points, that is, IBVP (1.1) has infinite many classical solutions. □

Theorem 3.4. *Assume that the second inequalities in (H1), (H5), and (H6) hold, moreover, one assumes the following.*

(H8) *There exists* $A_1 > 0$ *such that*

$$\frac{A_1}{2} > M_1 \sum_{k=1}^{p-1} e^{-L(t_k)} \left(b'_k M_1^{\gamma'_k} A_1^{\gamma'_k} + a'_k \right) + M_1 \left(r_2 M_1^{\mu} A_1^{\mu} + r_1 \right) \int_0^1 e^{-L(t)} dt. \tag{3.16}$$

Then IBVP (1.1) *has at least two classical solutions.*

Proof. We will use Theorems 2.1 and 2.2 to prove the main results. Firstly, we will show that φ satisfies the Palais-Smale condition. Similarly, as in the proof of Theorem 3.1, by (2.8), (2.9), (3.1), the second inequalities in (H1) and (H5), we have

$$\begin{aligned}\left(\frac{\beta}{2} - 1\right)\|u_n\|_X^2 &= \frac{\beta}{2}\|u_n\|_X^2 - \|u_n\|_X^2 \\ &= \beta\varphi(u_n) - \varphi'(u_n)u_n + \beta \sum_{k=1}^{p-1} e^{-L(t_k)} \int_0^{u_n(t_k)} I_k(s)ds + \beta \int_0^1 e^{-L(t)} G(t, u_n) dt \\ &\quad - \sum_{k=1}^{p-1} e^{-L(t_k)} I_k(u_n(t_k)) u_n(t_k) - \int_0^1 e^{-L(t)} g(t, u_n) u_n dt \\ &\leq \beta C_3 + M_1^2 C_3 \|u_n\|_X + \int_0^1 e^{-L(t)} dt \left(r_1 + r_2 M_1^{\mu} \|u_n\|_X^{\mu} \right) \\ &\quad + \beta \int_0^1 e^{-L(t)} dt \left(M_1 r_1 \|u_n\|_X + r_2 M_1^{\mu+1} \|u_n\|_X^{\mu+1} \right) \\ &\quad + \sum_{k=1}^{p-1} e^{-L(t_k)} \max_{u_n(t_k) \in [-M, M]} \left| \beta \int_0^{u_n(t_k)} I_k(s)ds - I_k(u_n(t_k)) u_n(t_k) \right|\end{aligned}$$

$$
\begin{aligned}
&= \beta C_3 + M_1^2 C_3 \|u_n\|_X + \int_0^1 e^{-L(t)}dt\left(r_1 + r_2 M_1^{\mu}\|u_n\|_X^{\mu}\right)\left(\beta M_1\|u_n\|_X + 1\right) \\
&\quad + \sum_{k=1}^{p-1} e^{-L(t_k)} \max_{u_n(t_k)\in[-M,M]} \left| \beta \int_0^{u_n(t_k)} I_k(s)ds - I_k(u_n(t_k))u_n(t_k) \right|.
\end{aligned}
\tag{3.17}
$$

It follows that $\{u_n\}$ is bounded in X. In the following, the proof of P. S. condition is the same as that in Theorem 3.1, and we omit it here.

Let $A > 0$, which will be determined later. Set $B_A := \{u \in X : \|u\|_X < A\}$, then $\overline{B}_A := \{u \in X : \|u\|_X \leq A\}$ is a closed ball. From the reflexivity of X, we can easily obtain that $\overline{B}_A$ is bounded and weakly sequentially closed. We will show that φ is weakly lower semicontinuous on $\overline{B}_A$. Let

$$
\begin{aligned}
\varphi_1(u) &= \frac{1}{2}\int_0^1 e^{-L(t)} p(t)\left|u'(t)\right|^2 dt + \int_0^1 e^{-L(t)} q(t)|u(t)|^2 dt, \\
\varphi_2(u) &= -\sum_{k=1}^{p-1} e^{-L(t_k)} \int_0^{u(t_k)} I_k(s)ds - \int_0^1 e^{-L(t)} G(t,u)dt.
\end{aligned}
\tag{3.18}
$$

Then $\varphi(u) = \varphi_1(u) + \varphi_2(u)$. By $u_n \rightharpoonup u$ on X we see that $\{u_n\}$ uniformly converges to u in $C([0,1])$. So φ_2 is weakly continuous. Clearly, φ_1 is continuous, which, together with the convexity of φ_1, implies that φ_1 is weakly lower semicontinuous. Therefore, φ is weakly lower semi-continuous on $\overline{B}_A$. So by Theorem 2.1, without loss of generality, we assume that $\varphi(u_0) = \inf_{u\in\overline{B}_A}\varphi(u)$. Now we will show that

$$
\varphi(u_0) < \inf_{u\in\partial B_A} \varphi(u).
\tag{3.19}
$$

For any $u \in \partial B_A$, by (H5) and (H6), we have

$$
\begin{aligned}
\varphi(u) &= \frac{1}{2}\|u\|_X^2 - \sum_{k=1}^{p-1} e^{-L(t_k)} \int_0^{u(t_k)} I_k(s)ds - \int_0^1 e^{-L(t)} G(t,u)dt \\
&\geq \frac{1}{2}\|u\|_X^2 - M_1 \sum_{k=1}^{p-1} e^{-L(t_k)} \left(b'_k M_1^{\gamma'_k} \|u\|_X^{\gamma'_k+1} + a'_k \|u\|_X\right) \\
&\quad - M_1 \int_0^1 e^{-L(t)}dt\left(r_2 M_1^{\mu}\|u\|_X^{\mu+1} + r_1\|u\|_X\right).
\end{aligned}
\tag{3.20}
$$

Hence

$$\inf_{u\in\partial B_A}\varphi(u) \geq \frac{1}{2}\|u\|_X^2 - M_1\sum_{k=1}^{p-1}e^{-L(t_k)}\left(b'_k M_1^{\gamma'_k}\|u\|_X^{\gamma'_k+1} + a'_k\|u\|_X\right) - M_1\int_0^1 e^{-L(t)}dt\left(r_2 M_1^{\mu}\|u\|_X^{\mu+1} + r_1\|u\|_X\right). \tag{3.21}$$

In view of (H8), take $A = A_1 > 0$, we have $\inf_{u\in\partial B_{A_1}}\varphi(u) > 0$, for any $u \in \partial B_{A_1}$. So $\varphi(u_0) < \varphi(0) = 0 < \inf_{u\in\partial B_{A_1}}\varphi(u)$.

Next we will verify that there exists a u_1 with $\|u_1\|_X > A_1$ such that $\varphi(u_1) < \inf_{u\in\partial B_{A_1}}\varphi(u)$. Let $\xi \in \mathbb{R}\setminus\{0\}, B(t) = 1$. Then by (3.10) and (H5), we have

$$\begin{aligned}\varphi(\xi B) &= \frac{\xi^2}{2}\int_0^1 e^{-L(t)}q(t)dt - \sum_{k=1}^{p-1}e^{-L(t_k)}\int_0^{\xi}I_k(s)ds - \int_0^1 e^{-L(t)}G(t,\xi)dt \\ &\leq \frac{\xi^2}{2}\int_0^1 e^{-L(t)}q(t)dt - C_6|\xi|^{\beta}\sum_{k=1}^{p-1}e^{-L(t_k)} + C_7\sum_{k=1}^{p-1}e^{-L(t_k)} \\ &\quad + \left(r_2|\xi|^{\mu+1} + r_1|\xi|\right)\int_0^1 e^{-L(t)}dt.\end{aligned} \tag{3.22}$$

Since $\beta > 2, 0 \leq \mu < 1$, we have $\lim_{|\xi|\to+\infty}\varphi(\xi B) = -\infty$. Therefore, there exists a sufficiently large $\xi_0 > 0$ with $\|\xi_0 B\|_X > A_1$ such that $\varphi(\xi_0 B) < \inf_{u\in\partial B_{A_1}}\varphi(u)$. Set $u_1 = \xi_0 B$, then $\varphi(u_1) < \inf_{u\in\partial B_{A_1}}\varphi(u)$. So by Theorem 2.2, there exists $u_2 \in X$ such that $\varphi'(u_2) = 0$. Therefore, u_0 and u_2 are two critical points of φ, and they are classical solutions of IBVP (1.1). □

Remark 3.5. Obviously, if g is a bounded function, in view of Theorem 3.4, we can obtain the same result.

Theorem 3.6. *Suppose that (H4) and (H5) hold. Then IBVP* (1.1) *has at least one solution.*

Proof. The proof is similar to that in [19], and we omit it here. □

Corollary 3.7. *Suppose that g and impulsive functions $I_k, k = 1,2,\ldots,p-1$ are bounded, then IBVP* (1.1) *has at least one solution.*

4. Some Examples

Example 4.1. Consider the following problem:

$$\begin{gathered}-u''(t) + u'(t) + u(t) = g(t,u(t)), \quad t \neq t_k, \text{ a.e. } t \in [0,1], \\ -\Delta\left(u'(t_k)\right) = I_k(u(t_k)), \quad k = 1,2, \\ u'(0^+) = u'(1^-) = 0,\end{gathered} \tag{4.1}$$

where $g(t,u) = 4u^3 + 6tu^5$, $I_k(u) = u^3$.

Obviously, $g(t,u), I_k(u)$ are odd on u. Compared to IBVP (1.1), $p(t) = 1$, $q(t) = 1$, $r(t) = 1$, $k = 2$. By simple calculations, we obtain that $M_1 = \sqrt{2e}$. Let $\beta = 3$, $M = 1$. Clearly, (H1), (H2) are satisfied. Applying Theorem 3.1, IBVP (4.1) has infinitely many classical solutions.

Example 4.2. Consider the following problem:

$$\begin{gathered} -u''(t) + u(t) = g(t,u(t)), \quad t \neq t_k, \text{ a.e. } t \in [0,1], \\ -\Delta\left(u'(t_k)\right) = I_k(u(t_k)), \quad k = 1, \\ u'(0^+) = u'(1^-) = 0, \end{gathered} \tag{4.2}$$

where $g(t,u) = (1/8)u^3 + (1/20)t\sin u, I_k(u) = (1/16)u^{1/3}\cos u$.

Obviously, $g(t,u), I_k(u)$ are odd on u. Compared to IBVP (1.1), $p(t) = 1$, $q(t) = 1$, $r(t) = 0$, $k = 1$. By simple calculations, we obtain that $M_1 = \sqrt{2}, e^{-L(t)} = 1$. Let $\beta = 3$, $M = 4$, $m = 1/20$, $n = 1/8$, $a_k = 0$, $b_k = 1/16$, $\gamma_k = 1/3$, $p_1 = 3$. Clearly, the first inequalities in (H1), (H3), and (H4) are satisfied. Take $A_0 = 1/2$, then (H7) is also satisfied. Applying Theorem 3.3, IBVP (4.2) has infinitely many classical solutions.

Example 4.3. Consider the following problem:

$$\begin{gathered} -u''(t) + u(t) = g(t,u(t)), \quad t \neq t_k, \text{ a.e. } t \in [0,1], \\ -\Delta\left(u'(t_k)\right) = I_k(u(t_k)), \quad k = 1, \\ u'(0^+) = u'(1^-) = 0, \end{gathered} \tag{4.3}$$

where $g(t,u) = (1/16)u^{1/3}\sin t$, $I_k(u) = (1/2)u^5 + \cos u$.

Compared to IBVP (1.1), $p(t) = 1$, $q(t) = 1$, $r(t) = 0$, $k = 1$. By simple calculations, we obtain that $M_1 = \sqrt{2}, e^{-L(t)} = 1$. Let $\beta = 3$, $M = 2$, $r_1 = 0$, $r_2 = 1/16$, $\mu = 1/3$, $a'_k = 1$, $b'_k = 1/2$, $\gamma'_k = 5$. Clearly, the second inequalities in (H1), (H5), and (H6) are satisfied. Take $A_1 = 1/2$, then (H8) is also satisfied. Applying Theorem 3.4, IBVP (4.3) has at least two classical solutions.

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